Can Actuarially Unfair Tontines be Optimal?

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joint work with
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The one-period tontine

- *Principle:* **collective** investment in which every participant enters with an initial contribution and only participants **surviving at maturity** obtain a payout (sharing the amount in the piggy bank).

Attractiveness of tontines

- Protect against longevity risk.
- As there is no guarantee, their cost (fees) should be not much higher than in the case of a direct investment in the underlying financial product.
- Attractive to anyone with no bequest motive (with respect to the amount invested)!

Optimality of Tontines

If the payout of the tontine is

$$\text{Value of Underlying Investment} + \text{Mortality Credit}$$

Then the tontine payout for survivors will always dominate the payout of the underlying (personal) investment, and a tontine is then always the **preferred** investment for investors **with no bequest motive.**

Proposition

Assuming a non-negative mortality credit, the payout of a tontine strictly dominates the payout of the underlying investment in the sense of first order stochastic dominance, conditional on survival.

The one-period tontine

Homogeneous Case : all *identical* participants:
- same age/gender
- same initial contribution c_0,

At maturity T, N_S survivors (set **S**), N_D premature deaths (set **D**)

$$V_i = \begin{cases} c + \frac{1}{N_S} c N_D & \text{if } i \in \mathbf{S}, \\ 0 & \text{if } i \notin \mathbf{S} \ \& \ \mathbf{S} \neq \emptyset, \\ c & \text{if } \mathbf{S} = \emptyset. \end{cases}$$

Each participant receives the value of her own investment $c = c_0(1+r)^T$ and an equal proportion of cN_D, the total amount of money left in the tontine by the N_D dead participants (cN_D shared equally among the N_S survivors).

General Tontine Payout (heterogeneous)

- Sharing is not anymore equal among all survivors: Let $\kappa_i^* > 0$ determine the sharing rule for participant i.
- **S**: set of survivors and **D**: set of premature deaths.

$$V_i = \begin{cases} c_i + \dfrac{\kappa_i^*}{\sum_{j \in \mathbf{S}} \kappa_j^*} \displaystyle\sum_{j \in \mathbf{D}} c_j & \text{if } \; i \in \mathbf{S} \; , \\ 0 & \text{if } \; i \notin \mathbf{S} \;\&\; \mathbf{S} \neq \emptyset \; , \\ c_i & \text{if } \; \mathbf{S} = \emptyset \; , \end{cases}$$

where

⇒ $c_i = c_{0,i}(1+r)^T$ is the value at time T of the underlying investment (we consider that the initial contribution $c_{0,i}$ accumulates at the risk-free rate r),

⇒ $\sum_{j \in \mathbf{D}} c_j$ is the amount of money to be shared among survivors.

Outline of the talk & Contributions

1. A characterization of the Sabin rule as the only sharing rule that is *robust to heterogeneity in mortality*.
2. A proposal for a new sharing rule aiming at equal utility for each participant.
3. Existence of such tontine requires the presence of a social planner imposing constraints : in a competitive market only the tontine with Sabin rule exists.

Some Literature Review

- Piggott, Valdez, Detzel (2005), Valdez, Piggott, Wang (2006)
- Stamos (2008)
- Sabin (2010), Sabin and Forman (2016), Füllmer and Sabin (2019)
- Milevsky and Salisbury (2015, 2016), Milevsky (2015, 2020, 2022)
- Donnelly, Guillen, Nielsen (2014), Donnelly (2015), Donnelly and Young (2017), Bernhardt and Donnelly (2019, 2021)
- Qiao and Sherris (2013)
- Denuit, Hieber and Robert (2022)
- An Chen and co-authors: Chen, Hieber, Klein (2019), Chen, Guillen and Rach (2021), Chen, Nguyen and Sehner (2022), Chen, Qian, Yang (2021).

Sharing rule

Assumption

V_i is proportional to the accumulated value c_i for all surviving participants.

Thus, each κ_i^* needs to be linear in the accumulated contribution c_i, and we can then use the re-parametrization $\kappa_i^* = \kappa_i c_i$ to make the linearity of the payout with c_i explicit. Upon survival of i,

$$V_i = c_i \left[1 + \frac{\kappa_i}{\sum_{j \in \mathbf{S}} \kappa_j \, c_j} \sum_{j \in \mathbf{D}} c_j\right].$$

Assumption

The sharing rule solely depends on κ_i, which is a **non-increasing function** of the survival probability p_i of the i-th participant.

Sabin sharing rule

Actuarial Fairness

An investment is actuarially fair if at maturity T an initial investment c_0 generates a payoff V_T such that

$$\mathrm{E}[V_T] = c_0(1+r)^T.$$

A natural candidate is then

$$\kappa_i := \frac{q_i}{p_i},$$

where $q_i := 1 - p_i$. We label it the “Sabin rule” (Sabin (2010)).

The Sabin Rule: properties (1/2)

- **Asymptotic Actuarial Fairness**
  In the limit case of a large number of participants, the tontine payout mimicks a one-period life annuity.

$$V_i \approx \frac{c_i}{p_i} I_i,$$

in which $I_i = 1$ if i survives (so that $E[I_i] = p_i$).

The Sabin Rule: properties (2/2)

- **Asymptotic indifference to heterogeneous mortality**

Proposition

Each participant in an heterogeneous tontine with the Sabin sharing rule gets asymptotically the same payout as if she was in a single cohort tontine (containing those participants who are having the same survival probability as she has). The payout of a single cohort tontine V_i^{SC} is such that κ_j is constant for all j:

$$V_i^{SC} = \begin{cases} c_i \left[\dfrac{\sum_{j=1}^{N} c_j}{\sum_{j=1}^{N} c_j \, I_j} \right] I_i & \text{if } \mathbf{S} \neq \emptyset \, , \\ c_i & \text{if } \mathbf{S} = \emptyset \, , \end{cases}$$

The Sabin Rule: properties (2/2)

- **Asymptotic indifference to heterogeneous mortality**

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The Sabin rule is the *only* sharing rule with this property!

The Sabin Rule: drawback

- **Variance of Benefits in a Sabin tontine**

Proposition

Neglecting the payout received when all participants die before maturity, the variance of the tontine payoff per unit investment satisfies

$$\mathrm{Var}\left[\frac{V_i}{c_i}\right] \approx p_i \left(\frac{q_i}{p_i}\right)^2 \mathrm{Var}\left[\frac{D_{tot}}{S_{tot}}\right] + p_i q_i \left(1 + \frac{q_i}{p_i} \mathrm{E}\left[\frac{D_{tot}}{S_{tot}}\right]\right)^2 ,$$

where the term $\frac{D_{tot}}{S_{tot}}$ is the same for all participants and D_{tot} is the amount left by premature deaths: $D_{tot} := \sum_{j \in \mathbf{D}} c_j$ and S_{tot} is defined by $S_{tot} := \sum_{j \in \mathbf{S}} \kappa_j c_j = \sum_{j=1}^{N} \kappa_j c_j I_j$.

- Sabin tontine for 10 groups of 100 participants each of age x_i between 65 and 74 (Belgian best estimate motality tables).

x_i	κ_i	$E[V_i/c_i]$	$Std[V_i/c_i]$	$E[U_E(V_i/c_i)]$
65	5.7	1	0.432	0.764
67	7.0	1	0.478	0.744
69	8.7	1	0.535	0.718
71	11.1	1	0.602	0.686
73	14.4	1	0.687	0.644

- 10^6 simulations to estimate expected value, standard deviation, and of exponential utility (U_E with $\lambda = 2$) of the payout per unit investment.
- **In the Sabin tontine, more risky payoffs do not earn a higher return and as a consequence yield lower utility. Is this fair?**

Equal Utility Tontine

We aim at designing a tontine in which all participants have the **same expected utility** solving $\mathrm{E}[U(V_i/c_i)] = \mathrm{E}[U(V_j/c_j)]$, $\forall$ i,j.

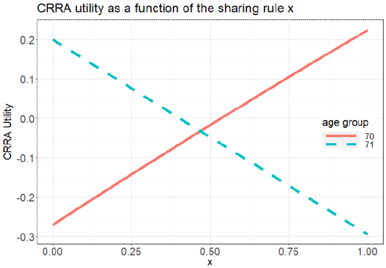
- more generally,

$$\min_{\kappa_i} \quad \sum_i \left(\mathrm{E}[U(V_i/c_i)] - \overline{U}\right)^2 \tag{1}$$

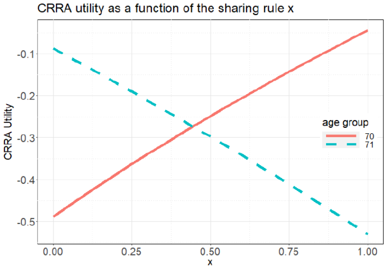
  where $\overline{U} = \sum_j \mathrm{E}[U(V_j/c_j)]/N$ is the average expected utility among participants for some utility function U that is increasing and continuous over $\mathbb{R}^+$.

Issue:

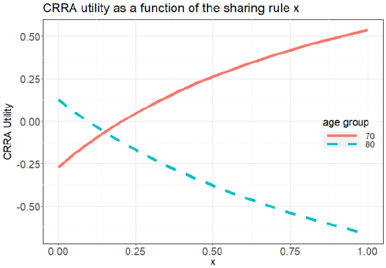
- Can be solved numerically, but the solution may (sometimes) exhibit negative κ_i : some cohorts will then get less than what they receive by investing in the underlying financial product (in our case the risk free investment)! Unacceptable...



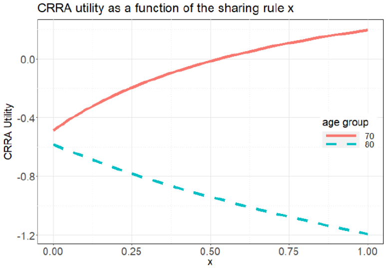
(a) $\gamma = 0.1$



(b) $\gamma = 0.5$



(c) $\gamma = 0.1$



(d) $\gamma = 0.5$

Figure 1: CRRA utility for a tontine scheme composed by two groups with different age difference of 1 year in (a) and (b), and 10 years in (c) and (d). The risk aversion parameter γ is 0.1 in (a) and (c) and 0.5 in (b) and (d).

Constrained Utility tontine

We aim at designing a tontine in which all participants have the **similar expected utility under some some constraints** so that we still ensure optimal participation in the tontine.

- Add a constraint on the values of the sharing rule κ_i: $a\kappa_i^{Sabin} \leqslant \kappa_i \leqslant b\kappa_i^{Sabin}$, where a and b can be freely set, for instance to $a = 0.9$ and $b = 1.5$.

$$\min_{\kappa_i} \quad \sum_i \left(\mathrm{E}\left[U(V_i/c_i)\right] - \overline{U} \right)^2$$
$$\text{s.t.} \quad a\kappa_i^{Sabin} \leqslant \kappa_i \leqslant b\kappa_i^{Sabin}.$$

Issue:

- **Cannot be offered in a competitive market in which a tontine with the Sabin rule is also offered**.

Market with Sabin tontine and Utility tontine

We assume that two tontines are offered on the market:

- A tontine that has the property that participants are asymptotically indifferent to mortality heterogeneity (in other words, they get same payoff as they would in a single cohort tontine), that is the Sabin tontine.
- Another tontine in which at least some participants benefit from pooling with other generations, e.g., the equal utility tontine.

Proposition

All participants in the market will ultimately choose the Sabin tontine (or will be indifferent to be in the Sabin tontine).

Conclusion

- The Sabin tontine does suffer from some **unfairness** in terms of risk exposure of each participant in the tontine.
- Nevertheless, ensuring equal risk exposure (equal expected utility, equal variance...) **can only be done by external intervention** (making the subscription in the tontine mandatory) and with the goal in mind to impose some subsidies from one generation to another.

Thank you for listening !

Do not hesitate to contact me to get updated working papers!

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