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Lifetime Asset Allocation with Idiosyncratic and Systematic Mortality Risks

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Abstract

This paper considers the lifetime asset allocation problem with both idiosyncratic and systematic longevity risks, in which the stochastic mortality model is given by a general diffusion process. A wage earner can invest in a zero-coupon bond, a stock and a longevity bond, consume part of his wealth and purchase life insurance or annuity so as to maximize the expected utility from consumption, terminal wealth and bequest. The problem is solved via the dynamic programming principle and the Hamilton-Jacobi-Bellman equation. General solutions and special solutions are derived for the general diffusion mortality model and the square-root mortality model, respectively. To illustrate our results, numerical examples based on special solutions are provided. It is shown that idiosyncratic mortality risk has significant impacts on the wage earner's investment, consumption, life insurance purchase and bequest decisions regardless of the length of the decision-making horizon, while systematic mortality risk only has significant impacts on the wage earner's investment in the zero-coupon bond and the longevity bond. Since systematic mortality risk can be hedged by trading the longevity bond, its impacts on consumption, purchase of life insurance and bequest are not significant, especially when the decision-making horizon is short.

Keywords: Investment-consumption, life insurance, stochastic mortality, longevity risk.

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1 Introduction

Asset allocation plays an important role in financial markets around the globe. Imagine an individual allocating his wealth and income so as to achieve a steady growth in wealth (investment), maintain a decent standard of living (consumption) and protect his family against financial distress due to his accidental death (life insurance purchase). Indeed, there has been a long history of research on asset allocation problems. Yaari (1965) investigated the demand for life insurance/annuities by the consumer taking into account a stochastic lifetime. Yaari was attributed to introducing individual lifetime uncertainty to asset allocation problems, which facilitated the formulation of the lifecycle model afterwards. Samuelson (1969) and Merton (1969, 1971) were the first to use the dynamic programming principle to study lifetime consumption and portfolio choice problems in a multi-period setting and a continuous-time setting, respectively. Richard (1979) considered optimal investment, consumption and life insurance rules for an individual with a bounded random lifetime. Bodie et al. (1992) examined the effect of labor-leisure choice on portfolio and consumption decisions over an individual's life cycle and found that labor and investment choices are intimately related. Other recent works on asset allocations problems include Cairns et al. (2006), Huang and Milevsky (2008), Huang et al. (2008) and others.

It is now widely accepted that stochastic mortality, the risk that aggregate mortality will be different from that anticipated, is an important risk factor in both life insurance and pensions. In particular, longevity risk (which is the risk aggregate mortality is lower than that anticipated) has become one of the biggest challenges in the 21st century. Various mortality models have been proposed in the last two decades. Examples include the Lee-Carter model (Lee and Carter, 1992), the CBD model (Cairns et al., 2006), the affine-type model (Schrager, 2006 and Blackburn and Sherris, 2013), the Markov aging process (Lin and Liu, 2007 and Su and Sherris, 2012), the Lévy model (Hainaut and Devolder, 2008) and the regime-switching model (Milidonis et al., 2011 and Shen and Siu, 2013), just to name a few. Although asset allocation problems with lifetime uncertainty have been well studied, the mortality rate is usually assumed to be deterministic over time. Allowing the mortality rate to vary stochastically over time and incorporating longevity risk in the modeling framework will provide additional more insights and implications for individuals and financial markets.

It is of interest to formally consider optimal investment, consumption and life insurance purchase with mortality risks, including systematic and idiosyncratic (unsystematic) mortality risks. Systematic mortality risk is the risk that arises from shocks to population-level mortality rates that apply to all individuals to a greater or lesser extent, whereas idiosyncratic mortality risk is uncertainty in individual survival given the population mortality rates (Hanewald et al., 2013). There is a longstanding literature on individual asset allocation problems with idiosyncratic mortality risk. See, for example, Yaari (1965), Richard (1979), Pliska and Ye (2007), Kwak et al. (2009), Pirvu and Zhang (2012), Kronborg and Steffensen (2013) and references therein. It is not until recently, however, that asset allocation with systematic mortality risk has attracted much needed attention. Menoncin (2008) investigated an optimal consumption and portfolio problem of an agent with a stochastic mortality investing in a financial market with a longevity bond. Huang et al. (2012) considered an optimal retirement consumption problem and compared the impact of stochastic versus deterministic mortality rates on the optimal consumption rate.

In this paper, we consider an optimal investment, consumption and life insurance purchase problem for an investor with a power utility, whose mortality evolves in a stochastic fashion. More specifically, the randomness in our modeling framework is given by a Brownian motion filtration and the force of mortality of the investor is assumed to follow a general diffusion process. Furthermore, we assume that the investor is a wage earner, receives a deterministic income flow before death and allocates his wealth among a zero-coupon bond, a stock, a longevity bond, consumption and purchase of life insurance/annuities so as to maximize the expected, discounted utilities derived from intertemporal consumption, terminal wealth and bequest over an uncertain lifetime horizon. Our modeling framework therefore incorporates both idiosyncratic and systematic mortality risks. To solve the problem, we employ the dynamic programming principle to derive a corresponding Hamilton-Jacobi-Bellman equation (HJB). Through solving this HJB equation, we obtain expectation representations of the optimal investment-consumption-insurance strategy and the value function for the general stochastic mortality model.

We further obtain closed-form solutions of the optimal strategy and the value function under a square-root stochastic mortality model. We then provide numerical examples to illustrate our results.

The rest of this paper is structured as follows. Section 2 introduces the model dynamics. In Section 3, we formulate the optimal investment, consumption and life insurance purchase problem with stochastic mortality. In Section 4, we use the dynamic programming principle to derive an HJB equation related to the problem and give general solutions to the problem through solving the HJB equation. Section 5 provides special solutions to the problem under the square-root stochastic mortality model. In Section 6, we present numerical examples to illustrate practical implementations of our results. Section 7 concludes the paper.

2 The model

In this section, we introduce the model dynamics to be used in this paper. Consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and a finite time horizon $\mathcal{T} := [0, T^*]$ containing all objects to be defined in our modeling framework. Here $\mathcal{P}$ is the real-world probability measure, or the reference probability measure, from which a family of equivalent probability measures can be generated. Denote by $\mathrm{E}[\cdot]$ the expectation under $\mathcal{P}$. Let $\{(W_S(t), W_r(t), W_\lambda(t))^\top | t \in \mathcal{T}\}$ be a three-dimensional, $(\mathbb{F}, \mathcal{P})$-standard Brownian motion, where $\mathbb{F} := \{\mathcal{F}(t) | t \in \mathcal{T}\}$ is the right-continuous, $\mathcal{P}$-complete natural filtration generated by $(W_S(\cdot), W_r(\cdot), W_\lambda(\cdot))^\top$. Let τ be a non-negative random variable on $(\Omega, \mathcal{F}, \mathcal{P})$, representing the remaining lifetime of an investor at time 0. By convention, we assume the Brownian motion $(W_S(\cdot), W_r(\cdot), W_\lambda(\cdot))^\top$ and the random variable τ are stochastically independent under $\mathcal{P}$.

Let $\mathcal{Q}$ denote the equivalent martingale measure, or the risk-neutral measure, which is defined by the Randon-Nikodym derivative $\{\Lambda(t) | t \in \mathcal{T}\}$ as follows

$$\begin{aligned} \left.\frac{d\mathcal{Q}}{d\mathcal{P}}\right|_{\mathcal{F}(T^*)} &= \Lambda(T^*) \\ &= \exp\left\{ -\frac{1}{2}\int_0^{T^*} |\theta(t)|^2 dt - \int_0^{T^*} \theta(t) dW(t) \right\}, \end{aligned}$$

where $\{\theta(t) | t \in \mathcal{T}\} := \{(\theta_S(t), \theta_r(t), \theta_\lambda)^\top | t \in \mathcal{T}\}$ is an $\Re^3$-valued, $\mathbb{F}$-adapted process such that the Novikov condition is satisfied

$$\mathrm{E}\left[\exp\left\{ \frac{1}{2}\int_0^{T^*} |\theta(t)|^2 dt \right\} \right] < \infty \ .$$

Here $\theta_S(\cdot)$, $\theta_r(\cdot)$ and $\theta_\lambda(\cdot)$ represent the market prices of the stock risk, the interest rate risk and the mortality risk, respectively, whose structure will be introduced subsequently in this section. By Girsanov's theorem, the process $\{W^{\mathcal{Q}}(t) | t \in \mathcal{T}\} = \{(W_S^{\mathcal{Q}}(t), W_r^{\mathcal{Q}}(t), W_\lambda^{\mathcal{Q}}(t))^\top | t \in \mathcal{T}\}$ defined by

$$W^{\mathcal{Q}}(t) = W(t) + \int_0^t \theta(s) ds \ .$$

is a three-dimensional standard Brownian motion under $\mathcal{Q}$. For convenience of the valuation of zero-coupon bond and longevity bond, we will introduce the model dynamics under $\mathcal{Q}$ and $\mathcal{P}$ sequentially.

In what follows, we introduce an arbitrage-free financial market consisting of four primitive assets, namely, a bank account, a zero-coupon bond, a stock and a longevity bond. The bank account is considered as a risk-free asset of the market, which allows instantaneous borrowing and lending at the risk-free rate. We assume that the evolution of the price process of the bank account $\{M(t) | t \in \mathcal{T}\}$ follows

$$dM(t) = r(t)M(t)dt \ , \quad M(0) = 1 \ . \tag{1}$$

Here $r(t)$ is the risk-free, instantaneous interest rate at time t. We further assume that the instantaneous interest rate process $\{r(t)|t \in \mathcal{T}\}$ is also stochastic and satisfies the following stochastic differential equation (SDE) under $\mathcal{Q}$

$$dr(t) = \mu_r(t, r(t))dt + \sigma_r(t, r(t))dW_r^{\mathcal{Q}}(t) \ , \quad r(0) = r_0 \ , \tag{2}$$

where $\mu_r(\cdot,\cdot) : \mathcal{T} \times \Re^+ \to \Re$ and $\sigma_r(\cdot,\cdot) : \mathcal{T} \times \Re^+ \to \Re^+$ are two deterministic functions such that the SDE (2) admits a unique solution $r(\cdot) : \mathcal{T} \to \Re^+$. Suppose that the market price of the interest rate risk at time t is given by $\theta_r(t) := \theta_r(t, r(t))$, where $\theta_r(\cdot,\cdot) : \mathcal{T} \times \Re^+ \to \Re$. Then under $\mathcal{P}$, the interest rate process satisfies

$$dr(t) = \big[\mu_r(t, r(t)) + \sigma_r(t, r(t))\theta_r(t, r(t))\big]dt + \sigma_r(t, r(t))dW_r(t) \ , \quad r(0) = r_0 \ . \tag{3}$$

Suppose that the dynamics of the stock price process $\{S(t)|t \in \mathcal{T}\}$ is governed by the following Geometric Brownian Motion (GBM) model:

$$dS(t) = S(t)\big[r(t)dt + \sigma_S(t)dW_S^{\mathcal{Q}}(t) + \sigma_{Sr}(t)dW_r^{\mathcal{Q}}(t)\big] \ , \tag{4}$$

where $\sigma_S(t) > 0$ and $\sigma_{Sr}(t) > 0$ are the volatilities of the stock at time t corresponding to random shocks from $W_S(\cdot)$ and $W_r(\cdot)$, respectively, and $\sigma_S(\cdot) : \mathcal{T} \to \Re^+$ and $\sigma_{Sr}(\cdot) : \mathcal{T} \to \Re^+$ are deterministic, uniformly bounded functions. Suppose that the market price of the stock risk is deterministic, i.e. $\theta_S(\cdot) : \mathcal{T} \to \Re$. Then under $\mathcal{P}$, the stock price process follows

$$dS(t) = S(t)\big[\mu_S(t)dt + \sigma_S(t)dW_S(t) + \sigma_{Sr}(t)dW_r(t)\big] \ , \tag{5}$$

with the appreciation rate

$$\mu_S(t) := r(t) + \sigma_S(t)\theta_S(t) + \sigma_{Sr}(t)\theta_r(t, r(t)) \ .$$

Consider a zero-coupon bond paying one dollar at time T_1. The price of the bond at time t is

$$\begin{aligned} P(t, T_1) &= \mathrm{E}^{\mathcal{Q}}[e^{-\int_t^{T_1} r(u)du}|\mathcal{F}^r(t)] \\ &= \mathrm{E}^{\mathcal{Q}}[e^{-\int_t^{T_1} r(u)du}|r(t)] \\ &= P(t, T_1, r(t)) \ , \end{aligned}$$

where $\mathrm{E}^{\mathcal{Q}}[\cdot]$ denotes the expectation under $\mathcal{Q}$ and $\mathcal{F}^r(t)$ is the σ-field generated by $r(\cdot)$ up to time t. Here the second equality holds since the interest rate process $r(\cdot)$ is Markovian with respect to its natural filtration $\mathbb{F}^r := \{\mathcal{F}^r(t)|t \in \mathcal{T}\}$. Given $r(t) = r > 0$, we can use the martingale method to derive the following partial differential equation (PDE) for P:

$$\frac{\partial P}{\partial t} + \mu_r(t, r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2(t, r)\frac{\partial^2 P}{\partial r^2} = rP \ , \quad P(T_1, T_1, r) = 1 \ . \tag{6}$$

In what follows, we write $P(t, T_1) = P(t, T_1, r(t))$ whenever no confusion arises. The dynamics of the bond price process $\{P(t, T_1)|t \in [0, T_1]\}$ evolves as:

$$\begin{aligned} dP(t, T_1) &= \left\{\left[\frac{\partial P}{\partial t} + \mu_r(t, r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma_r^2(t, r)\frac{\partial^2 P}{\partial r^2}\right]dt + \frac{\partial P}{\partial r}\sigma_r(t, r)dW_r^{\mathcal{Q}}(t)\right\}\Big|_{r=r(t)} \\ &= P(t, T_1)\left[r(t)dt + \nabla_r^P(t, T_1)\sigma_r(t, r(t))dW_r^{\mathcal{Q}}(t)\right] \ , \end{aligned} \tag{7}$$

where

$$\nabla_r^P(t, T_1) := \frac{\frac{\partial P}{\partial r}(t, T_1)\big|_{r=r(t)}}{P(t, T_1)} \ .$$

Under $\mathcal{P}$, the dynamics of the bond price process is governed by

$$dP(t,T_1) = P(t,T_1)\big[\mu_P(t,T_1)dt + \nabla_r^P(t,T_1)\sigma_r(t,r(t))dW_r(t)\big] \ , \tag{8}$$

with the appreciation rate

$$\mu_P(t,T_1) := r(t) + \nabla_r^P(t,T_1)\sigma_r(t,r(t))\theta_r(t,r(t)) \ .$$

We assume the hazard rate (intensity) process of τ or the force of mortality process of the investor $\{\lambda(t)|t \in \mathcal{T}\}$ is governed by the following general diffusion process

$$d\lambda(t) = \mu_\lambda(t,\lambda(t))dt + \sigma_\lambda(t,\lambda(t))dW_\lambda^{\mathcal{Q}}(t) \ , \quad \lambda(0) = \lambda_0 \ , \tag{9}$$

where $\mu_\lambda(\cdot,\cdot) : \mathcal{T}\times\Re^+ \to \Re$ and $\sigma_\lambda(\cdot,\cdot) : \mathcal{T}\times\Re^+ \to \Re^+$ are two deterministic functions such that the stochastic differential equation (9) admits a unique solution $\lambda(\cdot) : \mathcal{T} \to \Re^+$. Suppose that the market price of the mortality risk at time t is given by $\theta_\lambda(t) := \theta_\lambda(t,\lambda(t))$, where $\theta_\lambda(\cdot,\cdot) : \mathcal{T}\times\Re^+ \to \Re$. Then under $\mathcal{P}$, the mortality rate process is given by

$$d\lambda(t) = \big[\mu_\lambda(t,\lambda(t)) + \sigma_\lambda(t,\lambda(t))\theta_\lambda(t,\lambda(t))\big]dt + \sigma_\lambda(t,\lambda(t))dW_\lambda(t) \ , \quad \lambda(0) = \lambda_0 \ . \tag{10}$$

Next we introduce a mortality-linked security, namely, a longevity bond, which provides a hedge for mortality risk. Let $I(t)$ denote the survivor index, representing the proportion of a cohort surviving from time 0 until time t

$$I(t) = e^{-\int_0^t \lambda(s)ds} \ .$$

For each $T \geq t$, then $\frac{I(T)}{I(t)}$ represents the proportion of the cohort surviving from time t until time T. We now consider a zero-coupon longevity bond with a payoff at time T_2, which is equal to the proportion of the cohort surviving from time t until time T_2. The price of the longevity bond at time t is

$$\begin{aligned} L(t,T_2) &= \mathrm{E}^{\mathcal{Q}}[e^{-\int_t^{T_2} r(s)ds} e^{-\int_t^{T_2}\lambda(s)ds}|\mathcal{F}^{r,\lambda}(t)] \\ &= L(t,T_2,r(t),\lambda(t)) \ , \end{aligned} \tag{11}$$

where $\mathbb{F}^{r,\lambda} = \{\mathcal{F}^{r,\lambda}(t)|t \in \mathcal{T}\}$ is the natural filtration generated by r and λ. Again, we write $L(t,T_2) = L(t,T_2,r(t),\lambda(t))$ whenever no confusion arises. In the same vein as the derivation of the zero-coupon bond, we can see that given $r(t) = r$ and $\lambda(t) = \lambda$, L satisfies the following PDE

$$\frac{\partial L}{\partial t} + \mu_r(t,r)\frac{\partial L}{\partial r} + \mu_\lambda(t,\lambda)\frac{\partial L}{\partial \lambda} + \frac{1}{2}\sigma_r^2(t,r)\frac{\partial^2 L}{\partial r^2} + \frac{1}{2}\sigma_\lambda^2(t,\lambda)\frac{\partial^2 L}{\partial \lambda^2} = (r+\lambda)L \ , \quad L(T_2,T_2,r,\lambda) = 1 \ , \tag{12}$$

and the dynamics of the longevity bond price process $\{L(t,T_2)|t \in [0,T_2]\}$ is governed by the following SDE:

$$dL(t,T_2) = L(t,T_2)\big[(r(t)+\lambda(t))dt + \nabla_r^L(t,T_2)\sigma_r(t,r(t))dW_r^{\mathcal{Q}}(t) \ + \nabla_\lambda^L(t,T_2)\sigma_\lambda(t,\lambda(t))dW_\lambda^{\mathcal{Q}}(t)\big] \ , \tag{13}$$

where

$$\nabla_r^L(t,T_2) := \frac{\frac{\partial L}{\partial r}(t,T_2)\big|_{r=r(t)}}{L(t,T_2)} \ , \quad \nabla_\lambda^L(t,T_2) := \frac{\frac{\partial L}{\partial \lambda}(t,T_2)\big|_{\lambda=\lambda(t)}}{L(t,T_2)} \ .$$

Under $\mathcal{P}$, the longevity bond price process satisfies the following SDE

$$dL(t,T_2) = L(t,T_2)\big[\mu_L(t)dt + \nabla_r^L(t,T_2)\sigma_r(t,r(t))dW_r(t) + \nabla_\lambda^L(t,T_2)\sigma_\lambda(t,\lambda(t))dW_\lambda(t)\big] \ , \tag{14}$$

with the appreciation rate

$$\mu_L(t,T_2) := r(t) + \lambda(t) + \nabla_r^L(t,T_2)\sigma_r(t,r(t))\theta_r(t,r(t)) + \nabla_\lambda^L(t,T_2)\sigma_\lambda(t,\lambda(t))\theta_\lambda(t,\lambda(t)) \ .$$

In what follows, we denote by

$$\mu(t) := (\mu_S(t), \mu_P(t,T_1), \mu_L(t,T_2))^\top \in \Re^3 \ ,$$

and

$$\sigma(t) := \begin{pmatrix} \sigma_S(t) & \sigma_{Sr}(t) & 0 \\ 0 & \nabla_r^P(t,T_1)\sigma_r(t,r(t)) & 0 \\ 0 & \nabla_r^L(t,T_2)\sigma_r(t,r(t)) & \nabla_\lambda^L(t,T_2)\sigma_\lambda(t,\lambda(t)) \end{pmatrix} \in \Re^{3\times 3} \ ,$$

the appreciation rate vector and the volatility matrix of the risky assets, respectively. Then the vector of asset price processes can be written as

$$\left(\frac{dS(t)}{S(t)}, \frac{dP(t,T_1)}{P(t,T_1)}, \frac{dL(t,T_2)}{L(t,T_2)}\right)^\top = \mu(t)dt + \sigma(t)dW(t) \ .$$

Clearly, the risk premium vector of the assets is denoted by

$$B(t) := \mu(t) - r(t)\mathbf{1}_3 = (\mu_S(t) - r(t), \mu_P(t,T_1) - r(t), \mu_L(t,T_2) - r(t))^\top \in \Re^3 \ ,$$

Write the variance-covariance matrix of the risky assets as

$$\Sigma(t) := \sigma(t)\sigma(t)^\top \in \Re^{3\times 3} \ ,$$

By convention, we assume $\Sigma(\cdot)$ is uniformly non-singular with respect to t. From the first fundamental theorem of asset pricing, the no-arbitrage condition leads to the following relationship between the risk premium, the volatility and the market price of risk:

$$B(t) = \sigma(t)\theta(t)^\top \ .$$

3 Problem formulation

In this section, we introduce the lifetime asset allocation problem, where the investor can invest in different assets, consume and purchase life insurance. Note that the investor's life insurance purchase is related to his decision on bequest.

Let $T > 0$ denote the planned terminal time of asset allocation. Furthermore, suppose the asset allocation horizon and the terms of the zero-coupon bond and the longevity bond satisfy $[0,T] \subset [0,T_j] \subset [0,T^*]$, $j = 1,2$. Let $\pi(t) := (\pi_S(t), \pi_P(t), \pi_L(t))$ be the amount of the investor's wealth allocated into the risky assets at time t, where $\pi_S(t)$, $\pi_P(t)$ and $\pi_L(t)$ represent the amount of the wealth invested in the stock, the bond and the longevity bond, respectively, $c(t) \geq 0$ be the amount of the investor's wealth consumed at time t, and $D(t) \geq 0$ be the investor's bequest. Kronborg and Steffensen (2013) interpreted D as the sum insured to be paid upon death to investor's beneficiary upon death at time $t \in [0,T]$. We call $\{\pi(t)|t \in [0,T]\} = \{(\pi_S(t), \pi_P(t), \pi_L(t))^\top | t \in [0,T]\}$, $\{c(t)|t \in [0,T]\}$ and $\{D(t)|t \in [0,T]\}$ a portfolio process, a consumption process and a bequest process of the investor. Furthermore, let $X(t)$ be the investor's wealth associated with (π, c, D) at time t. Since the investor's wealth is $X(t)$ at time $t \in [0,T]$, he needs to make up the difference $D(t) - X(t)$ through purchasing the life insurance or annuity with infinitesimally small terms. If we consider the insurance premium rate $p(t)$, i.e., the amount of wealth that the investor is willing to pay for a life insurance or annuity, as a control variable, then the asset allocation problem can be formulated as an optimal investment, consumption, insurance purchase

problem (see Pliska and Ye, 2007). To simplify our analysis and make the problem mathematically tractable, we assume that the insurance market is frictionless. That is, the insurance company has full information of current mortality rate and does not charge any risk loading for providing the life insurance or annuity. Then the insurance premium rate paid by the investor at time $t \in [0,T]$ is $p(t) = \lambda(t)(D(t) - X(t))$. From the perspective of the investor, $D(t) - X(t) > 0$ corresponds to buying a life insurance and paying the premium rate $p(t) = \lambda(t)(D(t) - X(t))$ to the insurance company at time t, while $D(t) - X(t) < 0$ corresponds to buying an annuity, i.e. selling a life insurance, and receiving the instantaneous annuity income $p(t) = \lambda(t)(D(t) - X(t))$ from the insurance company at time t.

We further assume that the investor is a wage earner and receives a continuous income flow $i(t)$ at time t, where $i(\cdot)$ is a time-deterministic, uniformly bounded, positive function. Then given that the initial wealth $x_0 > 0$, the wealth process of the investor $\{X(t)|t \in [0,T]\}$ satisfies the following SDE:

$$\begin{aligned} dX(t) &= \pi_S(t)\frac{dS(t)}{S(t)} + \pi_P(t)\frac{dP(t,T_1)}{P(t,T_1)} + \pi_L(t)\frac{dL(t,T_2)}{L(t,T_2)} \\ &\quad + [X(t) - \pi_S(t) - \pi_P(t) - \pi_L(t)]\frac{dM(t)}{M(t)} + [i(t) - c(t) - \lambda(t)(D(t) - X(t))]dt \\ &= [r(t)X(t) + \pi(t)^\top B(t) + i(t) - c(t) - \lambda(t)(D(t) - X(t))]dt + \pi(t)^\top \sigma(t) dW(t) \ , \\ X(0) &= x_0 > 0 \ . \end{aligned}$$

The investor's problem is to choose an optimal portfolio-consumption-bequest strategy so as to maximize the expected, discounted utilities from the consumption during the period $[0, T \wedge \tau]$, from the bequest if he dies before time T, and from the terminal wealth if he survives till time T. Then the investor's performance functional is given by

$$\begin{aligned} &J(\pi, c, D) \\ &= \mathrm{E}\left[\int_0^{T\wedge\tau} e^{-\int_0^s \rho(u)du} U(c(s))ds + \alpha e^{-\int_0^\tau \rho(u)du} U(D(\tau))\mathbf{1}_{\{\tau<T\}} + \beta e^{-\int_0^T \rho(u)du} U(X(T))\mathbf{1}_{\{\tau\geq T\}}\right] , \end{aligned}$$

where $\alpha, \beta > 0$ are the weights on the investor's utilities derived from the bequest and the terminal wealth, and $\rho(\cdot) : \mathcal{T} \to \Re^+$ is a time-deterministic, uniformly bounded function, representing the investor's subjective discount rate. We assume that the investor's utility is modeled by a power function:

$$U(x) = \begin{cases} \dfrac{x^\gamma}{\gamma} \ , & \text{if } x > 0 \ , \\ -\infty \ , & \text{if } x \leq 0 \ , \end{cases}$$

where $\gamma < 1$ and $\gamma \neq 0$. Adopting the power utility makes the problem mathematically tractable. Here the investor has the constant relative coefficient aversion (CRRA) preference since the relative risk aversion coefficient $-\frac{xU''(x)}{U'(x)} = 1 - \gamma$ is a constant.

Although the performance functional involves a random time horizon, it can be transformed into one with deterministic horizon (see Pliska and Ye, 2007) as:

$$J(\pi, c, D) = \mathrm{E}\left[\int_0^T e^{-\int_0^s [\rho(u)+\lambda(u)]du}[U(c(s)) + \alpha\lambda(s)U(D(s))]ds + \beta e^{-\int_0^T [\rho(u)+\lambda(u)]du} U(X(T))\right] . \tag{15}$$

For ease of calculation, we consider a transformed portfolio process $\{u(t)|t \in [0,T]\}$ as follows

$$u(t) := \pi(t)^\top \sigma(t) \ . \tag{16}$$

With a little abuse of notation, let $X(t) := X^{(u,c,D)}(t)$ denote the total wealth of the investor at time t from adopting the portfolio-consumption-bequest strategy (u, c, D). Then the wealth process of the investor $\{X(t)|t \in [0,T]\}$ can be rewritten as

$$\begin{aligned} dX(t) &= [r(t)X(t) + u(t)\theta(t)^\top + i(t) - c(t) - \lambda(t)(D(t) - X(t))]dt + u(t)dW(t) \ , \\ X(0) &= x_0 > 0 \ . \end{aligned} \tag{17}$$

From (15) and (17), we can see both the interest rate process $r(\cdot)$ and the force of mortality process $\lambda(\cdot)$ are also state processes of the control problem. For simplicity, we write the two-dimensional state process $\{Z(t)|t \in \mathcal{T}\} := \{(r(t), \lambda(t))^\top | t \in \mathcal{T}\}$ as

$$\begin{aligned} dZ(t) &= \begin{pmatrix} dr(t) \\ d\lambda(t) \end{pmatrix} \\ &= \begin{pmatrix} [\mu_r(t, r(t)) + \sigma_r(t, r(t))\theta_r(t, r(t))]\, dt + \sigma_r(t, r(t))dW_r(t) \\ [\mu_\lambda(t, \lambda(t)) + \sigma_\lambda(t, \lambda(t))\theta_\lambda(t, \lambda(t))]dt + \sigma_\lambda(t, \lambda(t))dW_\lambda(t) \end{pmatrix} \\ &= \nu(t, Z(t))dt + \xi(t, Z(t))dW(t) \ , \quad Z(0) = z_0 \ , \end{aligned} \tag{18}$$

where

$$\nu(t, Z(t)) := (\mu_r(t, r(t)) + \sigma_r(t, r(t))\theta_r(t, r(t)), \mu_\lambda(t, \lambda(t)) + \sigma_\lambda(t, \lambda(t))\theta_\lambda(t, \lambda(t)))^\top \in \Re^2 \ ,$$

and

$$\xi(t, Z(t)) := \begin{pmatrix} 0 & \sigma_r(t, r(t)) & 0 \\ 0 & 0 & \sigma_\lambda(t, \lambda(t)) \end{pmatrix} \in \Re^{2\times 3} \ .$$

Denote by

$$\Xi(t, Z(t)) := \xi(t, Z(t))\xi(t, Z(t))^\top \in \Re^{2\times 2} \ .$$

Definition 3.1. *A portfolio-consumption-bequest strategy (u, c, D) is said to be admissible, if the following conditions hold*

1. *the portfolio-consumption-bequest process (u, c, D) is an $\Re^3 \times \Re^+ \times \Re^+$-valued, $\mathbb{F}$-progressively measurable process such that*

$$\int_0^T |u(t)|^2 dt < \infty \ , \quad \int_0^T c(t)dt < \infty \ , \quad \int_0^T D(t)dt < \infty \ , \quad \mathcal{P}\text{-a.s.} \ ;$$

2. *given that $x_0 > 0$, the SDE (17) admits a unique strong solution such that*

$$X(t) + b(t) \geq 0 \ , \quad t \in \mathcal{T} \ , \quad \mathcal{P}\text{-a.s.} \ ,$$

where

$$b(t) = E^{\mathcal{Q}}\bigg[\int_t^T i(s) \exp\bigg\{ - \int_t^s [\lambda(u) + r(u)]du \bigg\} ds \bigg| \mathcal{F}(t) \bigg] \ ,$$

can be interpreted as the actuarial present value of the future income at time t;

3.

$$E\bigg[\int_0^T e^{-\int_0^s[\rho(u)+\lambda(u)]du}[U^-(c(s))+\alpha\lambda(s)U^-(D(s))]ds+\beta e^{-\int_t^T[\rho(u)+\lambda(u)]du}U^-(X(T))\bigg]<\infty\ .$$

Here U^- denotes the negative part of the function U.

Write $\mathcal{A}$ for the space of all admissible triples (u,c,D).

We now formulate the lifetime asset allocation problem as follows:

$$\textbf{Problem:}\qquad \begin{cases} \sup\limits_{(u,c,D)\in\mathcal{A}} J(u,c,D)\ , \\ \text{subject to } (X(\cdot),Z(\cdot)) \text{ satisfying (17) and (18)}\ . \end{cases}$$

4 General solutions

In this section, we employ the dynamic programming principle to solve the lifetime asset allocation problem. We first provide a verification theorem for the HJB equation of the problem. Then we derive explicit solutions of the problem from the HJB equation when the interest rate and force of mortality are governed by general diffusion processes.

To pave the way for the dynamic programming principle, given that $X(t)=x$ and $Z(t)=z$, we consider the following dynamic version of the performance functional:

$$\begin{aligned} &J(t,x,z;u,c,D)\\ &=\mathrm{E}\bigg[\int_t^T e^{-\int_t^s[\rho(\zeta)+\lambda(\zeta)]d\zeta}[U(c(s))+\alpha\lambda(s)U(D(s))]ds+\beta e^{-\int_t^T[\rho(\zeta)+\lambda(\zeta)]d\zeta}U(X(T))\bigg|\mathcal{F}(t)\bigg]\\ &=\mathrm{E}_{t,x,z}\bigg[\int_t^T e^{-\int_t^s[\rho(\zeta)+\lambda(\zeta)]d\zeta}[U(c(s))+\alpha\lambda(s)U(D(s))]ds+\beta e^{-\int_t^T[\rho(\zeta)+\lambda(\zeta)]d\zeta}U(X(T))\bigg]\ , \end{aligned}$$

where $\mathrm{E}_{t,x,z}[\cdot]$ denotes the conditional expectation under $\mathcal{P}$ given that $X(t)=x$ and $Z(t)=z$, i.e. $\mathrm{E}[\cdot|X(t)=x,Z(t)=z]$. The third line in the above equation holds since the processes $\{X(t)|t\in\mathcal{T}\}$ and $\{Z(t)|t\in\mathcal{T}\}$ are jointly Markovian with respect to the filtration $\mathbb{F}$. Then the value function of the problem is given by

$$V(t,x,z)=\sup_{(u,c,D)\in\mathcal{A}} J(t,x,z;u,c,D)\ . \tag{19}$$

Since the dynamics of the state processes $\{X(t)|t\in\mathcal{T}\}$ and $\{Z(t)|t\in\mathcal{T}\}$ are Markovian, it is not unreasonable to take the optimal control processes to be Markovian (see, for example, Elliott, 1982 and Øksendal, 2003). In what follows, we restrict ourselves to consider only Markovian controls for the problem. Let $\mathcal{O}:=(0,T)\times(-\infty,+\infty)\times(0,+\infty)\times(0,+\infty)$ be the solvency region. Suppose that K_i, $i=1,2,3$, denotes the set such that $u(t)\in K_1$, $c(t)\in K_2$ and $D(t)\in K_3$. To restrict ourselves to Markovian controls, we assume that

$$u(t)=\overline{u}(t,X(t),Z(t))\ ,\quad c(t)=\overline{c}(t,X(t),Z(t))\ ,\quad D(t)=\overline{D}(t,X(t),Z(t))\ ,$$

for some functions $\overline{u}:\mathcal{O}\to K_1$, $\overline{c}:\mathcal{O}\to K_2$ and $\overline{D}:\mathcal{O}\to K_3$. In what follows, we do not distinguish notationally between (u,c,D) and $(\overline{u},\overline{c},\overline{D})$ whenever no confusion arises. So, we can simply identify the control processes with deterministic functions $u(t,x,z)$, $c(t,x,z)$ and $D(t,x,z)$, for each $(t,x,z)\in\mathcal{O}$. These are called the feedback controls.

Let $\phi(\cdot,\cdot,\cdot):\mathcal{O}\to\Re$ be a function such that $\phi(\cdot,\cdot,\cdot)$ is $\mathcal{C}^{1,2,2,2}(\mathcal{O})$. We define the partial differential generator $\mathcal{L}^{u,c,D}$ acting on $\phi\in\mathcal{C}^{1,2,2,2}(\mathcal{O})$ as

$$\mathcal{L}^{u,c,D}[\phi(t,x,z)] \quad=\quad -[\rho(t)+\lambda]\phi+\phi_t+[rx+u\theta(t)^\top+i(t)-c-\lambda(D-x)]\phi_x+\frac{1}{2}uu^\top\phi_{xx}$$

$$+\nu(t,z)^\top \phi_z + \frac{1}{2}\text{tr}[\Xi(t,z)\phi_{zz}] + u\xi(t,z)^\top \phi_{xz} \ , \tag{20}$$

where we denote by $\phi := \phi(t,x,z)$ and the corresponding partial derivatives by ϕ_t, ϕ_x, ϕ_z, ϕ_{xx}, ϕ_{zz}, ϕ_{xz}.

The following theorem is a verification theorem for the HJB equation to the asset allocation problem, which plays a central role in deriving general solutions of the problem.

Theorem 4.1. *Suppose that $\overline{\mathcal{O}}$ is the closure of the solvency region $\mathcal{O}$ and there exists a function ϕ such that $\phi(\cdot,\cdot,\cdot) \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\overline{\mathcal{O}})$ and a Markovian control $(u^*, c^*, D^*) \in \mathcal{A}$ such that*

1. *$\mathcal{L}^{u,c,D}[\phi(t,x,z)] + U(c) + \alpha\lambda U(D) \leq 0$, for all $(u,c,D) \in \mathcal{A}$ and $(t,x,z) \in \mathcal{O}$;*
2. *$\mathcal{L}^{u^*,c^*,D^*}[\phi(t,x,z)] + U(c^*) + \alpha\lambda U(D^*) = 0$, for all $(t,x,z) \in \mathcal{O}$;*
3. *for all $(u,c,D) \in \mathcal{A}$,*

$$\lim_{t \to T^-} \phi(t, X(t), Z(t)) = \beta U(X(T)) \ ;$$

4. *the family $\{\phi(\kappa, X(\kappa), Z(\kappa))\}_{\kappa \in \mathcal{K}}$ is uniformly integrable, where $\mathcal{K}$ denote the set of stopping times $\kappa \leq T$.*

Then,

$$\begin{aligned} \phi(t,x,z) &= V(t,x,z) \\ &= \sup_{(u,c,D)\in\mathcal{A}} J(t,x,z;u,c,D) \\ &= J(t,x,z;u^*,c^*,D^*) \ , \end{aligned}$$

and (u^*, c^*, D^*) is an optimal Markovian control of the problem.

Proof. The proof can be adapted to that of Theorem 3.1 in Øksendal and Sulem (2005). So we omit it here. □

Note that we can rearrange conditions in Theorem 4.1 as the following HJB equation:

$$\begin{cases} \sup_{(u,c,D)\in\mathcal{A}} \{\mathcal{L}^{u,c,D}[V(t,x,z)] + U(c) + \alpha\lambda U(D)\} = 0 \ , \\ V(T,x,z) = \beta U(x) \ . \end{cases} \tag{21}$$

From Theorem 4.1, it is clear that the value function is a classical solution of the HJB equation (21). Solving the problem (19) is simplified to deriving the solution of the optimal portfolio-consumption-bequest process (u^*, c^*, D^*) or (π^*, c^*, D^*) and the value function V from (21).

Theorem 4.2. *The explicit expressions for the optimal strategies and the value function of the problem are given by*

$$\begin{aligned} \pi^*(t,x,z) &= \frac{x+b(t,z)}{1-\gamma}\left[\Sigma(t)^{-1}\sigma(t)\theta(t)^\top + (1-\gamma)\Sigma(t)^{-1}\sigma(t)\xi(t,z)^\top \frac{h_z(t,z)}{h(t,z)}\right] \\ &\quad -\Sigma(t)^{-1}\sigma(t)\xi(t,z)^\top b_z(t,z) \in \Re^3 \ , \end{aligned} \tag{22}$$

$$c^*(t,x,z) = \frac{x+b(t,z)}{h(t,z)} \ , \tag{23}$$

$$D^*(t,x,z) = \alpha^{\frac{1}{1-\gamma}} \frac{x+b(t,z)}{h(t,z)} \ , \tag{24}$$

and

$$V(t,x,z) = \frac{[x+b(t,z)]^{\gamma}}{\gamma} \times [h(t,z)]^{1-\gamma} \ , \tag{25}$$

where $b(\cdot,\cdot), h(\cdot,\cdot) \in \mathcal{C}^{1,2,2}(\mathcal{T} \times \Re^{+} \times \Re^{+})$ *are given by*

$$b(t,z) = E^{\mathcal{Q}}_{t,z}\bigg[\int_t^T i(s) \exp\bigg\{\int_t^s -[\lambda(u)+r(u)]du\bigg\}ds\bigg] \ , \tag{26}$$

and

$$h(t,z) = \widetilde{E}_{t,z}\bigg[\beta\Gamma(t,T) + \int_t^T (1+\alpha^{\frac{1}{1-\gamma}}\lambda(s))\Gamma(t,s)ds\bigg] \ . \tag{27}$$

Here $\Gamma(t,s)$, $0 \le t \le s \le T$, *is defined by*

$$\Gamma(t,s) := \exp\bigg\{-\int_t^s \bigg[\frac{1}{1-\gamma}\rho(\zeta) + \lambda(\zeta) - \frac{\gamma}{1-\gamma}r(\zeta) - \frac{\gamma}{2(1-\gamma)^2}|\theta(\zeta)|^2\bigg]d\zeta\bigg\} \ , \tag{28}$$

and $E^{\mathcal{Q}}_{t,z}[\cdot]$ *and* $\widetilde{E}_{t,z}[\cdot]$ *denote the conditional expectations under* $\mathcal{Q}$ *and* $\widetilde{\mathcal{P}}$ *given that* $Z(t) = z$, *respectively, where* $\mathcal{Q}$ *is the risk-neutral measure and* $\widetilde{\mathcal{P}}$ *is a probability measure equivalent to* $\mathcal{P}$ *on* $\mathcal{F}(T)$ *as*

$$\frac{d\widetilde{\mathcal{P}}}{d\mathcal{P}}\bigg|_{\mathcal{F}(T)} = \exp\bigg\{-\frac{1}{2}\frac{\gamma^2}{(1-\gamma)^2}\int_0^T |\theta(t)|^2 dt + \frac{\gamma}{1-\gamma}\int_0^T \theta(t)dW(t)\bigg\} \ .$$

Proof. See the Appendix. □

Remark 4.1. *Here the functions* b *and* h *can be interpreted as the actuarial present value of the future income and the wealth-consumption ratio, respectively. Besides the given model parameters, the optimal strategies* (π^*, c^*, D^*) *and the value function* V *depend on the functions* b *and* h *as well as their derivatives. From the relation between* p *and* D, *we can see the optimal life insurance purchase strategy is given by*

$$p^*(t,x,z) = \lambda\bigg[\bigg(\frac{\alpha^{\frac{1}{1-\gamma}}}{h(t,z)} - 1\bigg)x + \frac{\alpha^{\frac{1}{1-\gamma}}}{h(t,z)}b(t,z)\bigg] \ . \tag{29}$$

Although systematic risk is present, if $0 < \frac{\alpha^{\frac{1}{1-\gamma}}}{h(t,z)} < 1$, *a similar insurance principle as in Pliska and Ye (2007) holds: the current wealth of the investor has a negative effect on his life insurance purchases, while the actuarial present value of future income has a positive effect on his life insurance purchases.*

5 Special solutions

In this section, we use square-root stochastic interest rate and mortality models to illustrate our results. To simplify our analysis, we assume that the coefficients in our model dynamics are time-constant. From Theorem 4.2, it is clear that

$$b(t,r,\lambda) = i\int_t^T \varphi_1(t,s,r,\lambda)ds \ , \tag{30}$$

and

$$h(t,r,\lambda) = \beta\Gamma_0(t,T)\varphi_2(t,T,r,\lambda) + \int_t^T \Gamma_0(t,s)[\varphi_2(t,s,r,\lambda) + \alpha^{\frac{1}{1-\gamma}}\varphi_3(t,s,r,\lambda)]ds \ , \tag{31}$$

where

$$\Gamma_0(t,s) := \exp\left\{ -\left[\frac{\rho}{1-\gamma} - \frac{\gamma}{2(1-\gamma)^2}|\theta_S|^2\right](s-t)\right\} , \tag{32}$$

and

$$\varphi_1(t,s,r,\lambda) = \mathrm{E}^{\mathcal{Q}}_{t,r,\lambda}\left[\exp\left\{ -\int_t^s [\lambda(u)+r(u)]du\right\}\right] , \tag{33}$$

$$\varphi_2(t,s,r,\lambda) = \widetilde{\mathrm{E}}_{t,r,\lambda}\left[\exp\left\{ -\int_t^s \left(\lambda(u) - \frac{\gamma}{1-\gamma}r(u) - \frac{\gamma}{2(1-\gamma)^2}[|\theta_r(u)|^2+|\theta_\lambda(u)|^2]\right)du\right\}\right] , \tag{34}$$

$$\varphi_3(t,s,r,\lambda) = \widetilde{\mathrm{E}}_{t,r,\lambda}\left[\lambda(s)\exp\left\{ -\int_t^s \left(\lambda(u) - \frac{\gamma}{1-\gamma}r(u) - \frac{\gamma}{2(1-\gamma)^2}[|\theta_r(u)|^2+|\theta_\lambda(u)|^2]\right)du\right\}\right] . \tag{35}$$

Therefore, once we can calculate (33)-(35) under specific interest rate and mortality models, we completely solve the optimal asset allocation problem.

In what follows, we consider the following square-root models for the short rate process and the force of mortality process:

$$dr(t) = \mu_r[\bar{r} - r(t)]dt + \sigma_r\sqrt{r(t)}dW_r^{\mathcal{Q}}(t) , \tag{36}$$

and

$$d\lambda(t) = \mu_\lambda\lambda(t)dt + \sigma_\lambda\sqrt{\lambda(t)}dW_\lambda^{\mathcal{Q}}(t) . \tag{37}$$

Suppose that the market prices of risks of the Brownian motions $W_r(\cdot)$ and $W_\lambda(\cdot)$ are given by

$$\theta_r(t) = \theta_r\sqrt{r(t)} , \quad \theta_\lambda(t) = \theta_\lambda\sqrt{\lambda(t)} . \tag{38}$$

To obtain closed-form expressions for the optimal strategies and the value function, we derive φ_i, for $i = 1,2,3$, in the following several propositions.

Proposition 5.1. *The function φ_1 is given by the following closed-form expression*

$$\varphi_1(t,s,r,\lambda) = \exp\left\{ L_r(t,s) - rK_r(t,s) - \lambda K_\lambda(t,s)\right\} , \tag{39}$$

where

$$\begin{aligned} K_r(t,s) &= \frac{2(e^{\eta_r(s-t)}-1)}{(\eta_r+\mu_r)(e^{\eta_r(s-t)}-1)+2\eta_r} , \quad \eta_r := \sqrt{\mu_r^2+2\sigma_r^2} , \\ K_\lambda(t,s) &= \frac{2(e^{\eta_\lambda(s-t)}-1)}{(\eta_\lambda-\mu_\lambda)(e^{\eta_\lambda(s-t)}-1)+2\eta_\lambda} , \quad \eta_\lambda := \sqrt{\mu_\lambda^2+2\sigma_\lambda^2} , \end{aligned}$$

and

$$L_r(t,s) = \frac{2\mu_r\bar{r}}{\sigma_r^2}\ln\left(\frac{2\eta_r e^{(\eta_r+\mu_r)(s-t)/2}}{(\eta_r+\mu_r)(e^{\eta_r(s-t)}-1)+2\eta_r}\right) .$$

Proof. See the Appendix. □

Proposition 5.2. *The function φ_2 is given by the following explicit expression*

$$\varphi_2(t,s,r,\lambda) = \exp\left\{ L_r^\gamma(t,s) - rK_r^\gamma(t,s) - \lambda K_\lambda^\gamma(t,s) \right\} , \tag{40}$$

where

$$\begin{aligned} K_r^\gamma(t,s) &= \frac{R_{r2}(s-t)}{R_{r1}(s-t)} , \\ K_\lambda^\gamma(t,s) &= \frac{R_{\lambda 2}(s-t)}{R_{\lambda 1}(s-t)} , \end{aligned}$$

and

$$L_r^\gamma(t,s) = \mu_r \bar{r} \int_t^s K_r^\gamma(u,s) du ,$$

where

$$\begin{pmatrix} R_{r1}(s-t) \\ R_{r2}(s-t) \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & \frac{1}{2}\sigma_r^2 \\ -\frac{\gamma}{1-\gamma} - \frac{\gamma\theta_r^2}{2(1-\gamma)^2} & -(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r) \end{pmatrix} (s-t) \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{41}$$

and

$$\begin{pmatrix} R_{\lambda 1}(s-t) \\ R_{\lambda 2}(s-t) \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & \frac{1}{2}\sigma_\lambda^2 \\ 1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} & \mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda \end{pmatrix} (s-t) \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{42}$$

Proof. See the Appendix. □

Proposition 5.3. *The function φ_3 is given by the following explicit expression*

$$\begin{aligned} \varphi_3(t,s,r,\lambda) &= \lambda \exp\left\{ L_r^\gamma(t,s) - rK_r^\gamma(t,s) - \lambda K_\lambda^\gamma(t,s) \right\} \\ &\quad \times \exp\left\{ \left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda \right)(s-t) - \sigma_\lambda^2 \int_t^s K_\lambda^\gamma(u,s) du \right\} , \end{aligned} \tag{43}$$

where K_r^γ, K_λ^γ and L_r^γ are defined in Proposition 5.2.

Proof. See the Appendix. □

Proposition 5.4. *Suppose that $\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2} < 0$ and $1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} > 0$. The functions K_r^γ, K_λ^γ and L_r^γ in Proposition 5.2 are given by the following closed-form expressions*

$$\begin{aligned} K_r^\gamma(t,s) &= -\left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2} \right] \times \frac{2(e^{\eta_r^\gamma(s-t)} - 1)}{(\eta_r^\gamma - \mu_r + \frac{1}{1-\gamma}\sigma_r\theta_r)(e^{\eta_r^\gamma(s-t)} - 1) + 2\eta_r^\gamma} , \\ K_\lambda^\gamma(t,s) &= \left[1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} \right] \times \frac{2(e^{\eta_\lambda^\gamma(s-t)} - 1)}{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(e^{\eta_\lambda^\gamma(s-t)} - 1) + 2\eta_\lambda^\gamma} , \end{aligned}$$

and

$$L_r^\gamma = \frac{2\mu_r\bar{r}}{\sigma_r^2} \ln\left\{ \frac{2\eta_r^\gamma e^{(\eta_r^\gamma - \mu_r + \frac{1}{1-\gamma}\sigma_r\theta_r)(s-t)/2}}{(\eta_r^\gamma - \mu_r + \frac{1}{1-\gamma}\sigma_r\theta_r)(e^{\eta_r^\gamma(s-t)} - 1) + 2\eta_r^\gamma} \right\} ,$$

where

$$\eta_r^\gamma := \sqrt{\left(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r\right)^2 - 2\sigma_r^2\left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2}\right]}\ ,$$

and

$$\eta_\lambda^\gamma := \sqrt{\left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)^2 + 2\sigma_\lambda^2\left[1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2}\right]}\ .$$

Proof. See the Appendix. □

Proposition 5.5. *Suppose that* $\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2} < 0$ *and* $1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} > 0$. *The function* φ_3 *is given by the following closed-form expression*

$$\begin{aligned}\varphi_3(t,s,r,\lambda) &= \lambda\exp\left\{L_r^\gamma(t,s) - rK_r^\gamma(t,s) - \lambda K_\lambda^\gamma(t,s) + \left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)(s-t)\right.\\ &\quad \left. -2\ln\left[\frac{2\eta_\lambda^\gamma e^{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(s-t)/2}}{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(e^{\eta_\lambda^\gamma(s-t)} - 1) + 2\eta_\lambda^\gamma}\right]\right\}. \qquad (44)\end{aligned}$$

where K_r^γ, K_λ^γ, L_r^γ *and* η_λ^γ *are defined in Proposition 5.4.*

Proof. See the Appendix. □

Remark 5.1. *We use square-root models (36)-(37) to illustrate our results for two reasons. Firstly, they are tractable and lead to closed-form solutions for the problem. Secondly, they are theoretically sound and practically meaningful since no controversial issue of negative interest rate and force of mortality will arise. An extension to other stochastic mortality models, such as the multi-factor model (see Blackburn and Sherris, 2013) which are less tractable would require the Monte Carlo method to calculate expectations (33)-(35).*

6 Numerical examples

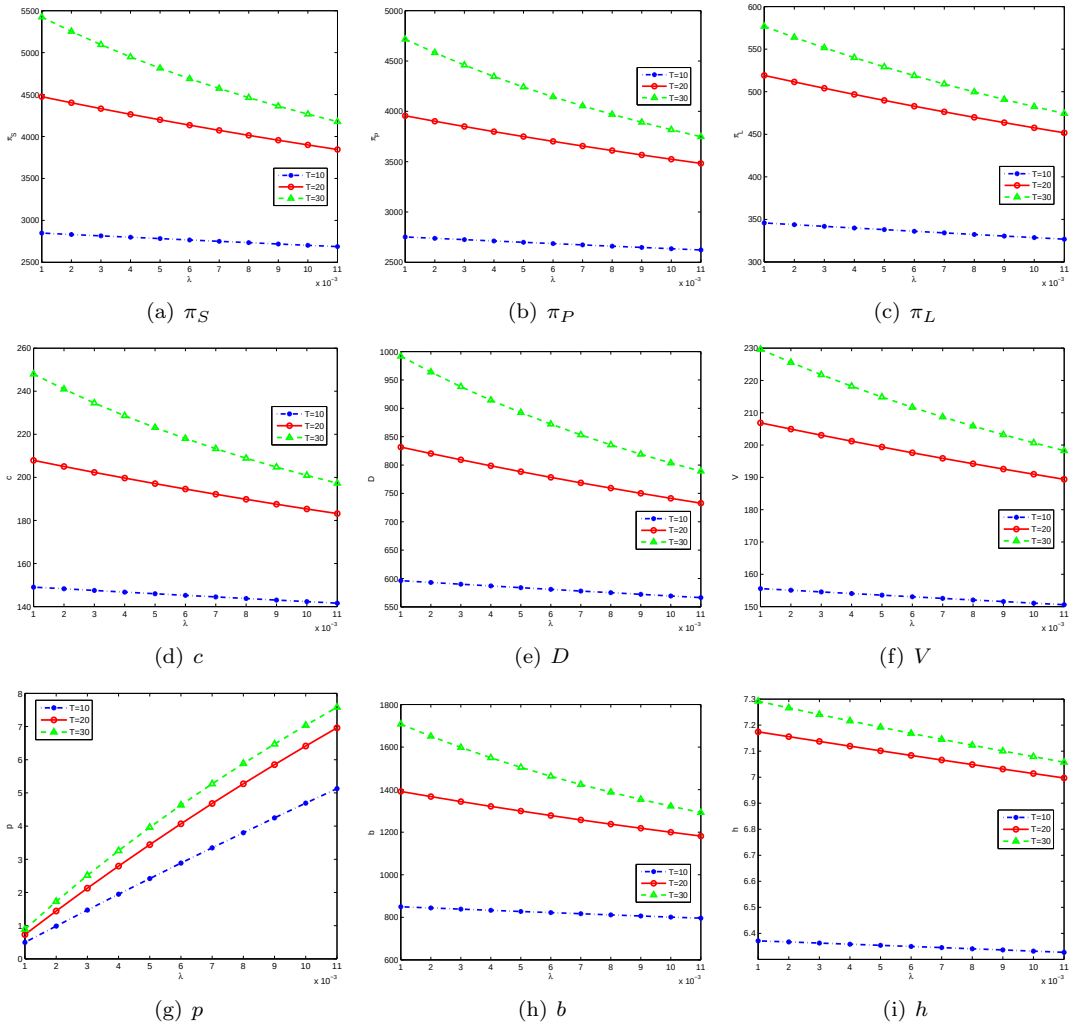
In this section, we provide numerical examples for special solutions of the optimal strategies and the value function given in the previous section. We are interested in the impacts of the different parameters, particularly those of the stochastic mortality model, on the optimal solutions, which is shown from the sensitivity analyses of these optimal solutions and other related quantities. We consider the following hypothetical values of the model parameters

$$\begin{gathered}t = 0\ ,\quad x = 100\ ,\quad i = 100\ ,\quad \alpha = 2\ ,\quad \beta = 2\ ,\quad \gamma = 0.5\ ,\quad \rho = 0.1\ ,\quad T_1 = 40\ ,\\ T_2 = 40\ ,\quad \lambda = 0.001\ ,\quad \mu_\lambda = 0.1\ ,\quad \sigma_\lambda = 0.001\ ,\quad \mu_r = 0.2\ ,\quad \sigma_r = 0.08\ ,\quad \bar{r} = 0.04\ ,\\ r = 0.02\ ,\quad \sigma_S = 0.20\ ,\quad \sigma_{Sr} = 0.05\ ,\quad \mu_S = 0.08\ ,\quad \theta_S = 0.3\ ,\quad \theta_r = -0.3\ ,\quad \theta_\lambda = -0.1\ .\end{gathered}$$

It is worth mentioning that the values of model parameters chosen are not calibrated from real data. Although we do not estimate the model parameters, we try to use reasonable values. In Huang et al. (2012), when the relative risk aversion coefficient is 0.5, the investor will consume less in a stochastic mortality environment compared with a deterministic one. We believe this is a reasonable behavior pattern of a rational human being since the uncertainty in the mortality rate may render the investor more conservative towards consumption. So we take $\gamma = 0.5$. In addition, the values of model parameters of stochastic interest rate and mortality models are close (or of a similar order of magnitude) to those estimated in related literature (see Chan et al., 1992 and

Luciano and Vigna, 2005) and the values of the stock model and market prices of risks are also meant to be representative. In what follows, we vary the value of one parameter of the stochastic mortality model each time and discuss the impacts of different parameters, including λ, μ_λ and σ_λ, on the optimal solutions.

Fig. 1 reports the optimal solutions of investment-consumption-bequest and the value function with the initial force of mortality $\lambda(t) = \lambda$ taking different values from $\{0.001, 0.002, \cdots, 0.011\}$. It can be seen that



(a) π_S (b) π_P (c) π_L

(d) c (e) D (f) V

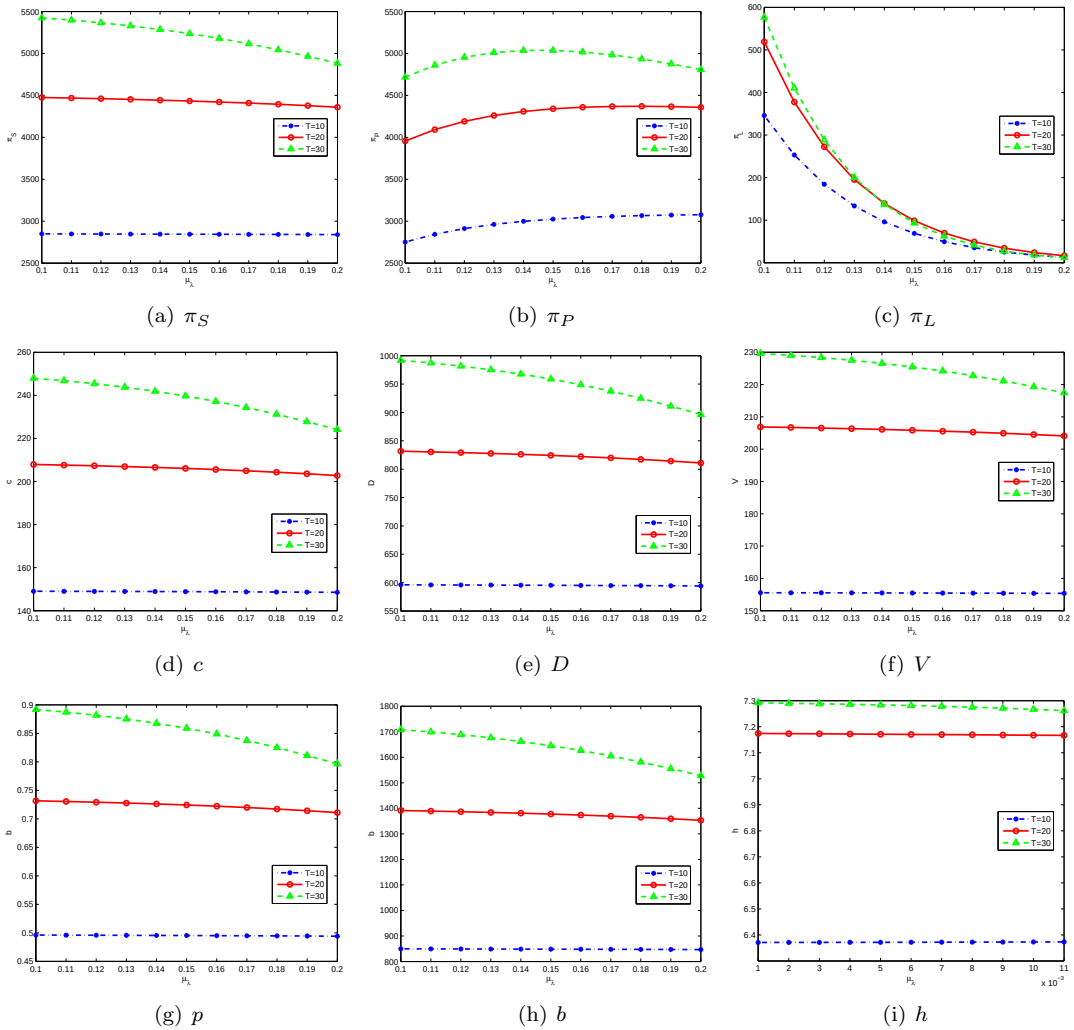
(g) p (h) b (i) h

Figure 1: Optimal solutions with different values of λ when $T = 10, 20, 30$

as the initial values of the force of mortality increases, the optimal strategies for the agent are to allocate less wealth in the stock, the bond and the longevity bond, consume less, leave less money as the bequest to his beneficiary and purchase more life insurance. In addition, the value function of the problem decreases with the initial value of the force of mortality. From Theorem 4.1, we can see both the optimal strategies (π, c, D) and the value function V have proportional relationships with the actuarial present value of the future income, b. Although (π, c, D) and V also depend on the wealth-consumption ratio function h, the change of h is insignificant compared with that of b (refer to Figs. 1(h) and (i)). Thus the value of b plays a dominant role in determining the trends of (π, c, D) and V. As the initial value of the force of mortality increases, the agent's life expectancy

becomes shorter and thus the actuarial present value of the future income becomes smaller (refer to Fig. 1(h)). Indeed, this can be also seen from the explicit expression for b (see Eqs. (30) and (39)). From Eq. (29), we can see the increase of the demand for life insurance is caused by the increase of the product of λ and b when λ becomes larger. Since λ increases with age, Figs 1 (g) and (h) show that the older investor has a higher demand of the life insurance although his future income is less.

From Fig. 2, we can see as μ_λ varies from 0.1 to 0.2, the optimal strategies and the value function have different trends. The larger is the exponential increasing parameter μ_λ, the smaller will be the survival probability of the investor in the future. The life expectancy of the investor will become shorter as μ_λ increases. If so, the actuarial present value of the future income b is smaller and the investor becomes more conservative in investment, consumption, bequest and purchase of life insurance. However, the impacts of μ_λ on the optimal investment in the stock, the optimal consumption and bequest, and the value function is not significant when the time horizon is 10 or 20 years. Only when the investment horizon is sufficiently long, say, 30 years, does the



(a) π_S (b) π_P (c) π_L

(d) c (e) D (f) V

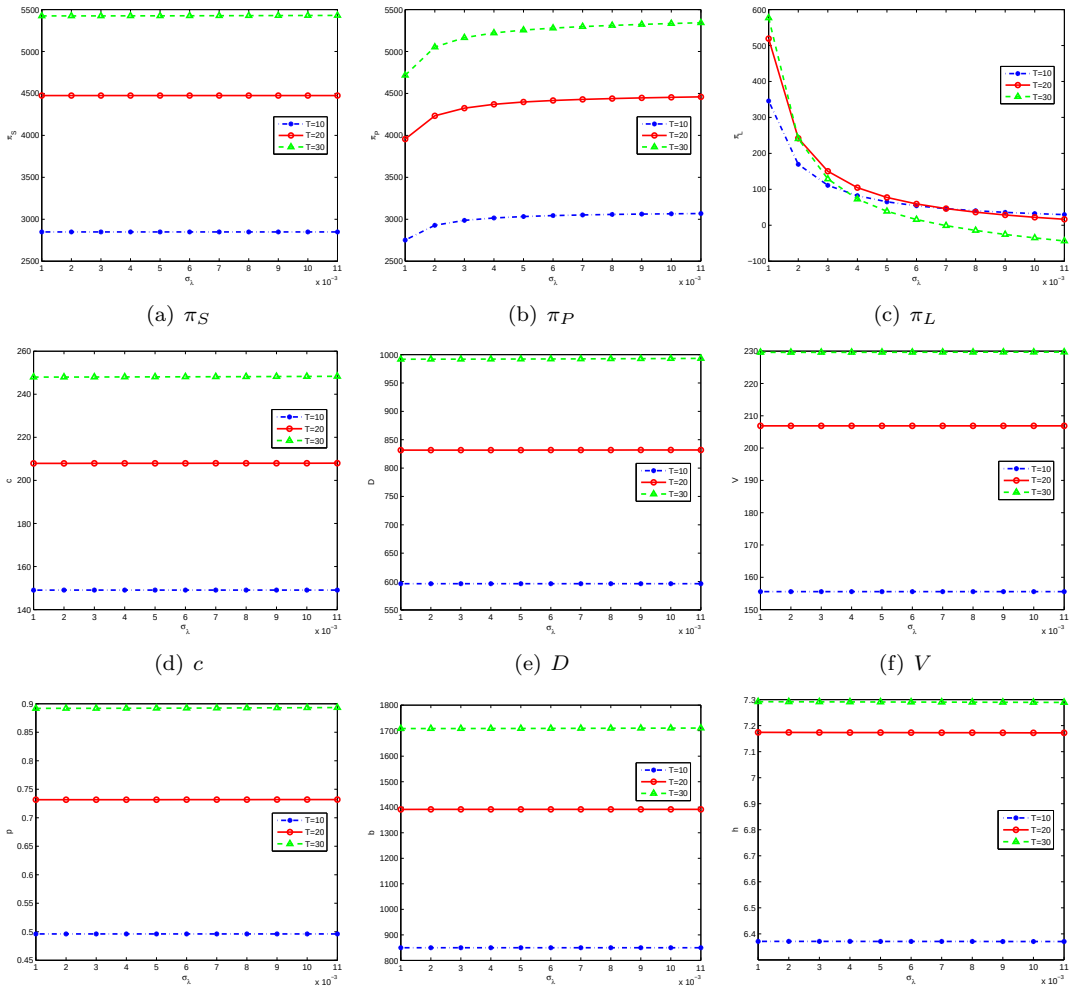
(g) p (h) b (i) h

Figure 2: Optimal solutions with different values of μ_λ when $T = 10, 20, 30$

impact of the increasing force of mortality or the decreasing life expectancy on the investor's decision making

become significant. On the other hand, since the investor's future life is shorter when μ_λ is larger, his demand for the longevity bond to hedge the longevity risk decreases. The increase in demand for the zero-coupon bond occurs since the dynamics of the ordinary and the longevity bond is correlated through the Brownian motion $W_r(\cdot)$. Although the investor has initially an increase in the zero-coupon bond, the increase of μ_λ results in a smaller b which offsets the effect of this. Therefore, it is interesting to note in Fig. 2 (b) that the optimal investment in the zero-coupon has an inverted U-shape, which is more noticeable when the investment horizon is 30 years.

In Fig. 3, we can see that the volatility of the force of mortality process has a significant impact on the optimal investment in the zero-coupon bond and the longevity bond while almost has no impact on stock holding, consumption or other values. As the volatility σ_λ increases from to 0.001 to 0.011, the force of mortality become



(a) π_S (b) π_P (c) π_L

(d) c (e) D (f) V

(g) p (h) b (i) h

Figure 3: Optimal solutions with different values of σ_λ when $T = 10, 20, 30$

more uncertain. However, varying the volatility σ_λ changes the random disturbance of the force of mortality, the impact of which is offset by adjusting the holding in the zero-coupon bond and the longevity bond. The longevity bond becomes more risky as the value of σ_λ increases. Whereas, the zero-coupon bond becomes

relatively safer due to the correlation of the dynamics of the longevity bond and the zero-coupon bond. So the optimal amounts of wealth invested in the zero-coupon bond has a negative relation, and optimal amounts of wealth invested in the longevity bond has a positive relation with respect to the value of σ_λ.

The initial value of the force of mortality λ determines the level of idiosyncratic mortality risk while the exponential increasing parameter μ_λ and the volatility of the mortality model σ_λ determine the level of systematic mortality risk. From Figs. 1-3, we can see that the longevity bond is an efficient tool to hedge the systematic mortality risk. This is not surprising since the longevity bond is linked to the survivor index for the whole population. From the perspective of the investor, the remaining idiosyncratic mortality risk is managed by purchasing life insurance.

7 Conclusion

We investigated an optimal investment, consumption and life insurance purchase problem under a stochastic mortality model. Both idiosyncratic and systematic mortality risks were incorporated in the modeling framework. Using the dynamic programming principle and the HJB equation, we derived explicit solutions of the problem when the interest rate and the force of mortality followed general diffusion models. Particularly, when general diffusion models had square-root structures, we provided closed-form expressions for the optimal strategies and the value function. Using numerical examples, we assess sensitivity of the results to different parameters of the stochastic mortality model. Longevity bonds and life insurance hedge the systematic and idiosyncratic mortality risk, respectively. The impact of systematic mortality risk on the investor's consumption-investment decisions are significant for longer investment horizons.

Appendix

Proof of Theorem 4.2. For all $(t,x,z) \in \mathcal{O}$ and $(u,c,D) \in \mathcal{A}$, denote by

$$\Psi(t,x,z;u,c,D) := \mathcal{L}^{u,c,D}[V(t,x,z)] + U(c) + \alpha\lambda U(D) \ .$$

For notational simplicity, we write

$$V := V(t,x,z) \ ,$$

whenever no confusion arises. To ensure there exists a regular interior maximum, the following set of sufficient conditions must be satisfied: (i) $\Psi_{uu} = V_{xx}\mathbf{I}_{3\times 3}$ is a negative-definite matrix; and (ii) $\Psi_{cc} = U_{cc}(c) < 0$; and (iii) $\Psi_{DD} = \alpha\lambda U_{DD}(D) < 0$. Otherwise, the problem has no solution. Note that the utility function U is strictly concave. Obviously, Conditions (ii)-(iii) are satisfied. Since the identity matrix $\mathbf{I}_{3\times 3}$ is uniformly positive-definite, Condition (i) is satisfied if and only if $V_{xx} < 0$. We assume $V_{xx} < 0$ at this stage and will verify this at the end of the proof.

Applying the first order conditions for maximizing $\Psi(t,x,z;u,c,D)$ with respect to (u,c,D) yields that

$$\begin{aligned}
\Psi_u &= \theta(t)^\top V_x + u^\top V_{xx} + \xi(t,z)^\top V_{xz} = 0 \ , && \text{(A1)}\\
\Psi_c &= -V_x + U_c = 0 \ , && \text{(A2)}\\
\Psi_p &= -\lambda V_x + \alpha\lambda U_D = 0 \ . && \text{(A3)}
\end{aligned}$$

Solving (A1)-(A3) gives that the optimal strategies are given by

$$\begin{aligned}
u^*(t,x,z) &= -\theta(t)\frac{V_x}{V_{xx}} - \xi(t,z)\frac{V_{xz}}{V_{xx}} \ , && \text{(A4)}\\
c^*(t,x,z) &= (U')^{-1}(V_x) \ , && \text{(A5)}
\end{aligned}$$

$$D^*(t,x,z) \quad = \quad (U')^{-1}\left(\frac{1}{\alpha}V_x\right) , \tag{A6}$$

where $(U')^{-1}$ denotes the inverse function of the first-order derivative of U. Using the relationship between the original portfolio process and the transformed process immediately yields

$$\pi^*(t,x,z) = -\Sigma(t)^{-1}\sigma(t)\theta(t)^\top \frac{V_x}{V_{xx}} - \Sigma(t)^{-1}\sigma(t)\xi(t,z)^\top \frac{V_{xz}}{V_{xx}} . \tag{A7}$$

Substituting (A4)-(A6) into the HJB equation (21) gives

$$\begin{aligned}
&-[\rho(t)+\lambda]V + V_t + [rx+i(t)]V_x + \nu(t,z)^\top V_z + \frac{1}{2}\mathrm{tr}\big[\Xi(t,z)V_{zz}\big] \\
&-\frac{1}{2}|\theta(t)|^2\frac{V_x^2}{V_{xx}} - \theta(t)\xi(t,z)^\top \frac{V_x V_{xz}}{V_{xx}} - \frac{1}{2}\frac{V_{xz}^\top \Xi(t,z) V_{xz}}{V_{xx}} + U\big((U')^{-1}(V_x)\big) \\
&+\alpha\lambda U\left((U')^{-1}\left(\frac{1}{\alpha}V_x\right)\right) - \left[(U')^{-1}(V_x) + \lambda\left((U')^{-1}\left(\frac{1}{\alpha}V_x\right) - x\right)\right]V_x = 0 .
\end{aligned} \tag{A8}$$

From the terminal condition of the value function, we try the following parametric form

$$V(t,x,z) = \frac{[x+b(t,z)]^\gamma}{\gamma} \times [h(t,z)]^{1-\gamma} . \tag{A9}$$

where $b(\cdot,\cdot), h(\cdot,\cdot) \in \mathcal{C}^{1,2,2}(\mathcal{T}\times\Re^+\times\Re^+)$. Substituting (A9) into (A4)-(A6) leads to (22)-(24). Furthermore, substituting (A9) into (A8) gives

$$\begin{aligned}
&[x+b]^{\gamma-1}h^{1-\gamma}\left\{b_t - (r+\lambda)b + \big[\nu(t,z)^\top - \theta(t)\xi(t,z)^\top\big]b_z + \frac{1}{2}\mathrm{tr}\big[\Xi(t,z)b_{zz}\big] + i(t)\right\} \\
&+\frac{1-\gamma}{\gamma}[x+b]^\gamma h^{-\gamma}\left\{h_t - \left[\lambda + \frac{\rho(t)-\gamma r(t)}{1-\gamma} - \frac{\gamma}{2(1-\gamma)^2}|\theta(t)|^2\right]h\right. \\
&\left.+\left[\nu(t,z) + \frac{\gamma}{1-\gamma}\theta(t)\xi(t,z)^\top\right]h_z + \frac{1}{2}\mathrm{tr}\big[\Xi(t,z)h_{zz}\big] + (1+\alpha^{\frac{1}{1-\gamma}}\lambda)\right\} = 0 .
\end{aligned} \tag{A10}$$

Therefore, letting the coefficients of $[y+b]^{\gamma-1}h^{1-\gamma}$ and $[y+b]^\gamma h^{-\gamma}$ equal to zeros gives that the functions $b(t,z)$ and $h(t,z)$ satisfy the following two parabolic partial differential equations (PDEs), respectively,

$$b_t - (r+\lambda)b + [\nu(t,z)^\top - \theta(t)\xi(t,z)^\top]b_z + \frac{1}{2}\mathrm{tr}\big[\Xi(t,z)b_{zz}\big] + i(t) = 0 , \tag{A11}$$

and

$$\begin{aligned}
&h_t - \left[\frac{1}{1-\gamma}\rho(t) + \lambda - \frac{\gamma}{1-\gamma}r - \frac{\gamma}{2(1-\gamma)^2}|\theta(t)|^2\right]h \\
&+\left[\nu(t,z)^\top + \frac{\gamma}{1-\gamma}\theta(t)\xi(t,z)^\top\right]h_z + \frac{1}{2}\mathrm{tr}\big[\Xi(t,z)h_{zz}\big] + (1+\alpha^{\frac{1}{1-\gamma}}\lambda) = 0 ,
\end{aligned} \tag{A12}$$

with the terminal conditions $b(T,z) = 0$ and $h(T,z) = \beta$.

To solve the PDEs (A11)-(A12), we employ the Feynman-Kac formula. From Girsanov's Theorem, we can see

$$\widetilde{W}(t) = W(t) - \frac{\gamma}{1-\gamma}\int_0^t \theta(s)^\top ds \tag{A13}$$

is a three-dimensional standard Brownian motion with respect to $\widetilde{\mathcal{P}}$. Then, the dynamics of the two-factor state process $\{Z(t)|t \in \mathcal{T}\}$ under $\mathcal{Q}$ and $\widetilde{\mathcal{P}}$ are governed by the following two SDEs

$$dZ(t) = [\nu(t, Z(t)) - \xi(t, Z(t))\theta(t)^\top]dt + \xi(t, Z(t))dW^{\mathcal{Q}}(t) \ , \tag{A14}$$

and

$$dZ(t) = \left[\nu(t, Z(t)) + \frac{\gamma}{1-\gamma}\xi(t, Z(t))\theta(t)^\top\right]dt + \xi(t, Z(t))d\widetilde{W}(t) \ . \tag{A15}$$

Therefore, using the Feynman-Kac formula to (A11)-(A12), we obtain that the solutions of $b(t,z)$ and $h(t,z)$ are given by the expectation representations (26)-(27), respectively.

From (26)-(27), it is clear that

$$b(t,z) > 0 \ , \quad h(t,z) > 0 \ , \tag{A16}$$

for all $(t,z) \in \mathcal{T} \times \Re^+ \times \Re^+$. So Assumption $V_{xx} < 0$ holds, i.e.

$$V_{xx} = (\gamma - 1)[y + b(t,z)]^{\gamma-1}[h(t,z)]^{1-\gamma} < 0 \ . \tag{A17}$$

Therefore, (22)-(24) are indeed the optimal feedback control processes of the problem. This completes the proof. □

Proof of Proposition 5.1. From the Feynman-Kac formula, we have that $\varphi_1(\cdot, s, \cdot, \cdot) \in \mathcal{C}^{1,2,2}((0,s) \times (0,\infty) \times (0,\infty))$, for each fixed $s \in \mathcal{T}$, is the solution of the following parabolic partial differential equation

$$\begin{cases} \dfrac{\partial \varphi_1}{\partial t} + \mu_r(\bar{r} - r)\dfrac{\partial \varphi_1}{\partial r} + \mu_\lambda \lambda \dfrac{\partial \varphi_1}{\partial \lambda} + \dfrac{1}{2}\sigma_r^2 r \dfrac{\partial^2 \varphi_1}{\partial r^2} + \dfrac{1}{2}\sigma_\lambda^2 \lambda \dfrac{\partial^2 \varphi_1}{\partial \lambda^2} = (r + \lambda)\varphi_1 \ , \\ \varphi_1(s, s, r, \lambda) = 1 \ . \end{cases} \tag{A18}$$

We try the following parametric form solution

$$\varphi_1(t, s, r, \lambda) = \exp\left\{L_r(t,s) - rK_r(t,s) - \lambda K_\lambda(t,s)\right\} \ . \tag{A19}$$

Substituting (A19) into (A18) gives two Ricatti ODEs

$$\frac{d}{dt}K_r(t,s) - \mu_r K_r(t,s) - \frac{1}{2}\sigma_r^2 K_r^2(t,s) + 1 = 0 \ , \quad K_r(s,s) = 0 \ , \tag{A20}$$

$$\frac{d}{dt}K_\lambda(t,s) + \mu_\lambda K_\lambda(t,s) - \frac{1}{2}\sigma_\lambda^2 K_\lambda^2(t,s) + 1 = 0 \ , \quad K_\lambda(s,s) = 0 \ , \tag{A21}$$

and one linear ODE

$$\frac{d}{dt}L_r(t,s) - \mu_r \bar{r} K_r(t,s) = 0 \ , \quad L_r(s,s) = 0 \ . \tag{A22}$$

Solving (A20)-(A22) yields the desired results. □

Proof of Proposition 5.2. The dynamics of $\{r(t)|t \in \mathcal{T}\}$ and $\{\lambda(t)|t \in \mathcal{T}\}$ under $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are

$$\begin{aligned} dr(t) &= [\mu_r \bar{r} - (\mu_r - \sigma_r\theta_r)r(t)]dt + \sigma_r\sqrt{r(t)}dW_r(t) \ , \\ d\lambda(t) &= (\mu_\lambda + \sigma_\lambda\theta_\lambda)\lambda(t)dt + \sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t) \ , \end{aligned}$$

and

$$\begin{aligned} dr(t) &= \left[\mu_r \bar{r} - \left(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r\right) r(t)\right] dt + \sigma_r \sqrt{r(t)} d\widetilde{W}_r(t) \ , \\ d\lambda(t) &= \left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)\lambda(t)dt + \sigma_\lambda\sqrt{\lambda(t)}d\widetilde{W}_\lambda(t) \ . \end{aligned}$$

Denote by

$$\begin{aligned} r^\gamma(t) &:= -\left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2}\right] r(t) \ , \\ \lambda^\gamma(t) &:= \left[1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2}\right]\lambda(t) \ . \end{aligned}$$

Therefore, (34) can be rewritten as

$$\varphi_2(t,s,r,\lambda) = \widetilde{\mathrm{E}}_{t,r,\lambda}\left[\exp\left\{-\int_t^s [\lambda^\gamma(u) + r^\gamma(u)]du\right\}\right] \ .$$

As in the proof of Proposition 5.1, we can derive that

$$\varphi_2(t,s,r,\lambda) = \exp\left\{L_r^\gamma(t,s) - rK_r^\gamma(t,s) - \lambda K_\lambda^\gamma(t,s)\right\} \ ,$$

where K_r^γ and K_λ^γ satisfy the following two Riccati equations

$$\begin{cases} \dfrac{d}{dt}K_r^\gamma(t,s) - \left(\mu_r - \dfrac{1}{1-\gamma}\sigma_r\theta_r\right)K_r^\gamma(t,s) - \dfrac{1}{2}\sigma_r^2(K_r^\gamma(t,s))^2 - \left[\dfrac{\gamma}{1-\gamma} + \dfrac{\gamma\theta_r^2}{2(1-\gamma)^2}\right] = 0 \ , \\ K_r^\gamma(s,s) = 0 \ , \end{cases} \tag{A23}$$

$$\begin{cases} \dfrac{d}{dt}K_\lambda^\gamma(t,s) + \left(\mu_\lambda + \dfrac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)K_\lambda^\gamma(t,s) - \dfrac{1}{2}\sigma_\lambda^2(K_\lambda^\gamma(t,s))^2 + \left[1 - \dfrac{\gamma\theta_\lambda^2}{2(1-\gamma)^2}\right] = 0 \ , \\ K_\lambda^\gamma(s,s) = 0 \ . \end{cases} \tag{A24}$$

and L_r^γ satisfies

$$\frac{d}{dt}L_r^\gamma(t,s) - \mu_r\bar{r}K_r^\gamma(t,s) = 0 \ , \quad L_r^\gamma(s,s) = 0 \ . \tag{A25}$$

In general, the last terms on the left hand sides of (A23)-(A24) may be negative. So the solutions of (A23)-(A24) do not have closed-form expressions as K_r and K_λ in Proposition 5.1. We consider the following parametric forms as solutions for K_r^γ and K_λ^γ

$$K_r^\gamma(t,s) = \frac{R_{r2}(s-t)}{R_{r1}(s-t)} \ ,$$

and

$$K_\lambda^\gamma(t,s) = \frac{R_{\lambda 2}(s-t)}{R_{\lambda 1}(s-t)} \ .$$

Let $\zeta := s - t$. By the product rule, we have

$$\begin{aligned}\frac{dR_{r2}(\zeta)}{d\zeta} &= -R_{r1}(\zeta)\frac{dK_r^\gamma(t,s)}{dt} + K_r^\gamma(t,s)\frac{dR_{r1}(\zeta)}{d\zeta} \\ &= -\left(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r\right)R_{r2}(\zeta) - \frac{1}{2}\sigma_r^2 R_{r2}(\zeta)K_r^\gamma(t,s) \\ &\quad -\left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2}\right]R_{r1}(\zeta) + K_r^\gamma(t,s)\frac{dR_{r1}(\zeta)}{d\zeta} \ . \end{aligned} \tag{A26}$$

Setting the coefficients of K_r^γ to be zeros in (A26) gives

$$\frac{dR_{r1}(\zeta)}{d\zeta} = \frac{1}{2}\sigma_r^2 R_{r2}(\zeta) \ ,$$

and

$$\frac{dR_{r2}(\zeta)}{d\zeta} = \left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2}\right]R_{r1}(\zeta) - \left(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r\right)R_{r2}(\zeta) \ ,$$

which are equivalent to the following matrix-valued, linear ODE:

$$\frac{d}{d\zeta}\begin{pmatrix} R_{r1}(\zeta) \\ R_{r2}(\zeta)\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\sigma_r^2 \\ -\frac{\gamma}{1-\gamma} - \frac{\gamma\theta_r^2}{2(1-\gamma)^2} & -(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r)\end{pmatrix} d\zeta \ , \quad \begin{pmatrix} R_{r1}(0) \\ R_{r2}(0)\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix} \ . \tag{A27}$$

Evidently, the solution of $(R_{r1}, R_{r2})^\top$ is given by the matrix exponential (41). Similarly, we can derive that $(R_{\lambda 1}, R_{\lambda 2})^\top$ satisfies the following matrix-valued, linear ODE:

$$\frac{d}{d\zeta}\begin{pmatrix} R_{\lambda 1}(\zeta) \\ R_{\lambda 2}(\zeta)\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\sigma_\lambda^2 \\ 1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} & \mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\end{pmatrix} d\zeta \ , \quad \begin{pmatrix} R_{\lambda 1}(0) \\ R_{\lambda 2}(0)\end{pmatrix} = \begin{pmatrix} 1 \\ 0\end{pmatrix} \ , \tag{A28}$$

whose solution is given by the matrix exponential (42). □

Proof of Proposition 5.3. Using the change of measure to (35) gives

$$\begin{aligned}\varphi_3(t,s,\lambda) &= \widetilde{\mathrm{E}}_{t,\lambda}\left[\lambda(s)\frac{e^{-\int_t^s \lambda^\gamma(u)du}}{\widetilde{\mathrm{E}}_{t,\lambda}[e^{-\int_t^s \lambda^\gamma(u)du}]}\right] \times \widetilde{\mathrm{E}}_{t,r,\lambda}\left[\exp\left\{-\int_t^s [\lambda^\gamma(u) + r^\gamma(u)]du\right\}\right] \\ &= \widehat{\mathrm{E}}_{t,\lambda}[\lambda(s)] \times \varphi_2(t,s,r,\lambda) \ , \end{aligned} \tag{A29}$$

where $\widehat{\mathrm{E}}_{t,\lambda}[\cdot]$ is the conditional expectation given $\lambda(t) = \lambda$ under a new probability measure $\widehat{\mathcal{P}}$ defined by

$$\left.\frac{d\widehat{\mathcal{P}}}{d\widetilde{\mathcal{P}}}\right|_{\mathcal{F}(s)} = \Lambda(s) := \frac{e^{-\int_0^s \lambda^\gamma(u)du}}{\widetilde{\mathrm{E}}[e^{-\int_0^s \lambda^\gamma(u)du}]} \ .$$

For each $t \in [0,s]$, denote by

$$\Lambda(t) := \widetilde{\mathrm{E}}[\Lambda(s)|\mathcal{F}(t)] = e^{-\int_0^t \lambda^\gamma(u)du}\frac{\varphi_2(t,s,\lambda(t))}{\varphi_2(0,s,\lambda_0)} \tag{A30}$$

where

$$\varphi_2(t,s,\lambda(t)) = \widetilde{\mathrm{E}}[e^{-\int_t^s \lambda^\gamma(u)du}|\lambda(t)] \ .$$

Differentiating both sides of (A30) yields

$$\begin{aligned}\frac{d\Lambda(u)}{\Lambda(u)} &= \frac{d\varphi_2(u,s,\lambda(u))}{\varphi_2(u,s,\lambda(u))} - \lambda^\gamma(u)du \\ &= -K_\lambda^\gamma(u,s)\sqrt{\lambda(u)}\sigma_\lambda d\widetilde{W}_\lambda(u) \ .\end{aligned}$$

Then

$$\Lambda(s) = \exp\left\{ -\frac{1}{2}\int_0^s (K_\lambda^\gamma(u,s))^2\lambda(u)\sigma_\lambda^2 du - \int_0^s K_\lambda^\gamma(u,s)\sqrt{\lambda(u)}\sigma_\lambda d\widetilde{W}_\lambda(u)\right\} \ .$$

By Girsanov's theorem, we have the process $\{\widehat{W}_\lambda(t)|t\in[0,s]\}$ defined by

$$\widehat{W}_\lambda(t) = \widetilde{W}_\lambda(t) + \int_0^t K_\lambda^\gamma(u,s)\sqrt{\lambda(u)}\sigma_\lambda du \ ,$$

is an $(\mathbb{F},\widehat{\mathcal{P}})$-standard Brownian motion. So the dynamics of $\{\lambda(t)|t\in\mathcal{T}\}$ under $\widehat{\mathcal{P}}$ is given by

$$d\lambda(u) = \left[\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda - \sigma_\lambda^2 K_\lambda^\gamma(u,s)\right]\lambda(u)du + \sqrt{\lambda(u)}\sigma_\lambda d\widehat{W}(u) \ . \tag{A31}$$

Then conditioning both sides of (A31) on $\lambda(t)=\lambda$ under $\widehat{\mathcal{P}}$ yields

$$d\widehat{\mathrm{E}}_{t,\lambda}[\lambda(u)] = \left[\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda - \sigma_\lambda^2 K_\lambda^\gamma(u,s)\right]\widehat{\mathrm{E}}_{t,\lambda}[\lambda(u)]du \ . \tag{A32}$$

Solving gives

$$\widehat{\mathrm{E}}_{t,\lambda}[\lambda(s)] = \lambda\exp\left\{\left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)(s-t) - \sigma_\lambda^2\int_t^s K_\lambda^\gamma(u,s)du\right\} \ . \tag{A33}$$

Combining (A29) and (A33) gives

$$\begin{aligned}\varphi_3(t,s,r,\lambda) &= \lambda\exp\left\{L_r^\gamma(t,s) - rK_r^\gamma(t,s) - \lambda K_\lambda^\gamma(t,s)\right\} \\ &\quad\times\exp\left\{\left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)(s-t) - \sigma_\lambda^2\int_t^s K_\lambda^\gamma(u,s)du\right\} \ .\end{aligned} \tag{A34}$$

This completes the proof. □

Proof of Proposition 5.4. Since $\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2} < 0$ and $1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} > 0$, we must have

$$\left(\mu_r - \frac{1}{1-\gamma}\sigma_r\theta_r\right)^2 - 2\sigma_r^2\left[\frac{\gamma}{1-\gamma} + \frac{\gamma\theta_r^2}{2(1-\gamma)^2}\right] \geq 0 \ ,$$

and

$$\left(\mu_\lambda + \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda\right)^2 + 2\sigma_\lambda^2\left[1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2}\right] \geq 0 \ .$$

Therefore, the desired results can be derived by either solving (A23)-(A25) directly or calculating two 2×2 matrix exponentials as in Proposition 5.2. □

Proof of Proposition 5.5. Since $1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2} > 0$, some tedious but manageable calculation gives

$$\begin{aligned}
&\int_t^s K_\lambda^\gamma(u,s)du \\
&= \left[1 - \frac{\gamma\theta_\lambda^2}{2(1-\gamma)^2}\right] \times \int_t^s \frac{2(e^{\eta_\lambda^\gamma(u-t)} - 1)}{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(e^{\eta_\lambda^\gamma(u-t)} - 1) + 2\eta_\lambda^\gamma} du \\
&= \frac{2}{\sigma_\lambda^2} \ln\left[\frac{2\eta_\lambda^\gamma e^{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(s-t)/2}}{(\eta_\lambda^\gamma - \mu_\lambda - \frac{1}{1-\gamma}\sigma_\lambda\theta_\lambda)(e^{\eta_\lambda^\gamma(s-t)} - 1) + 2\eta_\lambda^\gamma}\right].
\end{aligned}$$

Substituting this into (43) leads to the desired result. □

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