Optimal consumption and annuity equivalent wealth with mortality uncertainty

Yang Shen

School of Risk and Actuarial Studies and CEPAR
University of New South Wales, Sydney, Australia

Based on joint work with Zhengming Li (Purdue U) and Jianxi Su (Purdue U)

32nd Colloquium on Pensions and Retirement Research
27-28 November, 2024

Outline

Annuity Puzzle–Conflict between theory and practice

- Yaari (Econometrica, 1965): If there is no bequest motive, then the rational investor should convert all the savings into an actuarially fair annuity upon retirement.
- Davidoff et al. (AER, 2005): Under relaxed model assumptions, a substantial allocation of retirement savings to life annuities is expected.
- Benartzi et al. (JEP, 2011): Very few retirees choose to voluntarily annuitize a substantial portion of their retirement savings.

Annuity Puzzle–Conflict between theory and practice

- According to the Investment Company Fact Book published by the Investment Company Institute, as of year-end 2022, the U.S. retirement assets totaled $33.6 trillion, with less than 7% of assets held as annuity reserves.

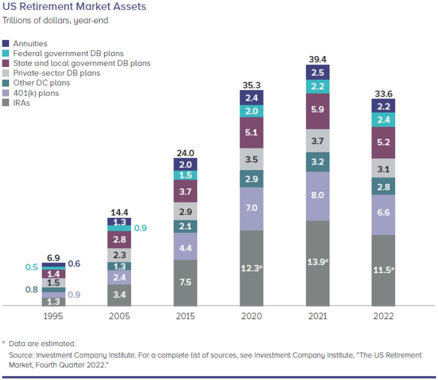

Figure: US Retirement Market Assets

Slogan: Buy annuity and have a good sleep!

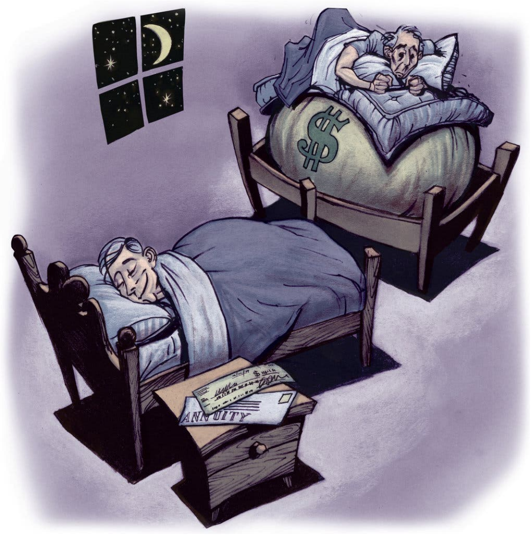

Figure: David G. Klein Illustration for *The Annuity Puzzle* by Richard Thaler

Explanations

- Decreased asset liquidity: Pang and Warshawsky (IME, 2010); Peijnenburg et al. (EJ, 2017)
- Lack of bequest motivates: Lockwood (RED, 2012)
- Incomplete annuity market: Horneff et al. (JEDC, 2008); Koijen et al. (RoF, 2011)
- Unfair annuity pricing: Mitchell et al. (AER, 1999)
- Default risk of annuity providers: Agnew et al. (AER, 2008)

Mortality risk, longevity risk, stochastic mortality, and mortality uncertainty

- Mortality risk: The remaining lifetime is random.
- Longevity risk: Risk of living longer than expected.
- Stochastic mortality: The mortality rate itself is stochastic.
    1. Individual level: Stochastic mortality caused by health shocks can partly explain annuity puzzle (Reichling and Smetters, AER, 2015).
    2. Population level: Longevity improvement has made annuity puzzle even more puzzling.
- Mortality uncertainty: No matter the mortality rate is stochastic or deterministic, it is uncertain due to limited knowledge, imperfect or unknown information. Recall Knightian uncertainty (Knight, 1921) and Ellsberg paradox (Ellsberg, 1962).

Motivating questions

1. Certainty: Would you buy annuity if you can certainly live to 100 years old?
2. Risk: Would you buy annuity if you can live to 100 years old with a known chance? (Known unknown)
3. Uncertainty: Would you buy annuity if you can live to 100 years old but with an unknown chance? (Unknown unknown)

Stochastic environment

- Stochastic basis: A complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{P}$ is a reference probability measure.
- Individual at the age of y, i.e., (y).
- Remaining lifetime of (y): τ_y (a nonnegative random variable).
- Force of mortality: $\lambda_{y+s} := \lambda_y(s) : \mathbb{R}_+ \to \mathbb{R}_+$ (a nondecreasing, deterministic function in $s \in \mathbb{R}_+$).
- Survival probability:

$$
{}_tp_y = \mathbb{P}[\tau_y > t] = \exp\left(-\int_0^t \lambda_{y+s} ds\right).
$$

- Death density:

$$
f_{\tau_y}(t) = \lambda_{y+t} \exp\left(-\int_0^t \lambda_{y+s} ds\right).
$$

Recursive utility

- Epstein-Zin recursive utility with mortality risk

$$J(t) = \mathbb{E}_t\left[\int_t^{\infty} f\left(c(s), \alpha_s, J(s)\right) ds\right]$$

with actuarial subjective discount factor

$$\alpha_t = \rho + \lambda_{y+t}$$

and normalized aggregator of consumption and utility

$$f(c, \alpha, v) = \frac{(1-\gamma)\, v}{1 - 1/\phi}\left[\left(\frac{c}{\left((1-\gamma)v\right)^{\frac{1}{1-\gamma}}}\right)^{1-1/\phi} - \alpha\right],$$

where $\gamma > 0$ is the relative risk aversion coefficient, and $\phi > 0$ is the elasticity of intertemporal substitution (EIS) coefficient.

- If $\phi = 1/\gamma$, then the recursive discounted utility reduces an additive discounted utility:

$$J(t) = \mathbb{E}_t\left[\int_t^{\infty} e^{-\int_t^s \alpha_u du} \times \frac{c(s)^{1-\gamma}}{1-\gamma} ds\right].$$

Complete annuity market

- In a complete annuity (CA) market which refers to the availability of a complete set of annuities at actuarially fair prices and with any maturities, the rational retiree will convert all the retirement savings into an annuity.
- Initial retirement saving:

$$x_0 = \mathbb{E}\left[\int_0^{\tau_y} e^{-rs} c_A(s) ds \right] = \int_0^{\infty} e^{-\int_0^s (r+\lambda_{y+u}) du} c_A(s)\, ds.$$

- Time-t actuarial present value of the future annuity payments:

$$X_A(t) = \mathbb{E}_t\left[\int_t^{\tau_y} e^{-r(s-t)} c_A(s)\, ds \right] = \int_t^{\infty} e^{-\int_t^s (r+\lambda_{y+u}) du} c_A(s)\, ds,$$

which satisfies

$$dX_A(t) = \big((r+\lambda_{y+t}) X_A(t) - c_A(t)\big)\, dt, \qquad X_A(0) = x_0.$$

Complete bond market

- In analogy to the CA market, it is the complete bond (CB) market wherein pure discount bonds are available for all maturities but annuities are absent.
- Initial retirement saving:

$$x_0 = \int_0^\infty e^{-r\,s}\, c_B(s)\, ds.$$

- Time-t actuarial present value of the future bond payments:

$$dX_B(t) = \big(rX_B(t) - c_B(t)\big)\, dt, \qquad X_B(0) = x_0.$$

Mortality uncertainty

- Alternative probability measures:

$$\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left\{ \int_0^{t\wedge\tau_y} \big[\log(\theta(s)) - \theta(s) + 1\big]\,\lambda_{y+s}\,ds + \int_0^t \log(\theta(s))\,dZ(s)\right\},$$

  where $Z(s) := \mathbf{1}_{\{\tau_y \le s\}} - \int_0^s \mathbf{1}_{\{\tau_y > u\}}\lambda_{y+u}du$ is a martingale associated with the single jump process $\mathbf{1}_{\{\tau_y \le s\}}$. Refer to Shen and Su (NAAJ, 2019).

- $\mathbb{Q}$ dynamics:
  1. Subjective mortality intensity: $\lambda^{\mathbb{Q}}_{y+t} = \theta(t)\,\lambda_{y+t}$.
  2. Actuarial subjective discount factor: $\alpha^{\mathbb{Q}}_t = \rho + \theta(t)\,\lambda_{y+t}$.
  3. Survival probability: ${}_tp^{\mathbb{Q}}_y := \mathbb{Q}[\tau_y > t] = \exp\left(-\int_0^t \theta(s)\lambda_{y+s}ds\right)$.

Relative entropy and penalty

- Relative entropy:

$$\mathcal{D}(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}\left[\log\left(\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right)\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t\wedge\tau_y} g(\theta(s))\lambda_{y+s}ds\right],$$

where $g(\theta) := \theta\,\log\theta - \theta + 1$.

- Penalty:

$$\frac{1}{\psi}\times\Gamma(t,\theta) = \frac{1}{\psi}\times\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty}\frac{1-\gamma}{(1-1/\phi)^2}\,J(s)\,g\big(\theta(s)\big)\,\lambda_{y+s}ds\right],$$

where $\psi > 0$ is the ambiguity aversion parameter on the subjective mortality model.

Statement of problem

- Objective:

$$J(c,\theta;\ t,x) := \mathbb{E}^{\mathbb{Q}}_{x,t}\left[\int_t^\infty f\left(c(s),\alpha^{\mathbb{Q}}_s,J(s)\right)ds\right] + \frac{1}{\psi}\times\Gamma(t,\theta).$$

- Problem: Finding optimal consumption strategy and perturbation function in the robust framework

$$V(t,x) = \max_{c\in\mathfrak{C}}\ \min_{\theta\in\mathfrak{T}} J(c,\theta;\ t,x), \qquad t>0 \text{ and } x>0,$$

where $\mathfrak{C}$ and $\mathfrak{T}$ are the admissible spaces for consumption strategies and perturbation functions.

Annuity equivalent wealth

- The annuity equivalent wealth (AEW) can be computed by solving

$$V_A(x_0) = V_B(\mathrm{AEW}),$$

where V_A and V_B denote the value functions under the CA and CB markets, respectively.

- AEW indicates the amount of initial wealth needed in order to compensate the absence of annuity in the CB market. It quantifies the utility indifference price of mortality/longevity risk pooling.
- Milevsky and Huang (NAAJ, 2018): the utility value of longevity risk pooling $\delta = \frac{\mathrm{AEW}}{x_0} - 1$.

Optimal strategies

Theorem 1.

The worst-case perturbation function associated with the optimization problem can be computed via

$$\theta_A^*(t\,;\psi) = \theta_B^*(t\,;\psi) \equiv \theta^*(\psi) = \exp\big(\psi(1 - 1/\phi)\big) \qquad \text{for all} \quad t > 0,$$

where $\psi > 0$ *is the robust preference parameter and* $\phi > 0$ *is the EIS coefficient. The robust optimal consumption strategy is*

$$c_\square^*(t;\psi) = \frac{X_\square^*(t)}{K_\square(t;\psi)} = c_\square^*(0;\psi) \times \exp\left\{ \int_0^t \left[\widetilde{G}_\square(\psi)\,\lambda_{y+u} - \phi(\rho - r)\right] du \right\},$$

where "□" can be either "A" or "B", $X_\square^*$ *denotes the optimal wealth associated with* $c_\square^*$ *and* $\theta_\square^*$*, and*

$$K_\square(t;\psi) = \int_t^\infty \exp\left\{ -\int_t^s \Big(\beta + G_\square(\psi)\,\lambda_{y+u}\Big) du \right\} ds,$$

Optimal strategies (cont'd)

Theorem 1 (cont'd).

is assumed to be bounded, for any $t \geq 0$. The parameter $\beta = (1-\phi)\, r + \phi\, \rho$ is the weighted average between the risk-free interest rate and subjective discount rate. Moreover, we have

$$\widetilde{G}_A(\psi) = 1 - G_A(\psi), \qquad \widetilde{G}_B(\psi) = -G_B(\psi),$$

$$G_A(\psi) = (1-\phi) + G_B(\psi), \qquad G_B(\psi) = \phi\, \theta^* + \frac{\phi^2}{\psi(1-\phi)}\, g(\theta^*).$$

The value function can be computed via

$$V_\square(t, x\,; \psi) = \Big[K_\square(t\,; \psi)\Big]^{-\frac{1-\gamma}{1-\phi}} \frac{x^{1-\gamma}}{1-\gamma}.$$

Implications

- Optimal perturbed mortality curve:

$$\lambda^*_{y+t} = \theta^* \times \lambda_{y+t} = e^{\psi(1-1/\phi)} \times \lambda_{y+t}.$$

- The larger the value of ψ is, the larger the value of $|\theta^* - 1|$ becomes.
- If the EIS $\phi < 1$ (empirically supported by Yogo, RES, 2004), then $\theta^* < 1$, meaning that the worst-case perturbed probability measure corresponds to an improved mortality (longevity risk) scenario.
- Otherwise, if the EIS $\phi > 1$ (empirically supported by Bansal and Yaron, JoF, 2004), then $\theta^* > 1$, meaning that the worst-case perturbed probability measure corresponds to a deteriorated mortality (mortality risk) scenario.

Implications (cont'd)

- If $1/\phi = \gamma$, then the optimal perturbation functions become

$$\theta^* = e^{\psi(1-\gamma)}.$$

  Shen and Su (NAAJ, 2019) show that if $\gamma > 1$ (resp. $\gamma < 1$) the worst-case perturbed mortality curve corresponds to a longevity risk (resp. mortality risk) scenario.
- When $\phi = 1/\gamma = 1$ which corresponds to the log utility case, the worst-case perturbation function equals unity, i.e. $\theta^* = 1$.

Subjective mortality model

We calibrate the Gompertz mortality model $\lambda_y^{\mathrm{GM}} = w_1 \exp(w_2\, y)$ into the 2015 - 2019 U.S. mortality table extacted from the Human Mortality Database. The parameters are estimated to be $\widehat{w}_1 = 5.01 \times 10^{-5}$ and $\widehat{w}_2 = 8.39 \times 10^{-2}$ for female, and $\widehat{w}_1 = 8.10 \times 10^{-5}$ and $\widehat{w}_2 = 8.25 \times 10^{-2}$ for male.

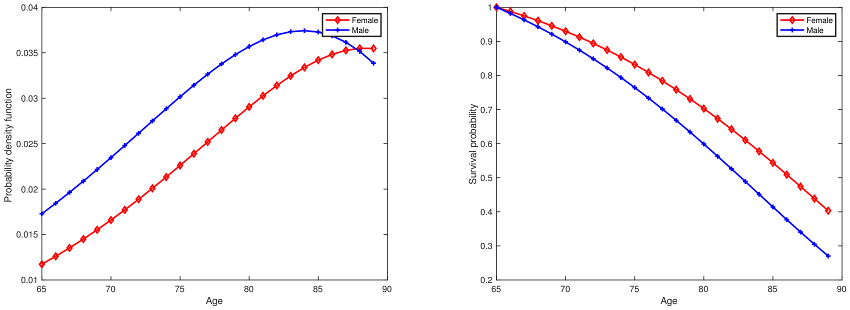

Figure: Probability density function (left panel) and survival probability function (right panel) of the retiree's remaining life time RV τ_{65}.

Implications (cont'd)

- Annuity equivalent wealth, $\mathrm{AEW}(\psi)$:
  1. Since $G_A(\cdot) = (1-\phi) + G_B(\cdot)$ implies $\left[K_A(\cdot)\right]^{1/(1-\phi)} \geq \left[K_B(\cdot)\right]^{1/(1-\phi)}$, $\mathrm{AEW}(\psi)$ is always greater than x_0.
  2. If the EIS $\phi < 1$ (resp. $\phi > 1$), then $\mathrm{AEW}(\psi)$ is increasing (resp. decreasing) in $\psi > 0$.
  3. Mortality uncertainty may be a potential contributor to the enduring annuity puzzle.

Sensitivity analysis of AEW in response to ψ

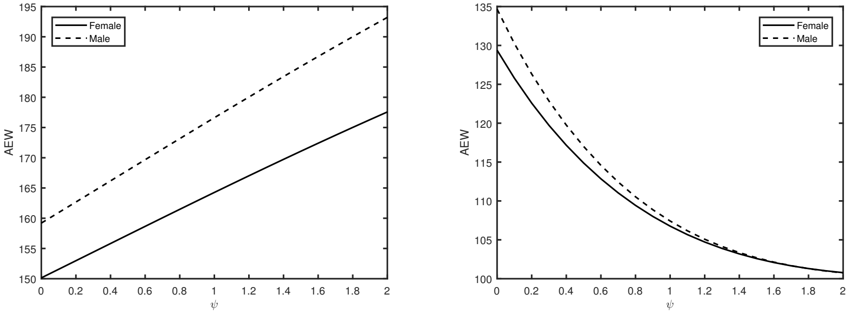

Figure: The AEW with varying ψ when $x_0 = 100$, the EIS $\phi = 0.5$ (left panel) and $\phi = 1.5$ (right panel).

Conclusions

- Proposed and studied a revamped LCM in which there is an incorporation of mortality model uncertainty.
- Derived the optimal robust consumption rules and worst-case mortality curves as well as the associated AEW in explicit forms.
- Key findings
  1. For a typical retiree having EIS smaller than one, the worst-case perturbed mortality curve corresponds to an improved mortality scenario.
  2. Mortality ambiguity aversion will lower the optimal consumption-to-wealth ratio.
  3. If mortality uncertainty is ignored by the retiree, then the value of annuity will be under-estimated, causing a lower than expected annuity demand.
  4. Our main results still hold under generalized model settings.
- Future research: i) stochastic mortality with uncertainty; ii) life annuity $\rightarrow$ deferred annuity/VA/tontine; iii) health model uncertainty.

Thank you for your attention

Q&A

Email: y.shen@unsw.edu.au

Homepage: www.unsw.edu.au/staff/yang-shen