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Abstract

Existing longevity indices commonly use age–based mortality rates or period life expectancy. We propose an alternative cohort–based value index for insurers and pension funds to manage longevity risk. This index is an expected present value of a longevity linked cash flow valued using a specified cohort mortality model and a commonly used interest rate model. Since interest rate and longevity risk are inherent with any longevity linked obligation and interest rate risk can be effectively hedged, this index will provide a better measure of the longevity risk than current indices. Current mortality models are largely age–period based, so we develop a cohort based stochastic mortality model with age–dependent model parameters that provides realistic cohort correlation structures as an underlying basis for the value index. We show how the model improves fitting performance compared to other cohort models, particularly for very old ages, and has a familiar model formulation for financial market participants. We also demonstrate the hedge effectiveness of the index.

Keywords: Cohort mortality, Value index, Mortality risk, Interest rate risk, Hedge efficiency

JEL Classification: C63, G22, G12

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1 Introduction

1.1 Background and Motivation

For defined benefit (DB) pension plans and insurance companies with significant annuity policies, uncertain future longevity improvement has become a risk that has to be quantified and managed. For example it is estimated that the value of UK pension liabilities increases by 3 to 4 percent with each additional year of life expectancy. Other factors have contributed to the increasing importance of longevity risk, including lower investment returns and regulatory changes (e.g. Solvency II capital requirement).

Longevity risk is now well recognized as a significant risk that has had limited success in being transferred into financial markets using standardized contracts. Traditional participants in the longevity market include DB pension funds, insurers and re–insurers. Pension funds have a negative exposure to longevity risk because the value of their liabilities increases with life expectancy. Life insurance companies with both life and annuity business have relatively flat exposure to longevity risk, with annuity portfolios offsetting insurance policies (Loeys et al. 2007). The market has overall negative exposure to longevity improvements. Re–insurers have a limited capacity and willingness to accept this risk (Wadsworth 2005). Capital markets have the depth, capacity and experience in risk hedging to hedge longevity risk effectively (Blake et al. 2009). Financial markets have a long history in innovative financial products to manage risk including equity, interest rate, credit and commodity risks.

Longevity risk management products have been developed by the capital markets since 2008[1], including customized, indemnity–based hedges and index–based hedges (e.g. q–forward, s–forward, longevity swaps). Indemnity–based hedges, such as longevity swaps offered by re–insurers, have been actively used by pension funds and life insurers. Basis risk arises for index based contracts because of differences between the mortality of the underlying lives being hedged and the index used for the hedge contract. This can result from differences in geographic location, gender, age or socio–economic class for the lives and the index. For indemnity–based hedges, the mortality experience of the lives is transferred and basis risk is zero.

Financial market contracts are based on standardized indices and basis risk is an inherent factor to consider. Index–based hedging contracts using publicly available national population data are suitable for financial market investors since this reduces the need for investors to have detailed knowledge of the lives in pension funds or insurance companies. Information asymmetry where different parties involved in a transaction do not have the same level of information, increases the effective costs and risk for an index–based hedge. Index–based hedges have the advantage of increased liquidity and relative ease to trade which can also reduce the hedging costs. Basis risk has been demonstrated to be limited because of the similarity in the mortality improvements of different populations. Coughlan et al. (2011), Cairns et al. (2014) show that despite differences in demographic profiles, basis risk is substantially reduced due to high correlations in mortality improvements, particularly for a long hedge horizon, between the underlying lives and the hedging population used for the index. Basis risk for longevity indices is also reduced by using a range of indices for different country, gender, cohort and/or socio–economic

[1]The first index–based hedge, a q–forward based on the J.P. Morgan's LifeMetrics longevity index, was executed in January 2008 by the UK pension insurer Lucida. The first indemnity–based longevity swap was entered in July 2008 by Canada Life with J.P. Morgan as the counterparty.

class. Ngai and Sherris (2011) also show how static hedging with longevity bonds and q–forwards based on Australian mortality can be effective for longevity risk for immediate life annuities, but also highlight the impact of interest rate risk.

Two important longevity indices are J.P. Morgan's LifeMetrics (launched in 2007 and transferred to Life and Longevity Markets Association in 2010) and Deutsche Börse Xpect–Club Vita Index since 2010. LifeMetrics consists of three underlying components: crude central mortality rates, graduated initial mortality rates and period life expectancy levels (Coughlan et al. 2007). The index data are classified in terms of country, gender and cohort. Deutsche Börse adds one more dimension to its Xpect Cohort index: the pension amount received, which aims to capture the heterogeneity of socio–economic classes within a particular population (Deutsche Börse 2012). The underlying component of the Xpect Cohort index is the number of survivors for a defined cohort group. These current longevity indices (i.e. LifeMetrics and Xpect–Club Vita Index) are based upon the mortality experience of a given population. They aim to provide a basis for hedging only mortality, or longevity, risk and to do this on a cash flow hedging basis by using the index to match the liability cash flow being hedged. Our motivation is to develop a value based longevity index, including a cohort based model of mortality for constructing the index. Longevity risks reflect in changes in the present value of future longevity linked payments. A value based index is an effective way to quantify longevity risk (Sherris and Wills 2008). Cash–flow hedges aim to match the hedger's liability cash flows and is suited to a static hedging approach. A value based hedging approach can be used in a dynamic approach to hedging. Value based hedging underlies the capital charge introduced by Solvency II, which is based on the change in net asset value. Interest–rate risk is also included in a value–based longevity index. Since interest rate risk is actively traded, this can be readily taken into account in a value based longevity hedging strategy using the value–based longevity index.

The paper is organized as follows. In Section 2 we outline the construction of the value based longevity index. Section 3 presents an analysis of cohort mortality data as a basis for the cohort mortality model proposed for the longevity index. We use Australian male population data to illustrate the implementation of the value–based cohort longevity index. Data for other countries and genders are available for many countries in the Human Mortality Database and this allows the calibration of the proposed index for all these countries. We then estimate and validate a cohort based mortality model in Section 4. This is used to estimate forecast survival probabilities up to the age of 120 for a cohort. The model allows for cohort dependence and age based trends and volatilities allowing consistent application to multiple cohorts. We also use a standard, well accepted model for the term structure of interest rates and this is presented and calibrated in Section 5. Section 6 presents an application of the value based cohort longevity index and shows its hedge effectiveness in hedging longevity linked liabilities. Finally Section 7 concludes.

2 The Value Based Index

The value–based longevity index proposed is the expected present value of a standardized annual payment of a unit of longevity indexed income to a group of lives currently 65–years old payable at the end of each year. Payments are based on the expected survivors in a cohort generated from a proposed stochastic cohort mortality model. The oldest age is assumed to be 120.

The index requires a model for mortality and interest rates to provide the expected number of survivors and discount factors used to calculate the PV of the immediate annuity as

$$PV_0^{65} = \sum_{i=1}^{\tau} n_i DF_i, \tag{1}$$

where the superscript refers to cohort aged 65 and $\tau = 55$. n_i is the expected number of survivors in the annuitant group at the end of year i and DF_i is the discount factor from the term structure model.

The index can also include the values of deferred annuities for a cohort aged 65. For example, PV_{15}^{65} denotes the index value for a deferred annuity that starts 15 years later for current 65–year olds. To compute these annuity values forward survival probabilities and interest rates are used from the initial mortality and interest rate model. The value index provides benchmark values of both an immediate annuity, as well as deferred annuities for a cohort aged 65. Other ages can be used for cohorts in the index since the mortality model used captures cohort dependence.

The value based index will also include interest rate risk. For longevity risk, interest rate risk in the index must be hedged using a series of interest rate swaps (IRS) with the notional amount adjusted each year according to a pre-specified schedule and fixed rate. In the IRS, a fixed rate is received and the floating rate is paid. Fixed rates are determined from the forward interest rates. This hedging leaves only longevity risk in index values. For the first IRS entered at time zero, the notional amount is simply PV_0^{65}. For the remaining hedging dates the notional for the IRS at the beginning of year i is

$$N_i = N_{i-1}(1 + k_i) - n_i, \qquad \forall i \in [1, 2, ..., \tau - 1], \tag{2}$$

where $N_0 = PV_0^{65}$ and k_i is the fixed rate in the swap.

The index uses the well accepted Affine Term Structure Model (ATSM) (Duffie and Kan 1996) as the basis for both mortality and interest rates. The ATSM provides analytical tractability, ease of implementation and the ability to determine forward mortality rates and interest rates for valuation. Special cases of ATSM, such as Vasicek model (Vasicek 1977) and Cox–Ingersoll–Ross (CIR) model (Cox et al. 1985), have analytical solutions for zero–coupon bond prices. Also, these models are designed to capture the time–series property of the term structures rather than the initial cross–sectional property (Bolder 2001). It is assumed that mortality risk and interest rate risk are independent. The index construction requires the calibration of the models to mortality and interest rate data.

Most mortality models are based on age–period data and trends. The value–based cohort index is based on a single cohort initially aged 65 and requires a cohort–based model with stochastic mortality intensity. A number of continuous–time, affine stochastic mortality models have been proposed for a single cohort, e.g. Dahl (2004), Biffis (2005), Schrager (2006) and Luciano and Vigna (2008). Age–period affine mortality models such as Blackburn and Sherris (2013) implicitly assume perfect correlations across multiple cohorts. The value–based longevity index requires an analysis of cohort based data to identify a suitable model for index construction. The main features of the cohort data of relevance are mortality trends and volatility by initial age and cohort correlations.

3 Analysis of Cohort Mortality Data

We use Australian male data from the Human Mortality Database (HMD) for the 1890, 1895, 1900 and 1905 cohorts and for ages 49 to 99. The trends and correlations structure can be readily estimated for other countries data where sufficient cohort data is available. For each cohort, we estimate the continuous–time mortality intensity $\mu(t, x)$ for an individual aged x at t with the crude death rate $m_c(t, x) = \frac{D(t,x)}{E(t,x)}$, where $D(t, x)$ and $E(t, x)$ respectively represents the number of deaths and average population exposure during calendar year t aged x last birthday. We consider mortality rates and changes in mortality rates, defined as $\triangle\mu(t, x) = \mu(t + 1, x + 1) - \mu(t, x)$, giving 50 observations of $\triangle\mu(t, x)$ for each cohort. $\triangle\mu(t, x)$ is required to de–trend the mortality rates.

3.1 Mortality Trends by Cohort

Drift parameters of cohort mortality intensities are expected to vary by age and cohort. To assess age dependence, we group the data into three age groups: 50–64, 65–84, 85–99. This allows us to consider the effects at initial ages 50, 65, and 85. We determine the average of $\triangle\mu(t, x)$ within each group.

Table 1: **Drift of Mortality Intensity By Cohort**

Initial Age	1890	1895	1900	1905
50	0.001425	0.001639	0.001977	0.001800
65	0.007501	0.005820	0.005335	0.005601
85	0.016177	0.017104	0.015258	0.017705

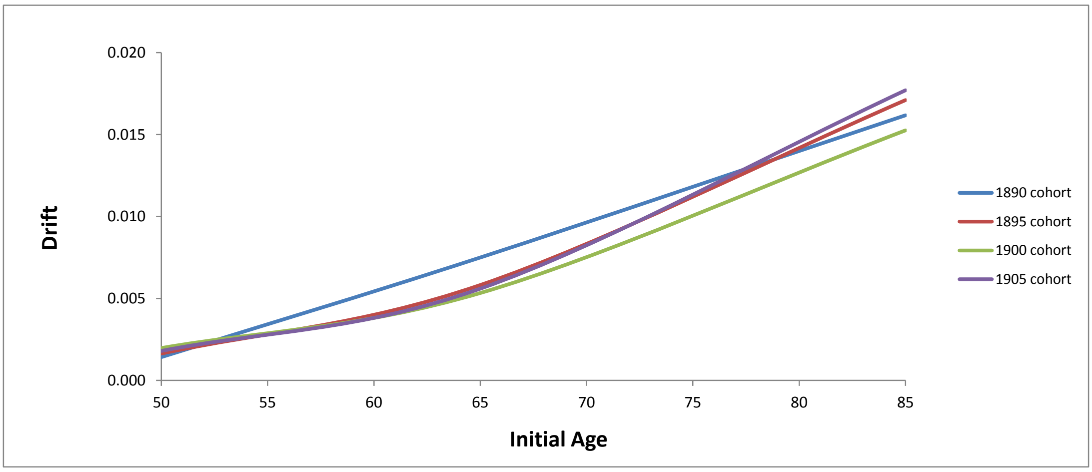

Figure 1: **Drift of Mortality Intensity By Cohort**

Table 1 and Figure 1 clearly show the initial–age dependence of the drift by cohort. Figure 1 plots the drift as a function of the initial age. The figure clearly shows that a linear function of age for the drift fits the cohort data well. The functional form $a + bx$ is proposed for the initial–age dependent drift, where a and b are constants and x is the initial age.

3.2 Volatility of Mortality Intensity

For each cohort, we calculate the standard deviation of $\triangle\mu(t,x)$ within each age group to approximate the continuous–time volatility of mortality intensity. Results summarized in Table 2 show the dependence of volatility on the initial age. From Figure 2 we see that the volatility increases approximately exponentially with age. The functional form $e^{(c+dx)}$ is proposed to represent the age–dependent volatility, where c and d are constants.

Table 2: **Volatility of Mortality Intensity By Cohort**

Initial Age	1890	1895	1900	1905
50	0.001031	0.000965	0.001748	0.001343
65	0.007080	0.005082	0.007610	0.005494
85	0.030253	0.031574	0.033655	0.024272

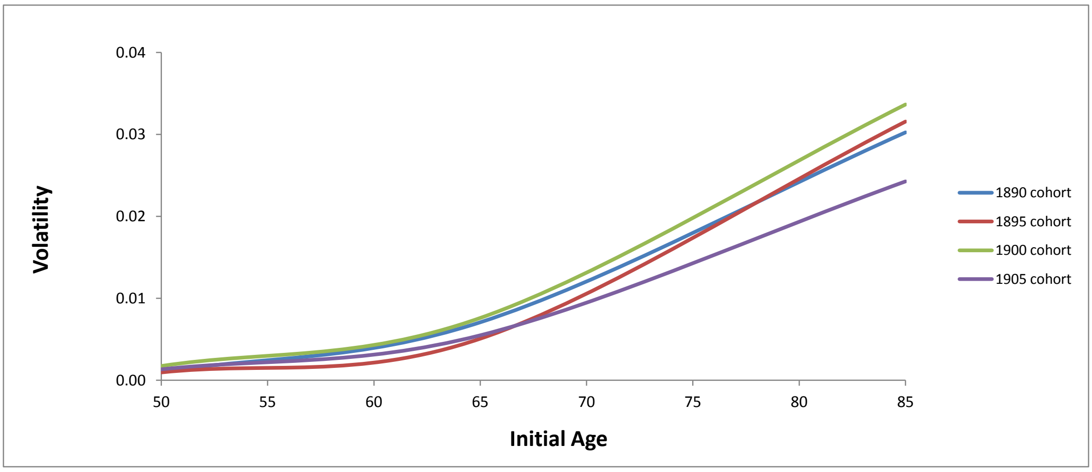

Figure 2: **Volatility of Mortality Intensity By Cohort**

3.3 Cohort Correlations

Cohort correlations are important in calibrating the model. We estimate correlations of $\triangle\mu(t,x)$ across cohorts starting from the same calendar time over a 20 year period. The correlation matrix estimates use a fixed calendar time and a fixed time horizon. We estimate the correlation matrix using 4 different calendar times – 1955, 1960, 1965 and 1970. The correlation matrices are presented in Table 3.

We see that the correlations vary with the calendar time, suggesting that correlations vary with the initial age of the cohort. Calculations can use different approaches with one based on a common calendar time period and the other using a fixed initial age. Empirically cohort correlations show dependence on the initial age.

From the analysis presented of the Australian mortality data we identify the need for age dependent drifts by cohort, with a linear function of age proposed, volatility that increases exponentially with age for any given cohort and correlations that vary by cohort.

Table 3: **20–Year Cohort Correlations**

Calendar Time 1955				
Cohort	1890	1895	1900	1905
1890	1.0000			
1895	0.6124	1.0000		
1900	0.5157	0.4758	1.0000	
1905	0.4271	0.1857	0.5568	1.0000
Calendar Time 1960				
1890	1.0000			
1895	0.4707	1.0000		
1900	0.1489	0.2728	1.0000	
1905	0.2103	0.1968	0.0949	1.0000
Calendar Time 1965				
1890	1.0000			
1895	0.6583	1.0000		
1900	0.2881	0.4585	1.0000	
1905	0.4376	0.4899	0.5200	1.0000
Calendar Time 1970				
1890	1.0000			
1895	0.4468	1.0000		
1900	0.3600	0.4117	1.0000	
1905	0.2401	0.7432	0.6260	1.0000

4 Cohort Based Mortality Model

This section presents details of the proposed mortality model for the value–based longevity index including the parameter calibrations. The data exploration in Section 3 supports the need for an age–dependent, cohort–based mortality model. Other model assumptions are based on those often used for mortality models. A Gaussian factor model is chosen for its analytical tractability and ease of implementation. The model is a two factor model[2]. Although a negative mortality intensity is theoretically possible for Gaussian models, in practice the probability is low (e.g. Brigo and Mercurio 2006).

Jevtic, Luciano and Vigna (2013) present a cohort–based affine mortality model with cohort dependence. The model uses a two–factor Ornstein–Uhlenbeck process with a common factor for all cohorts and a cohort–specific factor. An analysis of the model calibration shows that the cohort correlations are very high. The model also assumes that the correlations are constant and independent of the initial age. The calibrated model has large out–of–sample forecasting errors for old ages (age 80 and beyond). The drift and diffusion coefficients in the model are not age dependent. The common factor and lack of age dependence of the parameters results in high dependence across cohorts. The model does not satisfactorily fit observed survival probabilities over a long horizon and for older ages. The cohort based mortality model that we propose addresses these issues.

[2]We do so in order to keep the model tractable. Calibration results presented later show that the two–factor model fits the observed survival probabilities well.

4.1 Continuous Time Model

The continuous time model is developed for a probability space $(\Omega,\ \mathcal{F},\ \mathcal{F}_t(t \geq 0),\ P)$ which satisfies the usual hypothesis that the filtration $\mathcal{F}_t(0 \leq t \leq T)$ is right continuous with left limits and P is the real–world probability measure. $\mu^i(t,x)$ is a predictable process on this probability space and represents the mortality intensity for individuals aged x at calendar time t of cohort i. Within cohort i, an individual's death time is the first jump time of a Cox process with intensity $\mu^i(t,x)$. The two factors of $\mu^i(t,x)$ respectively follow the SDE's

$$d\mu_1^i(t,x) = \psi_1^i(t,x)\mu_1^i(t,x)dt + \sigma_1^i(t,x)dW_1^i(t), \tag{3}$$

$$d\mu_2^i(t,x) = \psi_2^i\mu_2^i(t,x)dt + \sigma_2^i dW_2^i(t), \tag{4}$$

where

$$\psi_1^i(t,x) = a + bx, \tag{5}$$

and

$$\sigma_1^i(t,x) = e^{(c+dx)}. \tag{6}$$

W_1^i and W_2^i are correlated Brownian motions on P and $dW_1^i dW_2^i = \rho^i dt$. Via the Cholesky decomposition of the correlation matrix of W_1^i and W_2^i we rewrite Eqn. (3) and (4) as

$$d\mu_1^i(t,x) = \psi_1^i(t,x)\mu_1^i(t,x)dt + \sigma_1^i(t,x)dZ_1(t), \tag{7}$$

$$d\mu_2^i(t,x) = \psi_2^i\mu_2^i(t,x)dt + \sigma_2^i\rho^i dZ_1(t) + \sigma_2^i\sqrt{1-(\rho^i)^2}dZ_2(t). \tag{8}$$

In Eqn. (7) and (8) Z_1 and Z_2 are two independent Brownian motions. Once the cohort index i and initial age x are specified, calendar time t is determined by $t = i + x$. As a result, the functional forms of $\psi_1^i(t,x)$ in Eqn. (5) and $\sigma_1^i(t,x)$ in Eqn. (6) implicitly capture dependence on t. ψ_2^i, σ_2^i and ρ^i are assumed to be constant for each cohort and independent of the initial age. We then have the instantaneous mortality intensity of each cohort as[3]

$$\begin{aligned} d\mu(t,x) &= d\mu_1(t,x) + d\mu_2(t,x) \\ &= (\psi_1(t,x)\mu_1 + \psi_2\mu_2)dt + (\sigma_1(t,x) + \sigma_2\rho)dZ_1(t) \\ &\quad + \sigma_2\sqrt{1-\rho^2}dZ_2(t). \end{aligned} \tag{9}$$

We make the assumption that ψ_1 and σ_1 are piecewise constant with respect to each age group and depend on the initial age of the group only. For instance, for the 50–64 age group, $\psi_1 = a + 50b$ and $\sigma_1 = e^{(c+50d)}$. Hence within each age group of each cohort, for each $s > t$, we let $\tau = s - t$ and integrate Eqn. (9) to get

$$\begin{aligned} \mu(s, x+\tau) &= \mu_1(t,x)e^{\psi_1(t,x)\tau} + \mu_2(t,x)e^{\psi_2\tau} + \sigma_1(t,x)\int_t^s e^{\psi_1(t,x)(s-u)}\,dZ_1(u) \\ &\quad + \sigma_2\rho\int_t^s e^{\psi_2(s-u)}\,dZ_1(u) + \sigma_2\sqrt{1-\rho^2}\int_t^s e^{\psi_2(s-u)}\,dZ_2(u). \end{aligned} \tag{10}$$

[3]We drop the cohort index i to ease the exposition.

The survival probability for an individual who is alive at t and aged x, from t to s, is given by[4]

$$
\begin{aligned}
P(t,s) &= E(e^{-\int_t^s \mu(u)\,d(u)} | \mathcal{F}_t) \\
&= \exp\left[\frac{1-e^{\psi_1(t,x)\tau}}{\psi_1(t,x)}\cdot\mu_1(t,x) + \frac{1-e^{\psi_2\tau}}{\psi_2}\cdot\mu_2(t,x) + \frac{1}{2}V(\tau)\right], \qquad (11)
\end{aligned}
$$

where

$$
\begin{aligned}
V(\tau) &= \frac{\sigma_1^2}{\psi_1^2}\left[\tau - \frac{2}{\psi_1}e^{\psi_1\tau} + \frac{e^{2\psi_1\tau}}{2\psi_1} + \frac{3}{2\psi_1}\right] \\
&= \frac{\sigma_2^2}{\psi_2^2}\left[\tau - \frac{2}{\psi_2}e^{\psi_2\tau} + \frac{e^{2\psi_2\tau}}{2\psi_2} + \frac{3}{2\psi_2}\right] \\
&= 2\rho\frac{\sigma_1\sigma_2}{\psi_1\psi_2}\left[\tau - \frac{e^{\psi_1\tau}-1}{\psi_1} - \frac{e^{\psi_2\tau}-1}{\psi_2} + \frac{e^{(\psi_1+\psi_2)\tau}-1}{\psi_1+\psi_2}\right]. \qquad (12)
\end{aligned}
$$

4.2 Calibration Method

We calibrate the model parameters by fitting the survival probabilities given by Eqn. (11) to the empirical survival survival probabilities. Within each age group, the empirical survival probability $\tilde{P}(t,s)$ is determined using

$$
\tilde{P}(t,s) = \prod_{i=1}^{\tau} e^{-\tilde{\mu}(t+i-1,x+i-1)}, \qquad (13)
$$

where $\tilde{\mu}(\cdot,\cdot)$ are mortality intensities approximated by the crude death rates $m_c(t,x)$. We then minimize the sum of weighted squared errors between $P(t,s)$ and $\tilde{P}(t,s)$ across all 4 cohorts and 3 age groups with $\tau = 15$ for the 50–64 group and 85–99 group and $\tau = 20$ for the 65–84 group. Parameters are selected to minimize the objective function

$$
G = \sqrt{\sum_{j=1}^{200} W_j (P_j(t,s) - \tilde{P}_j(t,s))^2}, \qquad (14)
$$

where W_j is the weight assigned to the jth squared error term. We fit to 200 actual survival probabilities in total. We use non–equal weights and assign highest weights to the 50–64 group, medium weights to the 65–84 group and the lowest weights to the 85–99 group. We use lower weights as the initial age increases because the volatility of mortality rates increases significantly with age, which can be seen from the data analysis in Section 3. An equal–weight calibration scheme will tend to over–fit the "noise". To determine the weights, we sum the inverse of the initial ages as

$$
\text{Sum} = \frac{1}{50}\times 15\times 4 + \frac{1}{65}\times 20\times 4 + \frac{1}{85}\times 15\times 4. \qquad (15)
$$

We then calculate each weight W_j as a proportion of this sum. For each age group across all cohorts, the weight is constant at $\frac{1}{\text{initial age}\times\text{Sum}}$. Hence $\sum_{j=1}^{200} W_j = 1$.

The cohort specific parameters include ψ_2^i, σ_2^i, ρ^i and initial values of the state variables $\mu_1^i(t,x)$ and $\mu_2^i(t,x)$. The parameters common to all cohorts are a, b, c and d. Since

[4] See Brigo and Mercurio (2006) for the detailed proof.

we consider three different initial ages, in total there are 40 parameters to estimate. We use nonlinear constrained optimization which is relatively computationally inexpensive. The drawback is that the estimators are sensitive to the initial conditions used in the optimization (see Cairns and Pritchard 2001). Therefore as the first step, we estimate the initial conditions of the model parameters with the observed drifts and volatilities in Section 3. The initial conditions for ψ_1 and σ_1 are $a = -0.0205$, $b = 0.0004$, $c = -10.32$ and $d = 0.0801$. The remaining initial conditions are presented in Table 4. We also impose the constraint that $-1 \leq \rho^i \leq 1$.

Table 4: **Initial Conditions for Optimisation**

Cohort	ψ_2	σ_2	ρ	$\mu_1(50)$	$\mu_1(65)$	$\mu_1(85)$	$\mu_2(50)$	$\mu_2(65)$	$\mu_2(85)$
1890	0.0032	0.0002	0.7660	0.0091	0.0305	0.1805	0.0091	0.0305	0.1805
1895	0.0011	-0.0001	0.9999	0.0080	0.0326	0.1490	0.0080	0.0326	0.1490
1900	0.0151	0.0017	0.9999	0.0079	0.0375	0.1442	0.0079	0.0375	0.1442
1905	0.0031	-0.0041	0.8377	0.0074	0.0344	0.1465	0.0074	0.0344	0.1465

4.3 Parameter Estimates

Parameter estimates for the age–dependent parameters are given in Table 5 and for the cohort specific parameters in Table 6. The model fit to the observed survival probabilities is measured by the absolute level of the percentage error $\left|\frac{\tilde{P}(t,s)-P(t,s)}{\tilde{P}(t,s)}\right|$. Figures 3, 4 and 5 show the fitting errors for age groups.

Table 5: **Initial Age Dependent Parameters**

a	b	c	d
0.2280	-0.0037	-10.3270	0.0343

Table 6: **Cohort Parameters**

Cohort	ψ_2	σ_2	ρ	$\mu_1(50)$	$\mu_1(65)$	$\mu_1(85)$	$\mu_2(50)$	$\mu_2(65)$	$\mu_2(85)$
1890	0.0721	-0.0001	0.7306	-0.0068	-0.0145	0.0227	0.0157	0.0460	0.1528
1895	0.0632	-0.0073	0.8710	-0.0368	-0.0292	-0.0343	0.0444	0.0607	0.1844
1900	0.0598	0.0000	0.9767	-0.0241	-0.0045	-0.0255	0.0321	0.0449	0.1815
1905	0.0817	-0.0000	0.8482	-0.0072	0.0106	-0.0011	0.0146	0.0256	0.1409

We see that for the age group 50–64 and 65–84, the fitting errors in general are below 1%. For the very old age group 85–99, the largest fitting error is 11.30% and only 5 out of 60 errors are above 5%. Such error levels are in line with the 3–factor age–period model results in Blackburn and Sherris (2013), who introduce a third factor to capture the variation in the survival curve for ages over 85. The out–of–sample forecasting in Jevtic et al. (2013) reports 26% fitting error at age 80 and sharp increases afterwards. The fitting performance of our 2–factor Gaussian cohort model is improved by the use of the age–dependent parameters based on the empirical data analysis. Both the drift and volatility of mortality rate changes increase with age in the data. The model in Jevtic et al. (2013) with constant parameters does not capture the dependence on initial age and does not perform well particularly at the older ages.

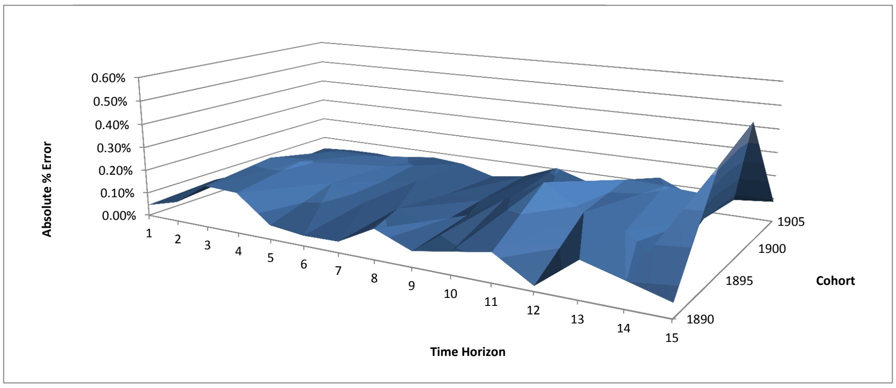

Figure 3: **Fitting Error for the Age Group 50-64**

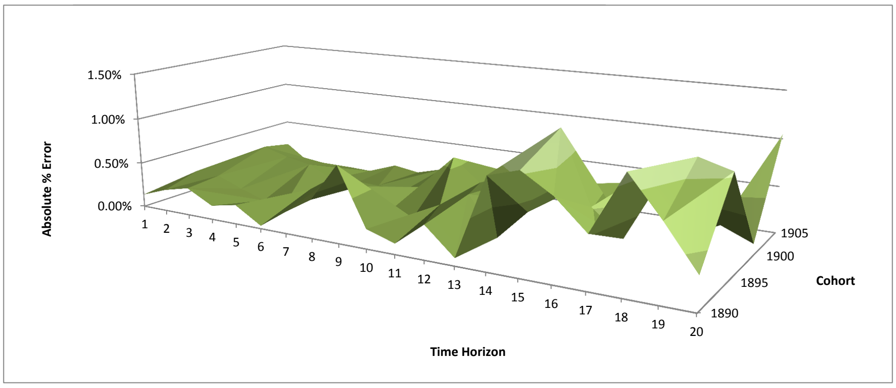

Figure 4: **Fitting Error for the Age Group 65-84**

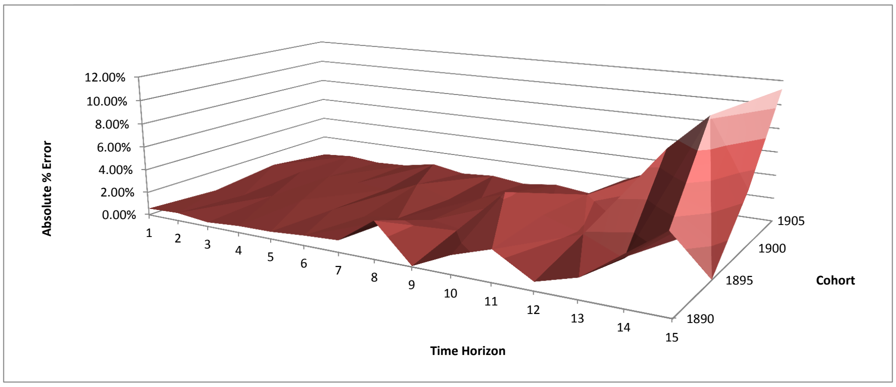

Figure 5: **Fitting Error for the Age Group 85-99**

Using the parameter estimates in Table 5 and 6, we simulate mortality rates in order to assess the model performances. There were no negative mortality rates in these simulations, even though in the optimisation scheme no positivity constraints of the mortality rates are imposed.

We compute the correlations from the simulated mortality rates and these are shown in Table 7.

Table 7: **Cohort Correlations with simulated mortality rates**

Calendar Time 1955				
Cohort	1890	1895	1900	1905
1890	1.0000			
1895	0.4710	1.0000		
1900	0.3427	0.4019	1.0000	
1905	0.5471	0.6195	0.5193	1.0000
Calendar Time 1960				
1890	1.0000			
1895	-0.1862	1.0000		
1900	0.3584	-0.1156	1.0000	
1905	0.6125	-0.2554	0.5283	1.0000
Calendar Time 1965				
1890	1.0000			
1895	0.2348	1.0000		
1900	0.6964	0.2709	1.0000	
1905	0.8403	0.2587	0.7707	1.0000
Calendar Time 1970				
1890	1.0000			
1895	0.4367	1.0000		
1900	0.7740	0.4174	1.0000	
1905	0.9324	0.4593	0.8174	1.0000

The 20–year correlations between cohorts show similar features to the empirical data using only a relatively small number of parameters. As noted already cohort correlations are much lower than those given in Jevtic et al. (2013), which are close to positive perfect correlations (i.e. 100%). The cohort mortality model generates a more realistic correlation structure than previous models. We provide a statistical test to determine whether the model correlations in Table 7 provide a good fit to the realized correlations in Table 3. We measure the sum of the fitting errors by the loss function

$$L = \sum_{i=1}^{K} \left[\max\left(\rho^m - \rho^u, 0\right) + \max\left(\rho^l - \rho^m, 0\right) \right]^2, \tag{16}$$

where $K = 24$ is the total number of correlations, ρ^m is the model correlation and (ρ^l, ρ^u) is the 95% confidence interval of the realized correlation. The standard error for the correlation estimate is $\sigma_r = \sqrt{\frac{1-r^2}{n-2}}$, where r is the estimated correlation and $n = 20$ is the sample size. The test statistic $t = \frac{r}{\sigma_r}$ follows a t–distribution with $n-2$ degrees of freedom. Eqn. (16) assumes that if ρ^m falls within the 95% confidence interval, the error is set to zero. Therefore we only have positive error terms if ρ^m is below the lower bound or above the upper bound of the interval, when we are 95% confident that the model correlation is significantly different from the estimated realized correlation. The fitting errors are shown in Table 8.

The total sum of fitting errors is 0.09 and the model fits 21 correlations. We measure the relative performance of our model against a 95% constant correlation[5]. The results show that our proposed model reduces the fitting errors by 85.05% from these much higher correlations, demonstrating how a model with age–dependent parameters is required to produce a good fit to the empirical correlations.

Table 8: **Fitting Errors of the Model Correlations**

Calendar Time 1955			
Cohort	1890	1895	1900
1895	0		
1900	0	0	
1905	0	0	0
Calendar Time 1960			
1895	0.0484		
1900	0	0	
1905	0	0	0
Calendar Time 1965			
1895	0.0026		
1900	0	0	
1905	0	0	0
Calendar Time 1970			
1895	0		
1900	0	0.0391	
1905	0	0	0

5 Interest Rate Model

5.1 Vasicek Model

We model interest rates with the Vasicek one–factor process with constant parameters. This is a well accepted interest rate model suitable for valuation of interest rate term structure based cash flows. Under the risk–neutral measure, the instantaneous spot rate $r(t)$ follows the SDE

$$dr(t) = k[\theta - r(t)]dt + \sigma dW^r(t), \qquad r(0) = r_0, \tag{17}$$

where k is the mean reversion speed, θ is the long term average rate, σ is the diffusion coefficient and $r(0)$ is the short rate at initiation. All parameters are positive constants. The stochastic integral equation for $r(t)$ is

$$r(t) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-u)}\, dW^r(u), \qquad s \leq t. \tag{18}$$

[5] 95% is the lowest correlation level calibrated by the 2–factor model of Jevtic et al. (2013). This provides an approximate, but conservative, estimate of the fitting errors of the correlations resulting from their model.

Conditional upon information up to s, $r(t)$ follows a normal distribution with mean

$$E(r(t)|\mathcal{F}_s) = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right), \tag{19}$$

and variance

$$Var(r(t)|\mathcal{F}_s) = \sigma^2 \int_s^t e^{-2k(t-u)}\, du = \frac{\sigma^2}{2k}\left(1 - e^{-2k(t-s)}\right). \tag{20}$$

The model also naturally fits into the ATSM framework, with the zero–coupon bond price given by

$$P(t,T) = A(t,T) \cdot e^{-B(t,T)r(t)}. \tag{21}$$

$A(t,T)$ and $B(t,T)$ are respectively given by

$$A(t,T) = \exp\left(\left(\theta - \frac{\sigma^2}{2k^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4k}B(t,T)^2\right), \tag{22}$$

and

$$B(t,T) = \frac{1}{k}\left(1 - e^{-k(T-t)}\right). \tag{23}$$

This allows zero–coupon prices for computing the value based index to be efficiently computed with analytical functions of the model parameters without the need for extensive simulations that might be the case for more complex models.

5.2 Data and Calibration

We calibrate the interest rate model parameters with Australian zero–coupon discount factors from the Reserve Bank of Australia (RBA). The discount factors are published by RBA daily with maturities ranging from 3–month up to 10–years at 3 month maturity intervals giving a discrete set of 40 maturities. We use a panel dataset from the 2nd of August, 2004 to 31st of July, 2014 giving 2527 days in the sample and 40 discount factors for each day. We then have 101,080 discount factors to fit with the model.

The model is calibrated by non–linear constrained optimisation, which minimizes the mean squared error between the model discount factor $P_{(r)}(0,t)$ and the actual discount factor $\tilde{P}_{(r)}(0,t)$. The objective function is

$$G_r = \sqrt{\frac{1}{N}\sum_{i=1}^{2527}\sum_{j=1}^{40}(P_{(r),i,j}(0,t) - \tilde{P}_{(r),i,j}(0,t))^2}, \tag{24}$$

where $N = 101,080$ is the total number of discount factors. The initial values of the model parameters we use in the optimisation are estimated from the dataset. Firstly, the mean reversion parameter k is estimated by the half–life of interest rates, which is the time it takes for the interest rate to move half the distance from $r(0)$ towards its long term average θ (Guimaraes 2005). Suppose $T(\frac{1}{2})$ is the half–life and we rearrange the deterministic part of Eqn. (17) and obtain

$$\frac{dr(t)}{\theta - r(t)} = kdt. \tag{25}$$

Integrate both sides of Eqn. (25) we get

$$\int_{r(0)}^{r(T(\frac{1}{2}))} \frac{dr(t)}{\theta - r(t)} = \int_0^{T(\frac{1}{2})} k\, dt, \tag{26}$$

where $r(T(\frac{1}{2})) - \theta = \frac{1}{2}(r(0) - \theta)$. Straightforward calculations then result in

$$k = \frac{ln(2)}{T(\frac{1}{2})}. \tag{27}$$

We then estimate k with Eqn. (27) and the sample data. We approximate the short rate $r(t)$ with the 3–month forward rate at t implied by the observed discount factors. We then calculate the average $r(t)$ for each maturity over the sample period. The sample average $r(T(\frac{1}{2}))$ is 5.05%, which corresponds to $T(\frac{1}{2}) = 5$. Hence our estimated k is $\frac{ln(2)}{5} = 0.1386$, which we use as the initial condition for the optimisation. The constraints we impose for the value of k in the optimisation are obtained by letting $T(\frac{1}{2})$ be respectively 0.25 and 10. We also estimate the initial conditions and constraints for θ and σ from the sample data. The results are shown in Table 9.

Table 9: **Inputs for the Interest Rate Model Calibration**

Inputs	k	θ	σ
Initial Value	0.1386	0.0542	0.0009
Upper Bound	2.7726	0.0660	0.0043
Lower Bound	0.0693	0.0375	0.0002

5.3 Calibration Results

The calibrated parameters are $k = 0.1781$, $\theta = 0.05$ and $\sigma = 0.0002$. The % fitting errors given by $\left|\frac{\tilde{P}_r(0,t)-P_r(0,t)}{\tilde{P}_r(0,t)}\right|$ are shown in Figure 6 for the whole sample period, the period before the Global Financial Crisis (GFC) and the post–GFC period.

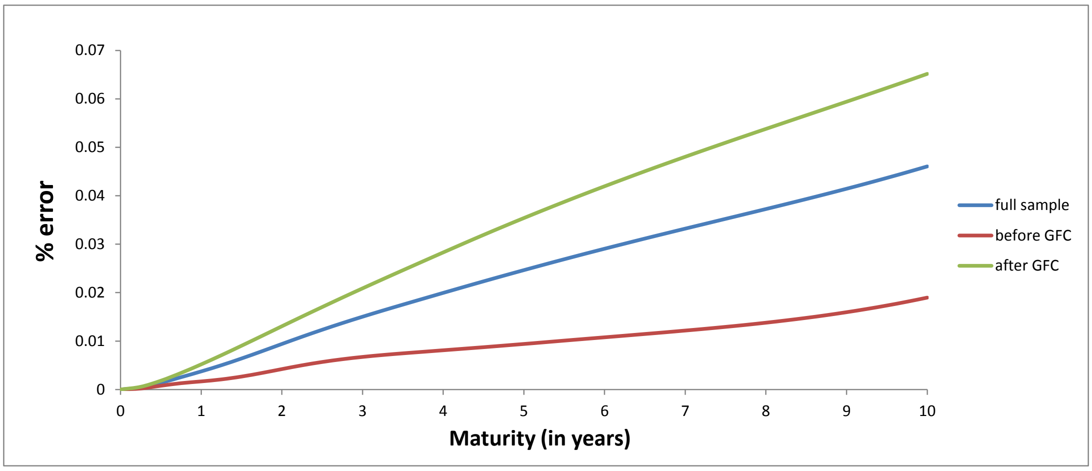

Figure 6: **Fitting Errors of Interest Rate Model**

We see from Figure 6 that the average fitting errors for each maturity are satisfactory, with the error increasing monotonically from 0.04% to 4.61%. We obtain interesting findings when we split the sample period into two sub–periods. We choose 16/09/2008 as the break date, which corresponds to the collapse of Lehman Brothers and is commonly considered the peak of the GFC. The fitting errors are much smaller before the crisis than after the crisis. Such results are not surprising given the post–GFC period is associated with greater market volatility and uncertainty.

We also check whether the model produces any negative interest rates. The discount curves constructed by the calibrated parameters on all sample days are monotonically decreasing, which implies strictly positive interest rates. Thus although negative rates are theoretically possible for Gaussian models, it is not a practical matter that would create issues in the construction of the value–based longevity index.

6 The Value Based Cohort Index

In this section we use the calibrated mortality and interest rate models to compute the value–based index for a 65 year old cohort for the Australian mortality data. The cohort mortality model is used to determine expected future survival probabilities for the cohort. These are combined with the term structure model to compute the index. To illustrate the practical application of the value based cohort index we assess its hedge efficiency using an index based mortality swap and s–forward contracts to hedge an immediate annuity.

6.1 Illustrative Index Calculation

The mortality model is used to determine cohort 65 survival probabilities. We determine expected survival probabilities from ages 65 to 120. These are shown in Figure 7. We use the interest rate model to determine the forward interest rates. With the expected survival probabilities and interest rates, we compute the PV for each index point. Figure 8 shows the value–based index for all future ages with PV_0 equal to 12.67.

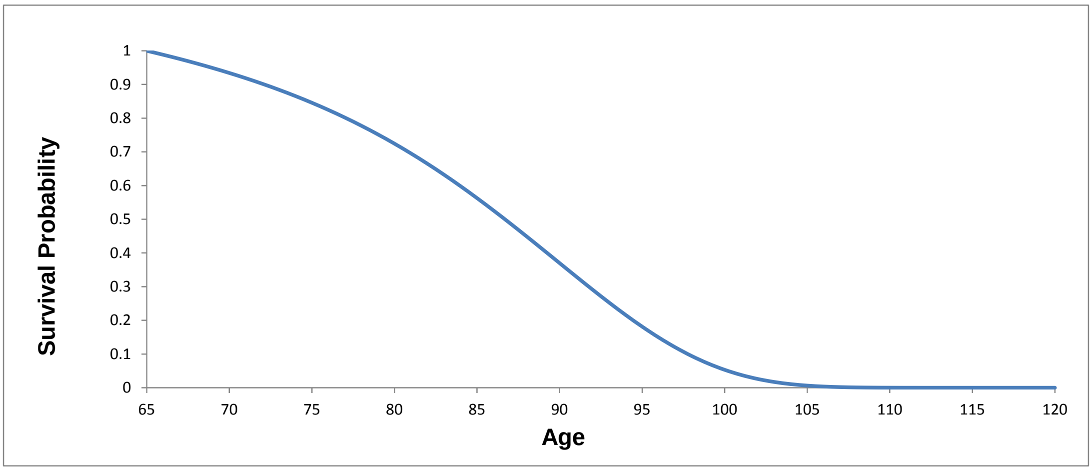

Figure 7: **Estimated Survival Probabilities for Cohort 65**

Each cohort will have an index value in practice and then current mortality and interest rate models are used to construct the index values. As a cohort ages the index value decreases. The decrease reflects that expected for an annuity as individuals age.

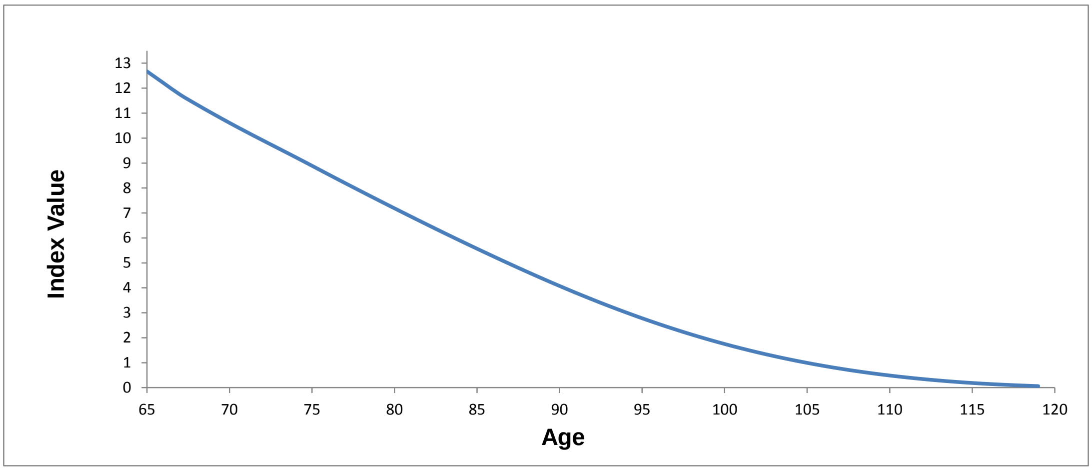

Figure 8: **Value Index**

At any given time cohorts differ by their current age and the index is constructed for varying initial ages for these different cohorts.

6.2 Hedge Efficiency

To show an application of the value–based longevity index we use it to compare the efficiency of two different hedge contracts in hedging an immediate life annuity portfolio. We do this for differing numbers of annuitants who are Australian males initially aged 65. The contracts used are an index based swap and an s–forward. For the swap the annuity provider pays the index value and receives the realized value, transferring both systematic longevity risk and interest rate risk. Idiosyncratic longevity risk reflecting differences between the annuitant portfolio and the index is not hedged. For the survivor–forward (LLMA 2010), or s-forward, the annuity provider pays the expected population survival rate of the cohort 65 and receives the realized population survival rate. The s–forward is also only designed to hedge the systematic longevity risk. However it will not hedge interest rate risk nor idiosyncratic longevity risk.

When evaluating hedge efficiency, most existing research has assumed that interest rates are constant or deterministic. This suggests that the hedge efficiency will be overestimated in studies such as Coughlan et al. 2011, Blackburn and Sherris (2014). Combining models for both stochastic interest rates and mortality rates provides a broader assessment of hedge performance.

Hedge efficiency is defined as

$$1 - \frac{\sigma_h}{\sigma_u}, \tag{28}$$

where σ_h and σ_u are respectively the standard deviation of the unexpected PV of the hedged position and the unhedged position.

We simulate the mortality rates and interest rates, and for each path m, the unexpected value (UV) for the unhedged position is

$$UV_u^m = PV_0 - SV^m, \tag{29}$$

where SV^m is the simulated PV of path m for the annuity portfolio. SV^m is calculated with the simulated survival probabilities of the portfolio. In order to generate idiosyncratic longevity risk for the portfolio, we follow Blackburn, Hanewald, Sherris and Olivieri

(2013) and determine the random death time for each individual in the portfolio by the first time the mortality hazard rate exceeding ϱ, an exponential random variable with parameter 1. For each simulation path m, we keep track of the number of accumulated deaths d_i^m at the end of each year i ($i = 1, 2, ..., 36$). If the initial number of annuitants of the portfolio is n_0, then the portfolio survival index at the end of each year i is

$$\frac{n_0 - d_i^m}{n_0}. \tag{30}$$

For the hedged position with the index swap, the annuity provider enters a single swap and we assume the swap is collateralized and not subject to default risk. In the index swap, the hedger pays the fixed index value PV_0 and receives the realized value. Then for each path m the UV is

$$UV_{hswap}^m = SIV^m - SV^m, \tag{31}$$

where SIV^m is the simulated PV of path m, based on the cohort 65 population experience. We simulate the population survival index with the calibrated mortality model parameters, which reflects systematic longevity risk only. For a large portfolio, the portfolio survival index should be close to the population survival index because the idiosyncratic longevity risk will be low.

For the hedged position with the s–forward, the UV of each simulation path m is

$$UV_{hforward}^m = PV_0 - SV^m + SIV^m - \overline{SIV}^m, \tag{32}$$

where $\overline{SIV}^m$ is the simulated PV at projected population survival probabilities. In general $\overline{SIV}^m \neq PV_0$ because in the s–forward the interest rate risk is not hedged. Table 10 shows the hedge efficiencies for these two hedging contracts. The portfolio size is varied to show the effect of idiosyncratic longevity risk on hedge effectiveness.

Table 10: **Hedge Efficiency**

Portfolio Size	200	1000	10000
Index Swap	18.52%	73.99%	97.15%
s–forward	16.14%	59.31%	69.17%

As expected, hedge efficiency increases as portfolio size increases for both contracts. We see that hedge efficiency improvement is significantly higher for the index swap than for the s–forward. For a group of 10,000 annuitants, the hedge efficiency is more than 95% for the swap, while less than 70% for the s–forward. A major difference between the index swap and the s–forward, is the hedging of interest rate risk in the index swap. There is little difference for smaller portfolio sizes. In these cases idiosyncratic mortality risk dominates.

7 Conclusion

Financial markets have been developing innovative approaches to manage longevity risk. There is yet to be a well accepted longevity index used for financial market contracts. Existing longevity indices are largely based on age–based mortality rates or life expectancy.

These indices are likely to be less effective in hedging longevity risk than a value–based index.

We propose such a value–based longevity index and show how it can be a more effective index for financial market participants since it is based on the present value of a standardized longevity linked cash flow valued using models for mortality and interest rates. This index combines both mortality risk and interest rate risk. The interest rate risk can be readily eliminated from the index since interest rate markets are deeper than longevity risk markets.

To support the value–based index we analyze cohort based mortality data and propose a cohort–based stochastic mortality model that includes age–dependent parameters that better capture trends, volatility and dependence between cohorts. Using Australian data we show that the model fitting performance improves over currently proposed cohort models, particularly at the older ages. We use the model along with a one–factor Vasicek short rate model calibrated to Australian bond yields to construct a value–based longevity index for this data.

We illustrate the effectiveness of the index by assessing the hedging of immediate annuities to 65 year-olds with an index based swap on the value–based index as the underlying compared to an s–forward, which hedges only systematic longevity risk based on survival rates.

Much remains to be done in supporting the development of a traded market in longevity risk. The long term nature of this risk requires new approaches and these are expected to be based on hedging values rather than future cash flows. Our aim has been to promote a different perspective on longevity risk hedging and to contribute a new cohort mortality model that is suited to the construction of a better longevity index.

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