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The determinants of mortality heterogeneity and implications for pricing underwritten annuities.

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The determinants of mortality heterogeneity and implications for pricing underwritten annuities

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Abstract It is widely accepted that mortality risk varies across individuals within age-sex bands of a population. This heterogeneity exposes insurers to adverse selection if only the healthiest lives purchase annuities, so standard annuities are priced with a mortality table that assumes above-average longevity. This makes standard annuities expensive for many individuals. To address this issue there has been a shift to underwriting annuities in order to offer lower prices to individuals with below-average longevity. While underwriting reduces heterogeneity, mortality risk still varies within each risk class due to unobservable individual-specific factors, referred to as frailty. This paper quantifies the financial impact of frailty on underwritten annuities. The heterogeneity implied by underwriting factors and frailty is quantified by fitting Generalized Linear Mixed Models to longitudinal data for a large sample of US males. The results show that heterogeneity remains after underwriting, creating significant variation in the fair value of underwritten annuities. We develop a method to adjust annuity prices to allow for frailty.

Keywords: mortality heterogeneity, frailty modeling, annuity pricing, longitudinal data, generalized linear mixed models.

JELL Classification Numbers: G22, G23, C23.

1 Introduction

Mortality risk varies across individuals due to many factors including age, sex, education and marital status (Brown and McDaid, 2003). This heterogeneity poses interesting and important problems for pricing annuities. The main problem is well known. If insurers offer the same price to both low-risk and high-risk individuals, then in a competitive insurance market either the insurer makes a loss (if the price is too low to cover payments to high-risk buyers) or low-risk lives will not buy insurance (if the price is too high by their assessment) (Rothschild and Stiglitz, 1976). Consistent with the Rothschild and Stiglitz (1976) theory, low-risk (below-average longevity) individuals buy fewer annuities than high-risk (above-average longevity) individuals when a single price is offered. Thus purchasers of standard annuities[1] have lower mortality than the general population (Mitchell and McCarthy, 2002). Insurers, therefore, price annuities with a mortality table that assumes above-average longevity and the resulting high prices limit the viability of private annuity markets (Su and Sherris, 2012). To address this issue, researchers and practitioners have extended risk classification of annuitants (Kwon and Jones, 2006) and there has been significant growth in the market for underwritten annuities, in order to offer lower prices to individuals with below-average expected longevity (Steinorth, 2012).

Underwriting factors are observable characteristics, such as smoking status, that explain mortality heterogeneity. In addition to observable factors, heterogeneity may be caused by unobservable individual-specific factors, referred to as frailty. A large body of research shows that frailty has a significant impact on estimates of life expectancy (Manton, Stallard, and Vaupel, 1986; Congdon, 1994; Su and Sherris, 2012). To price underwritten annuities, therefore, it is important to allow for both observable and unobservable factors.

[1]Standard annuities are offered at one price to all individuals of the same age and gender, with no underwriting; whereas underwritten or impaired annuities are offered at a range of prices to individuals of the same age and gender depending on underwriting information.

Most modelling approaches, however, exclusively allow for either observable individual characteristics or for frailty. For instance, empirical research mainly uses time-invariant observable individual characteristics, such as sex or race, to explain mortality risk (Brown and McDaid, 2003; Ding, Tian, Yu, and Guo, 2012). This is because it is complex to incorporate time-varying covariates within the Cox proportional hazards (PH) model (Fisher and Lin, 1999). Alternative survival models allowing for time-varying covariates observed at discrete time intervals have only recently been developed (Ding, Tian, Yu, and Guo, 2012). Furthermore the incorporation of unobservable factors (such as frailty) into Cox PH models poses considerable theoretical difficulties in the development of estimation and inference procedures (Therneau and Grambsch, 2000).

On the other hand, frailty models include unobservable factors but not observable factors. Frailty models allow for heterogeneity via an individual-level unobservable risk factor. Vaupel, Manton, and Stallard (1979) introduce the concept of frailty as an unobservable risk factor representing an individuals susceptibility to death that is fixed throughout his or her lifetime. Traditional frailty models jointly estimate the parameters for the standard force of mortality distribution and the frailty distribution from population data (Su and Sherris, 2012). The way in which frailty is modelled has a significant impact on estimates of life expectancy. Manton, Stallard, and Vaupel (1986) adopt Gamma and inverse Gaussian distributions for frailty and Gompertz, Makeham, Weibull and extended Weibull distributions for standard force of mortality. They find that estimates of life expectancy are more sensitive to the distribution of the standard force of mortality than to the distribution of frailty. There is no universal tendency for a reduction in life expectancy when frailty is included. The impact depends on the analytic form of the survivor function, the way that unobserved frailty is modelled, the amount of heterogeneity present and the extent and direction of the change in forecast mortality rates (Congdon, 1994). Ignoring frailty may therefore lead to underestimation of life expectancy and to underpricing of

annuities.

Markov aging models (MAMs) and generalized linear mixed models (GLMMs) can also be used to model frailty. Like frailty models, however, MAMs cannot include individual-level observable factors because they are fit to population-level data. MAMs are continuous-time multi-state models with states defined by physiological age (which is a measure of the degree of ageing, or the level of functionality, of the human body.) and death. The probability distribution of physiological age at calendar age t may be used to describe the heterogeneity in health status among the cohort of individuals at age t (Lin and Liu, 2007). Su and Sherris (2012) show that frailty models imply a reduction in heterogeneity at older ages, while MAMs imply higher heterogeneity at older ages. These differences influence standard annuity prices, with MAMs implying higher annuity prices than frailty models (Su and Sherris, 2012).

GLMMs are fit to individual-level data and can jointly model the effect of observable individual-level characteristics and frailty on mortality risk. GLMMs extend generalized linear models (GLMs) to incorporate random effects characterizing heterogeneity among subjects. Antonio and Beirlant (2007) show the importance of GLMMs as a tool in non-life actuarial applications. GLMMs have not previously been applied to life insurance problems, however they have been used for survival modelling in other contexts. For example, McGilchrist and Aisbett (1991) and Yau and McGilchrist (1998) apply a method of estimation of GLMMs to estimate parameters in a Cox (1972) PH model including frailty. The work by McGilchrist, however, focuses on estimation methods, model fitting and inference on the fixed effects regression coefficients. Second McGilchrist applies GLMMs to repeated events studies within a medical setting (e.g. number of kidney infections following a particular treatment) rather than to mortality risk studies.

Very few mortality studies allow for observable individual-level characteristics alongside frailty. It is not clear from existing research, therefore, whether heterogeneity is significant after allowing for commonly used underwriting factors, or how frailty affects the value of annuities after allowing for underwriting factors. These questions are central to fairly pricing and adequately reserving for underwritten annuities. The original contribution of this paper is to quantify the heterogeneity in mortality risk due to individual-level observable characteristics and frailty and its impact on annuity values.

First, the heterogeneity implied by underwriting factors and frailty is quantified by fitting GLMMs. Second we assess the financial implications of frailty for underwritten annuity prices. The results confirm that while underwriting explains a significant degree of heterogeneity, frailty is still significant even after allowing for a large set of common underwriting factors. Frailty has a significant impact on underwritten annuity prices, particularly for lives with below-average health.

The remainder of the paper is organized as follows. Section 2 introduces GLMMs for mortality modelling allowing for frailty. In Section 3 we formulate an annuity pricing framework based on longitudinal or interval censored survival data and underwriting factors, and describe the data from the United States (US) Health and Retirement Study (HRS), a longitudinal study that surveys a representative sample of the US population over the age of 50 every two years. Section 4 presents the model estimation results and the valuation of annuity contracts, with and without underwriting and/or frailty. Section 5 concludes the paper.

2 GLMMs for mortality data

This section reviews the basic concepts of GLMs and their extension to GLMMs. Full details on GLMs and GLMMs are presented in McCullagh and Nelder (1989) and McCulloch and Searle (2001) respectively.

2.1 Model assumptions

Assume the data set consists of N subjects, each observed over multiple time periods. Let n_i denote the number of observations for the i'th subject for whom we observe $\mathbf{Y_i} = [Y_{i1}, \cdots, Y_{in_i}]'$. In a GLM, $\mu_i = E[\mathbf{Y_i}|\mathbf{X_i}]$ is modelled as

$$g(\mu_i) = \mathbf{X_i}\beta', \tag{1}$$

where $\mathbf{X_i}$ is a (n_i x p) matrix of covariate values and β' is a (p x 1) vector of parameters. Standard GLMs require that $\mathbf{Y_i}$ is a sample of independent random variables. In many settings this assumption is not fulfilled. In particular when the data is repeated measurements on a group of individuals over time (i.e. longitudinal data), it is necessary to allow for dependence between repeated measurements on the same individual over time.

GLMMs extend the mean function in the GLM by adding a random effect. Including random effects is useful for modelling dependence between repeated measurements over time and for modelling individual response profiles rather than the population-averaged response. In a GLMM the mean function becomes

$$g(\mu_i) = \mathbf{X_i}\beta' + \mathbf{Z_i}\mathbf{b_i}, \tag{2}$$

where $\mathbf{Z_i}$ (n_i x k) is the design matrix for the k random effects and $\mathbf{b_i}$ (k x 1) is a vector of random effects specific to individual i.

2.2 Mortality modelling using GLMMs

We model and project the probability of death for individual i at time t, q_{it}, throughout. The GLMM framework is chosen as it is straightforward to include time varying individual-level factors and frailty, in contrast to other modelling frameworks (such as the Cox PH model) where it is difficult to include time varying individual-level factors and frailty (Fisher and Lin, 1999). A common basis for comparison across all models is established by using the complementary log log function to link q_{xt} to the linear predictor ($\mathbf{X_i}\beta'$).

Longitudinal data (i.e. repeated measurements over time) arises often in life and health insurance analysis (Antonio and Beirlant, 2007). The longitudinal response for individual i ($\mathbf{Y_i}$) is a set of binary indicators of death in each period. Define the discrete time hazard rate, which is equivalent to the conditional probability individual i dies in period t given the individual is alive at the start of the period with characteristics X_{it}, as:

$$q_{it} = Pr[T_i = t | T_i \geq t, X_{it}], \tag{3}$$

where T_i is the discrete random variable giving the uncensored time of death of individual i and X_{it} is a vector of covariate observations (or underwriting factors) for individual i at time t. Allison (1982) shows that if $\mathbf{Y_i} = [Y_{i1}, \cdots, Y_{in_i}]$ are assumed to be drawn from a proportional hazards model, the hazard rate depends on time and the covariates X_{it} as follows[2].

$$q_{it} = 1 - exp[-exp(X_{it}\beta')]. \tag{4}$$

[2]Alternatively the complementary log log model can be derived by assuming PH and defining a survival model for repeated measures of interval-censored data. Collett (1994) shows that for repeated measures of interval-censored data

$$log[-log(1 - q_{it})] = \eta_{it} + \gamma_t.$$

where q_{it} is the probability of death in interval t (i.e. $q_{it} = 1 - \frac{S_i(t_j)}{S_i(t_{j-1})}$), η_{it} is the linear predictor for subject i and may contain time-varying covariates, and γ_t, $t = 1, \cdots, T$, are constants associated with the T time intervals.

Therefore if we assume PH, the discrete time hazard function is given by the complementary log-log function of the linear predictor.

The specification of the GLMM is completed by assuming that the random effects, $\mathbf{b} = [\mathbf{b_i}, \cdots, \mathbf{b_N}]$ (k by N), are mutually independent and identically distributed (i.i.d.) with density function $f(\mathbf{b_i}|\alpha)$, where α is the set of unknown frailty parameters and that

$$q_{it} = E(Y_{it}|X_{it}) = 1 - exp[-exp(X_{it}\beta' + Z_i b_i)]. \tag{5}$$

As the response data is binary it is assumed that the variance function is given by $Var[Y_{it}|b_i] = \mu_i(1-\mu_i)$. Traditionally frailty is represented by a random intercept which is fixed throughout an individual's lifetime for each individual (Vaupel, Manton, and Stallard, 1979), corresponding to

$$q_{it} = E(Y_{it}|X_{it}) = 1 - exp[-exp(X_{it}\beta' + b_i)]. \tag{6}$$

where the frailty factors $\mathbf{b} = [\mathbf{b_1}, \cdots, \mathbf{b_N}]$ are assumed to be i.i.d. with $b_i \sim N(0, \sigma_b^2)$. Using a single random effect to model frailty means the model is more parsimonious, easier to interpret and less prone to overfitting (because models with multiple random effects are prone to overfitting) (Dion, 2011). Alternative frailty specifications are investigated in Appendix A, however the main results were robust to alternative specifications of frailty.

2.3 Model estimation

Let $f_{it}(y_{it}|b_i)$ denote the density function of Y_{it} given b_i. The marginal distribution of $\mathbf{Y_i}$ is then given by

$$f(y_i) = \int \prod_{j=1}^{n_i} f_{ij}(y_{ij}|b_i) f(b_i) db_i, \tag{7}$$

and the joint likelihood function for models with frailty and the unknown parameters β and α is

$$\begin{aligned} L(\beta, \alpha; \mathbf{y}) &= \prod_{i=1}^{N} f(y_i|\beta, \alpha) \\ &= \prod_{i=1}^{N} \int \prod_{j=1}^{n_i} f_{ij}(y_{ij}|b_i, \beta, \alpha) f(b_i|\alpha) db_i \end{aligned} \tag{8}$$

where the integral is with respect to the vector $\mathbf{b_i}$. For linear mixed models, closed-form expressions exist for the maximum likelihood estimator of the model parameters. For GLMMs, however, approximations to the likelihood or numerical integration techniques are required to maximize the likelihood function with respect to the unknown parameters. Statistical software (like SAS) contains in-built functions that estimate GLMMs via maximum likelihood techniques.

Alternatively, GLMMs can be estimated using Bayesian methods such as Markov Chain Monte Carlo (MCMC) simulations. Bayesian implementation of GLMMs enables the specification of more complicated structures for the linear predictor (for example, multiple, crossed and/or nested random effects) and for non-normal distributions to be used for the random effects. A draw-back of the Bayesian approach is that it treats all unknown parameters in the GLMM as random variables for which priors must be specified. This means that Bayesian approaches require more assumptions and are more computationally intensive than maximum likelihood estimation (Sargent, 1998). In applications without complex structures of random effects, or non-normal random effect distributions, maximum likelihood techniques require fewer assumptions and are easier to implement. Therefore as there is only one random effect in (6) we use maximum likelihood estimation. (See Antonio and Beirlant (2007) for more details on estimation techniques for GLMMs.)

2.4 Choice of explanatory variables

In order to test whether frailty has a significant impact on mortality rates and annuity values after underwriting, it is necessary to define a baseline model which has no underwriting and to establish the impact of frailty on mortality rates and annuity values in the baseline model.

Baseline model (no underwriting) The baseline model mimics the type of rating performed for a policy that is priced solely on age and gender. The model includes only year and age as covariates, and is fit to data for males only (i.e. gender specific mortality rates).

Survival model with underwriting Brown and McDaid (2003) identify 10 potential risk factors influencing mortality including education, marital status, income, occupation, race, health behaviour and religion. First, risk factors should only be used for underwriting if they are objective and readily measurable. Of the aforementioned factors, health behaviour is difficult to monitor and measure, however weight (or Body Mass Index (BMI)), smoking status and past medical history are typically used for underwriting as these factors can be reliably measured. Second, race and religion cannot be used as underwriting factors in the US (Brown and McDaid, 2003). Finally, education, occupation and income are highly collinear, since current income and wealth are linked to occupation which is strongly affected by education and health (Fong, 2011). Therefore income, occupation, race and religion are excluded from the set of predictors available, leaving education, marital status, BMI , smoking status and medical history to mimic a realistic underwriting setting.

3 Data

Longitudinal mortality data was obtained from the HRS. The HRS contains individual-level data collected every two years, in biennial survey 'waves' starting from 1992. Each subject contributes one set of observations over the measured variables per wave, provided they have not left the study (due to death or other exit reason). All models are fit to data for subjects born from 1931 to 1941, who entered the study in 1992, since this cohort represents the retired population with the greatest potential demand for annuity contracts and also has the longest time-series of data available. As death data may be delayed, the last year of deaths are not reliable as there may be incurred but not reported deaths. Therefore only data up to 2006 was used in order to ensure that all deaths within the study period were captured. The final dataset was an unbalanced panel of $4,592$ males, each with up to 8 biennial observations.

Statistics summarizing the dataset are shown in Table 1. Readily-measurable mortality risk factors available in HRS data include race, education, smoking status, marital status, prior health history, BMI and income. The average age within the panel was 62 years, and the average birth year of the individuals was 1936. Most individuals within the representative sample of the population of US males had some high school or college education, were married, overweight and had smoked in the past but not over the sample period (1992 to 2006). The most prevalent medical conditions among the sample were high blood pressure and arthritis.

4 Results

In order to quantify the impact of frailty before and after underwriting, models fit based only on gender, age and time are compared to models fit allowing for gender, age, time and underwriting factors. All models are fit with and without allowing for unobservable frailty.

Table 1: Summary statistics for the cohort of HRS Males

N	**4,592**	**Mean**	**Std. Dev.**
Age		61.8	5.4
Education	**Less than high school**	24%	
	GED, HS or some college	54%	
	College and above	21%	
Marital status	**Married**	79%	
	Married, spouse away	1%	
	Partnered	3%	
	Separated	2%	
	Divorced	7%	
	Separated or divorced	1%	
	Widowed	4%	
	Never married	3%	
BMI	**Underweight**	1%	
	Normal weight	27%	
	Overweight	47%	
	Obese	19%	
	Morbidly obese	6%	
Smoked ever		73%	
Smoked now		22%	
Health history	**High blood pressure**	45%	
	Diabetes	15%	
	Cancer	7%	
	Lung disease	7%	
	Heart problems	20%	
	Stroke	5%	
	Arthritis	42%	
	Psychiatric problems	8%	

Finally, we calculate annuity rates based on the estimates from each set of models, for a whole of life annuity and a deferred annuity paid from age 65.

4.1 Model diagnostics

To check that the functional form of the predictors in the baseline model was approximately linear, a generalized additive model (GAM) was fit to the (complementary log log) transformed data for HRS Males

$$log[-log(1 - Y_{it})] = c + f_1(a_{it}) + f_2(t)$$

where a_{it} is the age of individual i at time t. The GAM fits a non-parametric smoothing function to the relationship between the covariates and the response. Figure 1 shows that age and time effects are well described by a linear function when the response is transformed by the complementary log log link, so models (4) and (6) are reasonable.

4.2 Model estimation

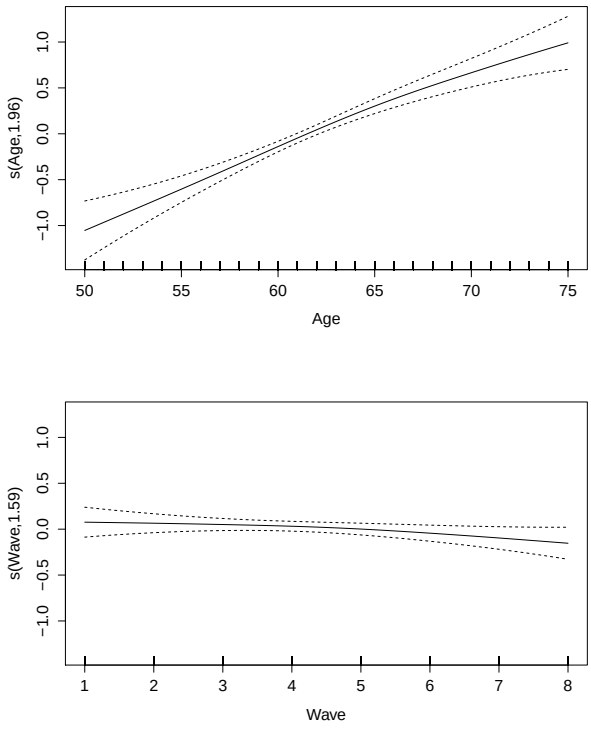
The estimated parameters for the standard model and the underwritten model, (4) and (6), with and without frailty are shown in Table 2. The frailty model is a GLMM including a random intercept as in (6). The main result is that frailty remains significant (σ_b remains greater than zero) even after controlling for a large number of underwriting factors.

The parameter estimates from the GLM (without frailty) and the GLMM (with frailty) are different for both the baseline and underwritten models. This is because GLM parameters capture the population-averaged effect of covariates on the response, while GLMM parameters capture the subject-specific effect of covariates. The population-averaged probability will differ from the subject-specific profiles more as the variation across subjects increases (Verbeke and Molenberghs, 2000). In general, parameter estimates obtained

Table 2: GLM and GLMM (Frailty) models of mortality rates: Baseline and Underwritten

	Baseline				**Underwritten**			
$\hat{\beta}$	**GLM**		**GLMM (Frailty)**		**GLM**		**GLMM (Frailty)**	
I(Wave=1)	-8.45	***	-11.88	***	-6.37	***	-8.83	***
I(Wave=2)	-8.36	***	-11.64	***	-6.29	***	-8.63	***
I(Wave=3)	-8.52	***	-11.70	***	-6.46	***	-8.69	***
I(Wave=4)	-8.35	***	-11.44	***	-6.29	***	-8.46	***
I(Wave=5)	-8.34	***	-11.32	***	-6.27	***	-8.34	***
I(Wave=6)	-8.57	***	-11.47	***	-6.54	***	-8.55	***
I(Wave=7)	-8.51	***	-11.33	***	-6.53	***	-8.54	***
I(Wave=8)	-8.67	***	-11.42	***	-6.76	***	-8.74	***
Age	0.08	***	0.12	***	0.06	***	0.08	***
Education:								
HS or some college					-0.12	.	-0.17	
College and above					-0.38	***	-0.51	***
Partnered					-0.35	***	-0.50	***
BMI:								
Normal weight					-1.08	***	-1.29	***
Overweight					-1.51	***	-1.84	***
Obese					-1.72	***	-2.14	***
Morbidly obese					-1.53	***	-1.97	***
Smoked ever					0.39	***	0.53	***
Smoked now					0.39	***	0.50	***
Health history:								
High blood pressure					0.32	***	0.42	***
Diabetes					0.73	***	0.91	***
Cancer					0.89	***	1.41	***
Lung disease					0.65	***	0.97	***
Heart problems					0.51	***	0.71	***
Stroke					0.55	***	0.90	***
Psychiatric problems					0.42	***	0.78	***
Std. Dev. Frailty (σ_b)			1.83				1.56	
AIC	9967		9809		8864		8737	
LL			-4894				-4342	

Figure 1: Plot of functional form of the predictors in the GAM $log[-log(1-\mu_{it})] = c + f_1(a_{it}) + f_2(t)$.



from a GLMM will be larger in absolute value than their GLM counterparts (Neuhaus, Kalbfleisch, and Hauck, 1991). If there is very little variation in subject-specific random effects, then the coefficients of the GLM will be closer to the GLMM[3]. In Table 2 the estimated coefficients of the baseline GLM are significantly different to those of the baseline GLMM, which implies that there is a significant degree of variation across the subject-specific random effects (frailty variables). The smaller difference between the GLM and GLMM coefficients when underwriting factors are included shows that the underwriting factors capture a significant degree of heterogeneity across individuals' mortality risk. Consistently, Table 2 shows that the underwriting factors are all statistically significant.

[3]A similar relationship exists when covariates are omitted from binary regression models. That is, the population averaged effect of covariate X will be closer to zero (no association) than the cluster specific effect of X, which is the true underlying effect of X when the "unobserved covariate" is included.

The results also show that frailty remains significant after allowing for a large number of underwriting factors. In particular, comparing the baseline and underwritten models shows that allowing for a large number of underwriting factors only reduces the variance of frailty 27% from 1.83^2 to 1.56^2. That is, heterogeneity in mortality risk due to frailty is still significant after underwriting. The financial impact of this change in unobservable heterogeneity on life expectancy and annuity values is discussed in Section 4.3.2. Finally the AIC of the Underwritten GLMM was lower than any of the other models[4]. This indicates that while underwriting factors capture a significant degree of variation in mortality risk, frailty creates additional variation in mortality risk after underwriting.

Figure 2 shows the fitted mortality rates from the baseline GLMM (allowing for frailty but not underwriting). Variation in the curves of individual mortality rates by age arises because of both time effects and frailty. To isolate variation due to frailty we produce the equivalent plot for the individual years corresponding to waves 2, 4 and 6 in Figure 3.

Figure 3 illustrates the significant heterogeneity in individual-level hazard rates, captured by the variation of individual mortality risk about the average mortality risk (dashed line). Different individuals of the same age have different hazard rates in the same year. A small number of individuals have hazard rates far above the average, while a large number of individuals have hazard rates a small amount less than the average.

4.3 Implications for annuity pricing

In this section we check whether frailty significantly impacts the actuarial present value of standard and underwritten annuities. The GLMs and GLMMs fit in the previous section are used to calculate expected values of annuities and the x% frailty value of annuities for a

[4]The likelihood numbers in the GLM and GLMMs are not comparable because different methods are used to approximate the likelihood functions in GLMMs (quasilikelihood rather than a true likelihood). Many statisticians warn that likelihood-based methods should not be used for inference with quasilikelihoods (Pinheiro and Chao, 2006; Bolker, Brooks, Clark, Geange, Poulsen, Stevens, and White, 2009).

Figure 2: Fitted mortality rates from the Baseline GLMM (frailty model). *The dashed line shows the mean predicted probability at age.*

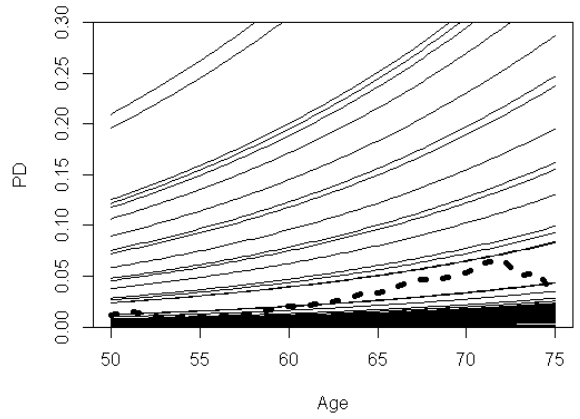
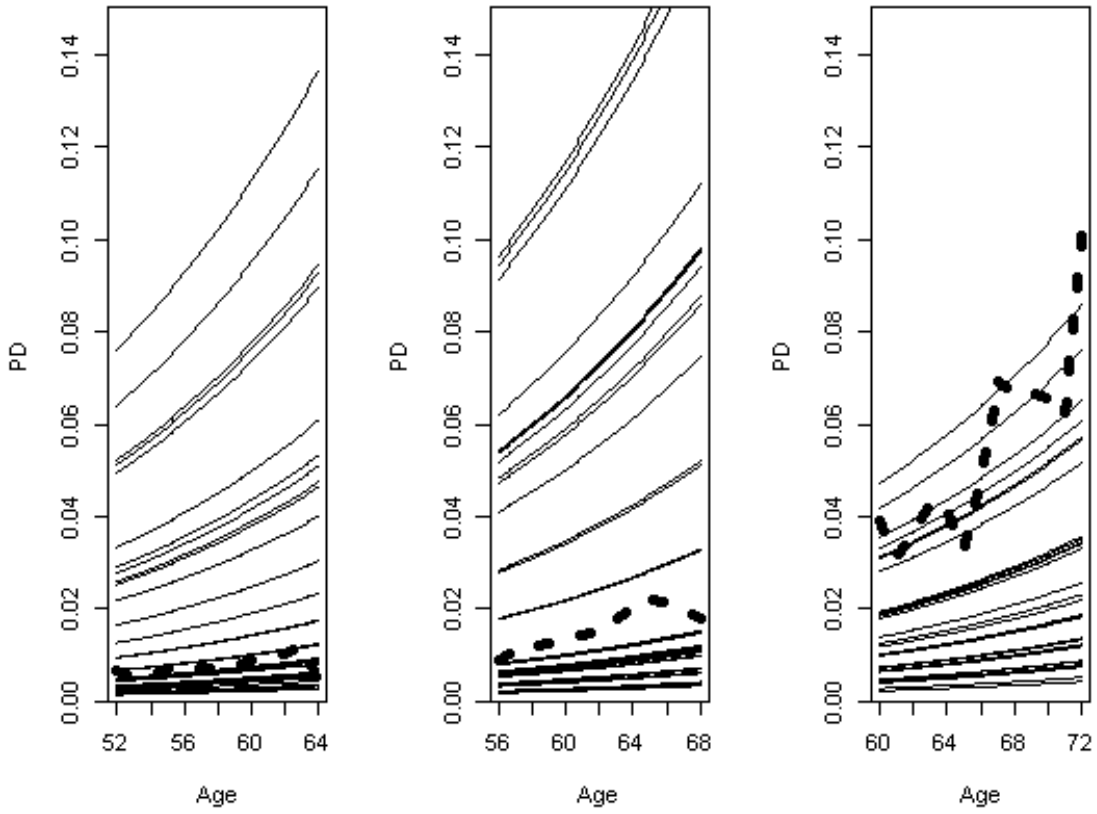


Figure 3: Fitted mortality rates from the Baseline GLMM (frailty model) in Waves 2, 4 and 6. *The dashed line shows the mean predicted probability at age.*



whole of life annuity and a 20 year deferred annuity paid from age 65. The x% frailty value is defined as the expected present value of an annuity sold to individuals with frailty in the bottom (1-x) percentile of the frailty distribution (that is, the x% least frail individuals). For example, say an insurer believed that the lives buying annuities were the least frail 15% of the general population, then they would charge the 85% frailty value of annuities.

4.3.1 Method

The HRS survey waves are every two years. Therefore the predictions from models (4) and (6) are predictions of the probability of death within the preceding two year interval. For a life aged x the model prediction $\hat{\mu}_x(t) = 1 - exp[-exp(X_{it}\hat{\beta}')] =_2 q_{x-2}(t)$. These biennial mortality rates can be decomposed into annual mortality rates as shown in Appendix B. For each model, we calculate the projected survival rates for a life aged 65 in 2006 (the final year of the data) using the cohort-based approach explained in Pitacco (2004). In future years, we assume that the average mortality improvement rate over the period 1992 to 2006 continues such that $\hat{\beta}_t = \beta_{Wave=8} + (t-2006)\sum_{n=2}^{8} \frac{\beta_{Wave=n} - \beta_{Wave=n-1}}{2}$.

Annuity values are calculated from the projected survival rates using discount rates of 4%p.a. (the UK government bond rate in March 2002) and 3%p.a. corresponding annuity values reported in Cannon and Tonks (2009) and Su and Sherris (2012) respectively. The annuity values are calculated to demonstrate the financial impact of including frailty and underwriting factors in mortality models and do not include any allowance for expenses, profit loading or frictional costs.

Frailty models do not provide a specific valuation of the mortality of an individual (Olivieri, 2006); however, these models provide an estimate of the expected subject-specific mortality rate and the distribution of frailty in the population. Estimates of the mortality of an individual are equal to the sum of the model predicted expected values and an

adjustment for frailty of $z_\alpha \sigma_b$, that is

$$\hat{q_{xt}}(\alpha) = 1 - exp[-exp(X_{it}\hat{\beta}' + z_\alpha \hat{\sigma_b})], \tag{9}$$

where z_α is the α percentile of the standard normal distribution and $\hat{\sigma_b}$ is the estimated variance of the frailty variable b_i. For example, in the model with no underwriting the predicted death rates for the 95'th frailty value are calculated by adding $z_{0.05}\hat{\sigma_b} = (1.645)(1.83)$ onto the linear predictor $\mathbf{X_{it}}\hat{\beta}$ and evaluating (6).

Profiles were developed for low, average, and high risk individuals based on the characteristics of the sample of the US population over 50 shown in Table 1. For example, low risk males had high levels of education, were married, had normal BMI and had never smoked or had any serious medical conditions. The profiles are summarized in Table 3. Estimates were obtained for each profile by evaluating (9) using the different covariates values corresponding to these profiles.

Table 3: Risk profiles for underwritten annuity purchases

	Low risk	**Avg risk**	**High risk**
Education	College or above	HS or some college	Not completed HS
Partnered	Yes	Yes	No
BMI	Normal	Overweight	Underweight
Smoker	Never	Before, not currently	Before and currently
Medical conditions	None	High BP	High BP and Diabetes

4.3.2 Results

First, the expected value of \$1 per annum for life paid to a male aged 65 at 2006 was calculated using the projected mortality rates implied from each model, discounted at 4% (in order to obtain comparable values to those listed in Cannon and Tonks (2009)).

The predicted mortality rates from the GLM yield an annuity value (12.33) that is just 4% higher than the average market price of an annuity for a male aged 65 listed in Cannon

Table 4: Expected value of $1 for life for a male aged 65, at 4%

	Baseline	**Underwritten**		
	Standard	Low	Average	High
GLM	12.33	18.57	17.29	4.20
Frailty - 50%	12.28	18.66	17.03	3.57
Frailty - 75%	15.95	19.92	19.15	6.75
Frailty - 85%	17.54	20.25	19.78	8.85
Frailty - 95%	19.36	20.54	20.34	12.55
Frailty - 99.5%	20.44	20.68	20.63	17.33
UK Avg. March 2002	**11.89**	-	-	-

Table 5: Expected value of 20 year deferred annuity for a male aged 65, at 4%

	Baseline	**Underwritten**		
	Standard	Low	Average	High
GLM	1.77	6.04	5.16	0.03
Frailty - 50%	1.27	5.86	4.57	0.00
Frailty - 75%	3.52	6.90	6.26	0.09
Frailty - 85%	4.78	7.19	6.78	0.39
Frailty - 95%	6.36	7.43	7.26	1.70
Frailty - 99.5%	7.33	7.55	7.51	4.80

and Tonks (2009) of 11.89. The 50% frailty value from the frailty model (12.28) is also comparable to the market price of annuities. For underwritten annuities, annuities sold to low risk or average risk individuals (as per Table 3) had much larger expected values than annuities sold to high risk individuals, reflecting the shorter expected lifetime for high risk lives. The 95% frailty value of an underwritten annuity for high risk lives (12.55) was comparable to the average market price of standard UK annuities reported in Cannon and Tonks (2009) (11.89).

The variation in the frailty values of underwritten annuities for low risk and average risk lives (18.66-20.68 and 17.03-20.63 respectively) was lower than the variation in the frailty values of standard annuities (12.28-20.44). The largest degree of variation in annuity values due to frailty was for high risk lives (3.57-17.33). This is because any differences in projected survival rates are more noticeable due to the shorter discounting period. Therefore it is important to allow for frailty when pricing underwritten

Table 6: UK annuity price of $1 for life for men of aged x on March 2002 from Cannon and Tonks (2009)

Company	**60**	**65**	**70**	**75**
AMP NPI	14.88	12.67	10.42	8.26
AXA Sun Life	14.33	12.50	10.52	8.62
B & CE Insurance	13.30	11.44	9.49	7.61
BRS Smoker	12.72	10.95	9.07	7.27
BRS Plus	12.52	10.66	8.73	6.93
BRS Special	12.17	10.21	8.23	6.44
Canada Life	14.10	12.36	10.50	8.65
Friends Provident	13.97	12.24	10.52	8.86
GE LIFE	15.34	13.76	11.96	10.02
GE LIFE (special)	12.71	10.80	8.85	7.00
Legal and General	13.85	12.24	10.56	8.92
MGM (Select)	12.77	11.12	9.40	7.59
Norwich Union	14.64	13.09	11.09	9.29
Pension Annuity FS	11.70	9.97	8.54	6.71
Prudential	13.81	11.96	10.01	8.10
Royal Liver	14.66	12.77	10.74	8.72
Scottish Equitable	14.37	12.50	10.65	8.83
Scottish Widows	13.83	12.05	10.16	8.31
Standard Life	14.49	12.69	10.81	8.93
Average	**13.69**	**11.89**	**10.01**	**8.16**
Max	**15.34**	**13.76**	**11.96**	**10.02**
Min	**11.70**	**9.97**	**8.23**	**6.44**

annuities for high risk lives because frailty can make a significant difference to annuity values.

We also compare the annuity prices implied by the GLM and GLMMs to the corresponding results in Su and Sherris (2012). Su and Sherris (2012) quantify heterogeneity and its financial implications for annuity prices using frailty models and a Markov ageing model (originally proposed by Lin and Liu (2007)) calibrated to Australian population mortality data. They use a valuation interest rate of 3%. The results from Su and Sherris (2012) and the results from the GLM and GLMMs with and without frailty and underwriting factors are compared for a whole of life annuity and a 20 year deferred annuity in Table 7 and Table 8 respectively.

The baseline GLMM (frailty model with no underwriting) produces reasonable annuity values compared to the frailty models fit to Australian population data in Su and Sherris (2012). The MAMs, however, imply higher annuity values than those from either the GLM or GLMMs fit in this paper or the frailty models in Su and Sherris (2012). This is because they imply greater heterogeneity at older ages Su2012.

Table 7: Expected value of $1 for life for a male aged 65, at 3%

3%	**Standard**	***Su Sherris (2012)***		**Underwritten**		
		Markov	***Frailty***	Low	Average	High
GLM	13.68			21.71	20.06	4.40
Frailty - 80.6%	19.20	***19.44***	***18.12***	23.69	22.87	8.37
Frailty - 61.7%	15.60	***18.31***	***16.36***	22.65	21.02	5.06
Frailty - 43.1%	12.24	***16.83***	***14.38***	21.03	18.55	3.06
Frailty - 23.5%	8.36	***15.63***	***11.49***	18.11	14.87	1.65
Frailty - 13.3%	5.87	***14.44***	***9.18***	15.48	12.05	1.16
Frailty - 0.41%	1.07	***11.15***	***2.59***	5.25	3.17	0.94

Table 8: Expected value of 20 year deferred annuity for a male aged 65, at 3%

3%	**Standard**	***Su Sherris (2012)***		**Underwritten**		
		Markov	***Frailty***	Low	Average	High
GLM	2.26			8.07	6.87	0.03
Frailty - 80.6%	5.45	***5.34***	***4.32***	9.47	8.75	0.26
Frailty - 61.7%	2.79	***4.55***	***2.94***	8.57	7.19	0.01
Frailty - 43.1%	1.05	***3.63***	***1.66***	7.20	5.22	0.00
Frailty - 23.5%	0.14	***2.99***	***0.47***	4.89	2.72	0.00
Frailty - 13.3%	0.01	***2.46***	***0.08***	3.09	1.29	0.00
Frailty - 0.41%	0.00	***1.50***	***0.00***	0.02	0.00	0.00

5 Conclusion

We modelled the heterogeneity implied by underwriting factors and frailty and quantified the financial impact of frailty on underwritten annuities. The relative importance of frailty declines when underwriting factors are included in the model, however frailty creates significant heterogeneity in mortality rates even after allowing for a large set of underwriting factors. It is therefore important to allow for frailty in applications that call for estimates of individual specific (rather than population averaged) estimates of mortality risk, even after underwriting.

In terms of the financial impact of frailty, two key results hold. First, underwriting reduces the level of heterogeneity and its impact on the fair value of annuities. Second, however, significant variation in the fair value of annuities remains after underwriting due to frailty. Finally, the impact of frailty on annuity values is greatest for lives with below-average longevity because of the shorter discounting period.

These results have important implications for the pricing and risk management of annuities and other life-contingent insurance products. Namely, frailty needs to be taken into account when pricing annuities, even if they are underwritten. Incorporating frailty in the pricing of underwritten annuities means that an insurer can offer lower annuity prices

to lives with life expectancy below the population average, whilst also protecting itself against adverse selection by lives whose life expectancy is above the average for a given risk class. Neglecting to adjust annuity prices for frailty may result in under-pricing and under-reserving which could threaten the sustainable growth of the much needed annuity market.

A Alternative frailty models

In the original model of frailty proposed by Vaupel, Manton, and Stallard (1979) it is assumed that frailty is constant throughout an individual's lifetime. Frailty may vary over time, however, for example when the study period is long or when the study period is short but the disease or conditions under consideration means that individuals are likely to become rapidly more frail (Yau and McGilchrist, 1998). Time variation in frailty may occur at a different rate for each individual, and is distinct from systematic time effects (mortality improvements) that are the same across all individuals.

Within the GLMM framework it is straightforward to allow for time-varying frailty. We fit three GLMMs (M1, M2 and M3) in order to illustrate the significance of the following alternative specifications of frailty before and after underwriting:

- M1: fixed time effect[5] and a random intercept
- M2: fixed time effect, a random slope on time and a random intercept
- M3: fixed time effect, a random slope on time, a random slope on age and a random intercept.

We fit models M1, M2 and M3 and test assumptions regarding the correlation between random effects and the models residuals, and the robustness of the predictions from different model specifications via MCMC sampling. The second column of Table 2 contains the results for the baseline GLM (with no allowance for frailty). The GLMMs correspond to the GLM (4) with a random intercept (M1), with a random intercept and a random slope for time (M2), or with a random intercept and a random slope on time and age (M3).

The coefficient estimates for all models showed mortality risk decreasing over time

[5]In order to fit mortality models to longitudinal data, it is necessary to allow for a factor variable representing time (γ_t in (4)).

Table 9: GLM and GLMM models of mortality risk as a function of time and age

	GLM		M1		M2		M3	
Random effects (Std. Dev.):								
ID intercept			1.83		1.64		0.63	
Time slope					1.15		1.70	
Age slope							0.04	
Fixed effects:								
I(Wave=1)	-8.45	***	-11.88	***	-9.04	***	-9.60	***
I(Wave=2)	-8.36	***	-11.64	***	-8.30	***	-8.44	***
I(Wave=3)	-8.52	***	-11.70	***	-7.55	***	-8.14	***
I(Wave=4)	-8.35	***	-11.44	***	-7.87	***	-7.76	***
I(Wave=5)	-8.34	***	-11.32	***	-7.98	***	-7.86	***
I(Wave=6)	-8.57	***	-11.47	***	-8.43	***	-8.47	***
I(Wave=7)	-8.51	***	-11.33	***	-8.85	***	-8.74	***
I(Wave=8)	-8.67	***	-11.42	***	-9.01	***	-8.88	***
Age	0.08	***	0.12	***	0.05	**	0.03	
AIC	9967		9809		9371		9004	
LL			-4894		-4673		-4487	

(corresponding to the coefficients on Wave become more negative over time) and increasing at older ages. Frailty (the random intercept for each individual) and the response of mortality risk to time effects both varied significantly across individuals, as indicated by the non-zero variability of the random ID intercept and random time slope. However, the response of mortality risk to age effects did not vary significantly across individuals, as indicated by the near-zero variability of the random age slope (0.04^2). Compared to the GLM, when a random intercept is included (M1) the fixed coefficient estimates become more pronounced; that is, the coefficients on the wave indicators and age have greater absolute value in M1, M2 and M3 than in the GLM. Neuhaus, Kalbfleisch, and Hauck (1991) show that the coefficient estimates under a GLM approach are closer to zero than those of the GLMM approach (allowing for individual frailty) when there is significant correlation between repeated observations of the same individual. These results suggest that individual frailty creates a significant degree of heterogeneity in mortality risk in age-gender risk classes.

Given the difficulties with likelihood-based methods of model selection noted above,

we use the criteria proposed by Cairns, Blake, and Dowd (2008) to select between the different mortality models:

- The model should be relatively parsimonious.
- Long-term dynamics under the model should be biologically reasonable.
- Parameter estimates and model forecasts should be robust relative to the period of data and range of ages employed.
- Forecast levels of uncertainty and central trajectories should be plausible and consistent with historical trends and variability in mortality data.
- The model should be straightforward to implement using analytical methods or fast numerical algorithms.

We follow the traditional assumption that random effects have a multivariate normal distribution with mean zero and a parameterized variance-covariance matrix (of the unconditional distribution) that is block diagonal. When there is more than one random effect per individual, however, the random effects can be correlated (Bates, Maechler, and Bolker, 2012) and it is important to check that the correlation between random effects in the models with multiple random effects (M2 and M3) is not too high. In M2 the correlation between the random intercept and random time slope was 0.63. Values under 0.8 are generally considered to be reasonable. The positive correlation in M2 implies that individuals who have high frailty are also highly sensitive to changes in mortality risk over time; and vice versa, individuals with low frailty are less sensitive to changes in mortality risk over time. In M3, however, the correlation between the random intercept and random

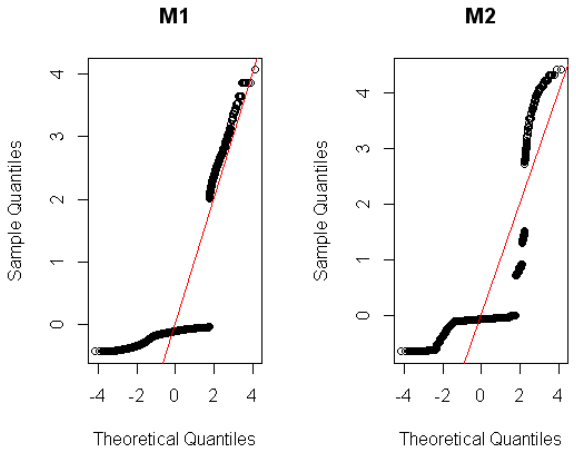
Table 10: The variance and correlation of random effects in M3

	Variance	Std.Dev.	Corr	
ID (int)	0.39	0.63		
Age (slope)	**0.00**	**0.04**	**1.00**	
Wave (slope)	2.90	1.70	0.22	0.22

age slope was 1, suggesting that the results of M3 should be interpreted with caution as high correlation may cause problems in estimation. Furthermore the variance explained by the age slope was close to zero (0.00) suggesting that random age effects are not significant over and above random time effects. The results of M2 and M3, however, support the finding that there is a significant degree of variation in individuals' rate of mortality improvement (as measured by the response of mortality risk to stochastic time effects). This effect is positively correlated with frailty; such that the individuals with low frailty have higher rates of mortality improvement over time, while individuals with high frailty have lower rates of mortality improvement over time. In order to further understand this effect however, longer time series would be required in order to fit individual-level time-series models to the rate of mortality improvement, similar to the approach of Lee and Carter (1992).

Checks on the normality of the model residuals indicated that M1 had a slightly better fit than M2[6]. In addition, there were some issues with convergence in the random

Figure 4: Qqplot of the residuals from M1 and M2



intercept and slope specification in M2. Specification tests generally use simulation to check robustness of results. We use cross-validation via Markov Chain Monte Carlo (MCMC)

[6]Due to the discrete nature of the data there were strong patterns in the plots of residuals versus fitted values that made these plots hard to interpret. Instead qqplots and histograms of the residuals were used for model fit analysis

sampling, graphical analysis and practical considerations to compare M1 and M2. Chains were run with a burn-in of 20,000 simulations was followed by another 50,000 simulations. As reported by many authors (e.g. Antonio and Beirlant (2007); Zhao, Staudenmayer, Coull, and Wand (2006)) centering of covariates greatly improves mixing of the chains and speed of simulation, therefore stochastic time effects were centered.

The fixed and random effects estimates and the predictions from M1 and M2 were estimated via MCMC sampling. The M1 estimates were more robust to estimation methodology, as the estimates via MCMC sampling were closer to the likelihood-based results (see Table 11) for M1 than for than the M2. The model predictions are similar in

Table 11: Parameters estimates for M2* and M3* via REML likelihood-based estimation vs. MCMC sampling

	M1	**MCMC**	**Diff**	**M2**	**MCMC**	**Diff**
ID intercept	1.83			1.64	0.93	
Time slope				1.15	0.02	
I(Wave=1)	-11.88	-9.59	-19%	-9.04	-10.52	16%
I(Wave=2)	-11.64	-9.47	-19%	-8.30	-10.26	24%
I(Wave=3)	-11.70	-9.64	-18%	-7.55	-10.33	37%
I(Wave=4)	-11.44	-9.47	-17%	-7.87	-10.06	28%
I(Wave=5)	-11.32	-9.44	-17%	-7.98	-9.96	25%
I(Wave=6)	-11.47	-9.69	-16%	-8.43	-10.17	21%
I(Wave=7)	-11.33	-9.62	-15%	-8.85	-10.07	14%
I(Wave=8)	-11.42	-9.77	-14%	-9.01	-10.23	13%
Age	0.12	0.09	-21%	0.05	0.10	93%

both models (as shown in Figure 5 and 6). Finally, models with multiple random effects are harder to interpret and to use for forecasting[7]. However, the model results showed that there is significant frailty in the mortality risk of the sample. Therefore, M1 was chosen over the other models for parsimony, ease of interpretation and implementation, and because it produced the most robust estimates under different estimation procedures.

[7]For example, in a model with two independent random effects, forecasting requires simulating draws from a bivariate normal distribution in order to project the random effects. The forecast levels of uncertainty are larger in models with multiple random effects than only a single random effect.

Figure 5: Model predicted probabilities from a MCMC fit of M1

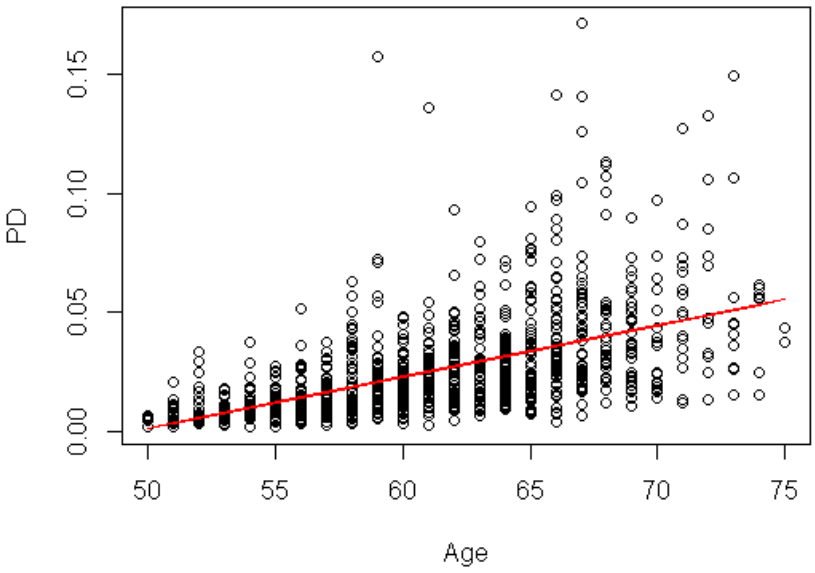
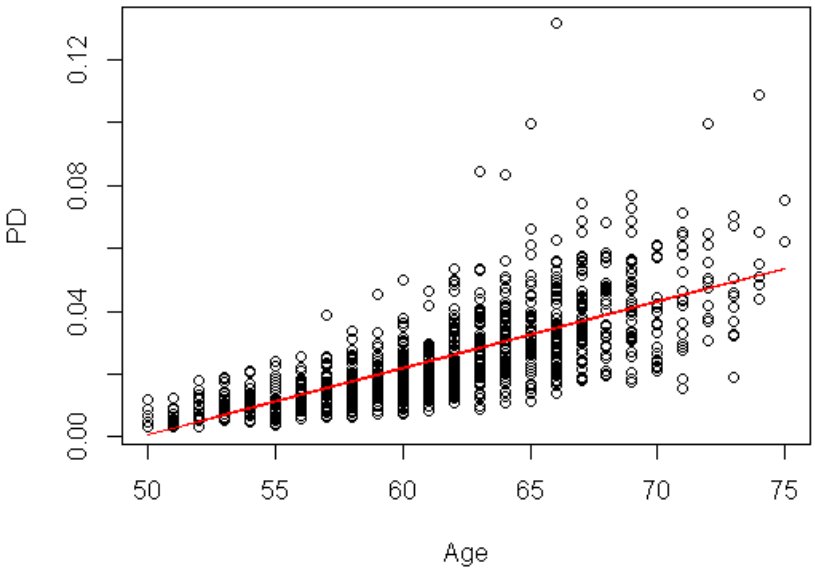


Figure 6: Model predicted probabilities from a MCMC fit of M2



B Projecting annual survival rates from a GLM fit to biennial data

The HRS survey waves are every two years. Therefore the predictions from the models fit to this data are predictions of the probability of death within the preceding two year interval; that is, for a life aged x at time t the model prediction $\hat{y}_x =_2 q_{x-2}$. These biennial mortality rates can be split into annual mortality rates as follows

$$ {}_2q_{x-2} \quad = q_{x-2} + (1 - q_{x-2})q_{x-1} \quad \approx q_{x-2} + q_{x-1} \tag{10} $$

Therefore if we know the starting rate of mortality q_{x-2}, then all subsequent annual mortality rates can be calculated as

$$ q_{x+n} =_2 q_{x+n-1} - q_{x+n-1}. \tag{11} $$

The starting rate of mortality is q_{65}. We estimate this value using the estimated biennial probability of death for a 65 year old in the last year of the survey data used (2006). Assuming that $q_{66} = cq_{65}$ and $q_{67} = cq_{66}$ then

$$ {}_2q_{65} \quad \approx \quad (1 + c)q_{65} \tag{12} $$

and

$$ {}_2q_{66} \quad \approx \quad c(1 + c)q_{65} \tag{13} $$

it follows that

$$ q_{65} \quad \approx \quad \frac{{}_2q_{65}}{1 + \frac{{}_2q_{66}}{{}_2q_{65}}}. \tag{14} $$

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