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Abstract

Systematic improvements in mortality increases dependence in the survival distributions of insured lives. This is not accounted for in standard life tables and actuarial models used for annuity pricing and reserving. Furthermore, systematic longevity risk undermines the law of large numbers; a law that is relied on in the risk management of life insurance and annuity portfolios. This paper applies a multivariate Tweedie distribution to incorporate dependence, which it induces through a common shock component. Model parameter estimation is developed based on the method of moments and generalized to allow for truncated observations. The estimation procedure is explicitly developed for various important distributions belonging to the Tweedie family, and finally assessed using simulation.

Keywords: systematic longevity risk, dependence, multivariate Tweedie, lifetime distribution

JEL Classifications: G22, G32, C13, C02

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1 Introduction

This paper generalizes an approach explored in Alai *et al.* (2013), where a multivariate gamma distribution was used to model the dependence of lifetimes. The property of gamma random variables generates a multivariate gamma distribution using the so-called multivariate reduction method; see Chereiyan (1941) and Ramabhadran (1951). This constructs a dependency structure that is natural for modelling lifetimes of individuals within a pool. The method uses the fact that a sum of gamma random variables with the same rate parameter follows a gamma distribution with that rate parameter. However, the claim that lifetimes follow a gamma distribution is too restricted; although previously applied in, for example, Klein and Moeschberger (1997).

We presently model a portfolio of lifetimes using a multivariate construction of the Tweedie family of distributions, which is an important subclass of the exponential dispersion family (EDF) that is as rich as it is popular in actuarial science; see, for example, Aalen (1992), Jørgensen and De Souza (1994), Smyth and Jørgensen (2002), Wüthrich (2003), Kaas (2005), and Furman and Landsman (2010). Recall that the random variable X is said to belong to the EDF of distributions in the additive form if its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q_λ and can be represented as follows for some function $\kappa(\theta)$ called the cumulant:

$$dP_{\theta,\lambda}(x) = e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x);$$

see Jørgensen (1997), Section 3.1; for a recent reference see Landsman and Valdez (2005). The parameter θ is named the canonical parameter belonging to the set

$$\Theta = \{\theta \in \mathbb{R} \, | \kappa(\theta) < \infty\}.$$

The parameter λ is called the index or dispersion parameter belonging to the set of positive real numbers $\Lambda = (0, \infty) = \mathbb{R}_+$. We denote by $X \sim ED(\theta, \lambda)$ a random variable belonging to the additive EDF.

In Furman and Landsman (2010) it was shown that the multivariate reduction method can construct the multivariate EDF distribution only for an important subclass of the EDF, the so-called Tweedie class. To define this class we notice that for regular EDF, see definition in Landsman and Valdez (2005), cumulant $\kappa(\theta)$ is a twice differentiable function and for the additive form, the expectation is given by

$$\mu = \lambda\kappa'(\theta).$$

Moreover, function $\kappa'(\theta)$ is one-to-one map and there exists inverse function

$$\theta = \theta(\mu) = (\kappa')^{-1}(\mu).$$

Then function $V(\mu) = \kappa''(\theta(\mu))$ is called the unit variance function and provides the classification of members of the EDF. In particular, the Tweedie subclass is the class of EDF with power unit variance function; introduced in Tweedie (1984).

$$V(\mu) = \mu^p,$$

where p is called the power parameter. Specific values of p correspond to specific distributions, for example when $p = 0, 1, 2, 3$, we recover the normal, overdispersed Poisson, gamma, and inverse Gaussian distributions, respectively. The cumulant $\kappa_p(\theta) = \kappa(\theta)$ for a Tweedie subclass has the form

$$\kappa(\theta) = \begin{cases} e^{\theta}, & p = 1, \\ -\log(-\theta), & p = 2, \\ \frac{\alpha-1}{\alpha}(\frac{\theta}{\alpha-1})^{\alpha}, & p \neq 1, 2, \end{cases}$$

where $\alpha = (p-2)/(p-1)$. Furthermore, the canonical parameter belongs to set Θ_p, given by

$$\Theta_p = \begin{cases} [0, \infty), & \text{for } p < 0, \\ \mathbb{R}, & \text{for } p = 0, 1, \\ (-\infty, 0), & \text{for } 1 < p \leq 2, \\ (-\infty, 0], & \text{for } 2 < p < \infty. \end{cases}$$

We denote by $X \sim Tw_p(\theta, \lambda)$ a random variable belonging to the additive Tweedie family.

Remark 1 *Although we deal with the additive form of the EDF, the reproductive form can easily be obtained by the transformation $Y = X/\lambda$, yielding probability measure $P^*_{\theta,\lambda}$, absolutely continuous with respect to some measure Q^*_λ,*

$$dP^*_{\theta,\lambda}(y) = e^{\lambda[\theta y - \kappa(\theta)]} dQ^*_\lambda(y).$$

Organization of the paper: Section 2 defines the multivariate Tweedie dependence structure for survival models for a pool of lives. Section 3 provides the estimation of the parameters of the model by method of moments. We consider the case when samples are given both with and without truncation. The former is essentially more complicated, but required in practice. In Section 4 we apply the estimation procedure to various distributions that fall under the Tweedie family. Section 5 concludes the paper.

2 Multivariate Tweedie Survival Model

The model is applied to individuals within a pool of lives. We assume M pools of lives. The pools can, in general, be of individuals with the same age or other characteristics that share a common risk factor. Let $T_{i,j}$ be the survival time of individual $i \in \{1, \ldots, N_j\}$ in pool $j \in \{1, \ldots, M\}$. Although the number of lives in each pool need not be identical, we presently make this assumption for simplicity and continue with $N_j = N$ for all j. We assume the following model for the individual lifetimes:

$$T_{i,j} = Y_{0,j} + Y_{i,j},$$

where

- $Y_{0,j}$ follows an additive Tweedie distribution with power parameter p, canonical and dispersion parameters θ_j and λ_0, $Tw_p(\theta_j, \lambda_0)$, $j \in \{1, \ldots, M\}$,
- $Y_{i,j}$ follows an additive Tweedie distribution with power parameter p, canonical and dispersion parameters θ_j and λ_j, $Tw_p(\theta_j, \lambda_j)$, $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, M\}$,
- The $Y_{i,j}$ are independent, $i \in \{0, \ldots, N\}$ and $j \in \{1, \ldots, M\}$.

Hence, there is a common component $Y_{0,j}$ within each pool j that impacts the survival of the individuals of that pool (i.e. $Y_{0,j}$ captures the impact of systematic mortality dependence between the lives in pool j). The parameters λ_j and θ_j may jointly be interpreted as the risk profile of pool j.

From the properties of the additive EDF it follows that the survival times $T_{i,j}$ are also Tweedie distributed with power parameter p, canonical parameter θ_j, and dispersion parameter $\tilde{\lambda}_j = \lambda_0 + \lambda_j$.

3 Parameter Estimation

In this section we consider parameter estimation using the method of moments. For an excellent reference we suggest, for example, Lindgren (1993) (Ch. 8, Theorem 6).

Notation

Before we undertake parameter estimation, we provide some necessary notation concerning raw and central, theoretical and sample, moments. Consider arbitrary random variable X. We denote with $\alpha_k(X)$ and $\mu_k(X)$ the k^{th}, $k \in \mathbb{Z}_+$, raw and central (theoretical) moments of X, respectively. That is,

$$\begin{aligned} \alpha_k(X) &= E[X^k], \\ \mu_k(X) &= E[(X - \alpha_1(X))^k]. \end{aligned}$$

Next, consider random sample $\mathbf{X} = (X_1, \dots, X_n)'$. The raw sample moments are given by

$$a_k(\mathbf{X}) = \frac{1}{n} \sum_{i=1}^{n} X_i^k, \quad k \in \mathbb{Z}^+.$$

For $X_1, \dots, X_n$ identically distributed, the raw sample moments are unbiased estimators of the corresponding raw moments of X_1:

$$E[a_k(\mathbf{X})] = \alpha_k(X_1).$$

Finally, we define the *adjusted* second and third central sample moments as

$$\begin{aligned} \widetilde{m}_2(\mathbf{X}) &= \frac{1}{n-1} \sum_{i=1}^{n} (X_i - a_1(\mathbf{X}))^2, \\ \widetilde{m}_3(\mathbf{X}) &= \frac{n}{(n-1)(n-2)} \sum_{i=1}^{n} (X_i - a_1(\mathbf{X}))^3. \end{aligned}$$

For $X_1, \dots, X_n$ independent and identically distributed, these (adjusted) central sample moments are unbiased and consistent estimators of the corresponding central moments of X_1:

$$E[\widetilde{m}_2(\mathbf{X})] = \mu_2(X_1) \qquad \text{and} \qquad E[\widetilde{m}_3(\mathbf{X})] = \mu_3(X_1).$$

3.1 Parameter Estimation for Lifetime Observations

We assume we are given samples, $\mathbf{T}_1, \dots, \mathbf{T}_M$, from the pools, where $\mathbf{T}_j = (T_{1,j}, \dots, T_{N,j})'$. This assumption requires the data used for calibration to be based on observed lifetimes for past lives. We allow for truncation which is addressed in the paper and leave allowance for censoring for future research.

We begin by considering the $\mathbf{T}_j$ separately in order to estimate corresponding parameters λ_j and θ_j, as well as predict the value of $Y_{0,j}$. Subsequently, we combine the obtained predictions of $Y_{0,1}, \dots, Y_{0,M}$ in order to estimate λ_0.

In our estimation procedure, we utilize the first raw sample moment and the second and third central sample moments. Define $\mathbf{Y}_j = (Y_{1,j}, \dots, Y_{N,j})'$. For the first raw sample moment, we obtain

$$a_1(\mathbf{T}_j) = \frac{1}{N} \sum_{i=1}^{N} T_{i,j} = \frac{1}{N} \sum_{i=1}^{N} Y_{0,j} + \frac{1}{N} \sum_{i=1}^{N} Y_{i,j} = Y_{0,j} + a_1(\mathbf{Y}_j). \tag{1}$$

For the second central sample moment, we obtain

$$\begin{aligned}
\widetilde{m}_2(\mathbf{T}_j) &= \frac{1}{N-1}\sum_{i=1}^{N}(T_{i,j}-a_1(\mathbf{T}_j))^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\Big(Y_{0,j}+Y_{i,j}-Y_{0,j}-a_1(\mathbf{Y}_j)\Big)^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}(Y_{i,j}-a_1(\mathbf{Y}_j))^2 = \widetilde{m}_2(\mathbf{Y}_j).
\end{aligned}$$

Similarly, for the third central sample moment, we obtain

$$\widetilde{m}_3(\mathbf{T}_j) = \widetilde{m}_3(\mathbf{Y}_j).$$

We take expectations of our sample moments in order to formulate a system of equations. Since each pool contains only one realization from the $Tw_p(\theta_j, \lambda_0)$ distribution, namely, $Y_{0,j}$, it is not prudent to take its expected value. Therefore, we condition on $Y_{0,j}$. Since $Y_{1,j}, \ldots, Y_{N,j}$ are identically distributed, the first raw sample moment is an unbiased estimator of the first raw moment of $Y_{1,j}$. Consequently, we have

$$E[a_1(\mathbf{T}_j)|Y_{0,j}] = Y_{0,j} + E[a_1(\mathbf{Y}_j)] = Y_{0,j} + \alpha_1(Y_{1,j}) = Y_{0,j} + \lambda_j\kappa'(\theta_j).$$

Furthermore, since $Y_{1,j}, \ldots, Y_{N,j}$ are also independent, the (adjusted) second and third central sample moments are unbiased estimators of the second and third central moments of $Y_{1,j}$, respectively. As a result, we obtain

$$\begin{aligned}
E[\widetilde{m}_2(\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_2(\mathbf{Y}_j)] = \mu_2(Y_{1,j}) = \lambda_j\kappa''(\theta_j), && (2)\\
E[\widetilde{m}_3(\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_3(\mathbf{Y}_j)] = \mu_3(Y_{1,j}) = \lambda_j\kappa'''(\theta_j). && (3)
\end{aligned}$$

Note that the above central sample moments do not depend on $Y_{0,j}$. As a result, equations (2) and (3) can be used to estimate λ_j and θ_j. Let us notice that from (1), it follows that for $N \to \infty$,

$$a_1(\mathbf{T}_j) \xrightarrow{P} Y_{0,j} + \lambda_j\kappa'(\theta_j),$$

and we cannot estimate parameters of $Y_{0,j}$ from one pool (j-pool). However, the estimators of λ_j and θ_j can be substituted into equation (1) to yield a prediction of $Y_{0,j}$. The system composed of equations (1)-(3) holds for all p; for illustrative purposes, we end this section with a presentation of the solution for $p \neq 0, 1$.

It is convenient to note the derivates of $\kappa(\theta)$. We have that

$$\kappa'(\theta) = \begin{cases} e^{\theta}, & p = 1, \\ (\frac{\theta}{\alpha-1})^{\alpha-1}, & p \neq 1, \end{cases} \qquad \kappa''(\theta) = \begin{cases} e^{\theta}, & p = 1, \\ (\frac{\theta}{\alpha-1})^{\alpha-2}, & p \neq 1. \end{cases}$$

We obtain

$$\begin{aligned} \widehat{\theta}_j &= (\alpha-2)\frac{\widetilde{m}_2(\mathbf{T}_j)}{\widetilde{m}_3(\mathbf{T}_j)}, \\ \widehat{\lambda}_j &= \left(\frac{\alpha-1}{\alpha-2}\right)^{\alpha-2}\frac{\widetilde{m}_3(\mathbf{T}_j)^{\alpha-2}}{\widetilde{m}_2(\mathbf{T}_j)^{\alpha-3}}. \end{aligned}$$

By applying $\widehat{\theta}_j$ and the $\widehat{\lambda}_j$, we can predict $Y_{0,j}$,

$$\widehat{Y}_{0,j} = a_1(\mathbf{T}_j) - \frac{\widehat{\theta}_j^{\alpha-1}\widehat{\lambda}_j}{(\alpha-1)^{\alpha-1}}.$$

Finally we estimate λ_0 using the predicted values of $Y_{0,j}$. We use the expectation:

$$E[Y_{0,j}] = \lambda_0\Big(\frac{\theta_j}{\alpha-1}\Big)^{\alpha-1}.$$

We obtain

$$\widehat{\lambda}_0 = \Big(\frac{\widehat{\theta}_j}{\alpha-1}\Big)^{1-\alpha} a_1(\widehat{\mathbf{Y}}_0),$$

where $\widehat{\mathbf{Y}}_0 = (\widehat{Y}_{0,1}, \ldots, \widehat{Y}_{0,M})$.

Summarizing, when considering only one pool j, the parameters λ_j and θ_j can be estimated and the random variable $Y_{0,j}$ predicted. In order to estimate λ_0, multiple pools are required.

3.2 Parameter Estimation for Truncated Observations

The results of the previous section cannot be directly used for calibration of parameters of the proposed model, because, in fact, we deal with truncated lifetime data. In this section we consider truncated observations ${}_{\tau_j}T_{i,j} = T_{i,j}|T_{i,j} > \tau_j$ with known truncation point τ_j. We assume all pools are subject to the same truncation point, that is $\tau_j = \tau$ for all j. This assumption is predominantly made for ease of presentation and simplicity.

We begin by constructing a useful proposition regarding the theoretical moments of truncated variables.

Proposition 1 *Consider $Y \sim ED(\theta, \lambda)$ with probability density and survival function denoted $f(y, \theta, \lambda)$ and $\overline{F}(y, \theta, \lambda)$, respectively. Define the associated truncated random variable ${}_\tau Y = Y|Y > \tau$, where $\tau \geq 0$. The first raw moment and the second and third central moments of ${}_\tau Y$ are given by*

$$\begin{aligned}
\alpha_1({}_\tau Y) &= \alpha_1(Y) + g_1(\tau), \\
\mu_2({}_\tau Y) &= \mu_2(Y) + g_2(\tau) - g_1(\tau)^2, \\
\mu_3({}_\tau Y) &= \mu_3(Y) + g_3(\tau) - 3g_2(\tau)g_1(\tau) + 2g_1(\tau)^3,
\end{aligned}$$

where

$$g_k(\tau) = g_k(\tau; \theta, \lambda) = \frac{1}{\overline{F}(\tau, \theta, \lambda)} \frac{\partial^k \overline{F}(\tau, \theta, \lambda)}{\partial \theta^k}, \quad k = 1, 2, 3.$$

Proof. The density function of ${}_\tau Y$ is given by

$$f_{{}_\tau Y}(y) = \frac{f(y, \theta, \lambda)}{\overline{F}(\tau, \theta, \lambda)}, \qquad y > \tau.$$

Notice that $\alpha_1(Y) = \lambda\kappa'(\theta)$ and $\mu_k(Y) = \lambda\kappa^{(k)}(\theta)$, $k = 2, 3$. Since the considered EDF, namely Y, is regular, we can differentiate its survival function with respect to θ.

$$\begin{aligned}
\frac{\partial \overline{F}(\tau, \theta, \lambda)}{\partial \theta} &= \frac{\partial}{\partial \theta} \int_\tau^\infty e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) \\
&= \int_\tau^\infty (x - \lambda\kappa'(\theta)) e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) \\
&= \int_\tau^\infty x e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) - \lambda\kappa'(\theta)\overline{F}(\tau, \theta, \lambda) \\
&= (\alpha_1({}_\tau Y) - \alpha_1(Y))\overline{F}(\tau, \theta, \lambda).
\end{aligned}$$

A trivial rearrangement yields the expression for the truncated first raw moment of Y. In order to obtain the truncated second central moment, we further differentiate the survival function.

$$\begin{aligned}
\frac{\partial^2 \overline{F}(\tau, \theta, \lambda)}{\partial \theta^2} &= \frac{\partial}{\partial \theta} \int_\tau^\infty (x - \lambda\kappa'(\theta)) e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) \\
&= \int_\tau^\infty (x - \lambda k'(\theta))^2 e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) - \lambda k''(\theta)\overline{F}(\tau, \theta, \lambda) \\
&= (E[({}_\tau Y - \alpha_1(Y))^2] - \lambda k''(\theta))\overline{F}(\tau, \theta, \lambda).
\end{aligned}$$

A final rearrangement, noting that

$$E[({}_\tau Y - \alpha_1(Y))^2] = \mu_2({}_\tau Y) + (\alpha_1({}_\tau Y) - \alpha_1(Y))^2$$

and

$$g_1(\tau) = (\alpha_1({}_\tau Y) - \alpha_1(Y)),$$

yields the expression for the truncated second central moment of Y. Finally, we differentiate the survival function a third time.

$$\begin{aligned}
\frac{\partial^3 \overline{F}(\tau,\theta,\lambda)}{\partial\theta^3} &= \frac{\partial}{\partial\theta}\Big\{\int_\tau^\infty (x-\lambda k'(\theta))^2 e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x) - \lambda k''(\theta)\overline{F}(\tau,\theta,\lambda)]\Big\} \\
&= \int_\tau^\infty (x-\lambda k'(\theta))^3 dP_{\theta,\lambda}(x) - 2\lambda k''(\theta)\int_\tau^\infty (x-\lambda k'(\theta)) dP_{\theta,\lambda}(x) \\
&\quad - \lambda k''(\theta)\frac{\partial \overline{F}(\tau,\theta,\lambda)}{\partial\theta} - \lambda k'''(\theta)\overline{F}(\tau,\theta,\lambda).
\end{aligned}$$

Then, dividing by $\overline{F}(\tau,\theta,\lambda)$, we obtain,

$$g_3(\tau) = E[({}_\tau Y - \alpha_1(Y))^3] - 3\mu_2(Y)g_1(\tau) - \mu_3(Y).$$

A final rearrangement, noting that

$$E[({}_\tau Y - \alpha_1(Y))^3] = \mu_3({}_\tau Y) + 3\mu_2({}_\tau Y)g_1(\tau) + g_1(\tau)^3$$

and

$$\mu_2({}_\tau Y) = \mu_2(Y) + g_2(\tau) - g_1(\tau)^2,$$

yields the expression for the truncated third central moment of Y. ■

In the above proposition, g_k can be interpreted as an *additive truncation adjustment*, one that is required for transforming un-truncated into truncated moments.

We explore the truncated lifetime ${}_\tau T_{i,j}$ by separating it into its component parts: the systematic, untruncated $Y_{0,j}$ and the idiosyncratic, truncated $Y_{i,j}$. We obtain

$${}_\tau T_{i,j} = Y_{0,j} + {}_{\tau'} Y_{i,j},$$

where $\tau' = \tau - Y_{0,j}$. The truncation on $Y_{i,j}$ must account for the value of the systematic component and hence differs from the relatively simple truncation imposed on $T_{i,j}$.

We consider the general case and a simplified case and obtain systems of equations for both. For the simplified case, we provide algorithms that facilitate parameter estimation.

3.2.1 The General Case

In this section, we follow the same method utilized in parameter estimation for untruncated observations. That is, we aim to use the first raw sample moment, and the second and third central sample moments. Consider given truncated samples ${}_{\tau}\mathbf{T}_1, \ldots, {}_{\tau}\mathbf{T}_M$, where ${}_{\tau}\mathbf{T}_j = ({}_{\tau}T_{1,j}, \ldots, {}_{\tau}T_{N,j})'$. From each pool j, we aim to estimate θ_j and λ_j, and predict the value of $Y_{0,j}$. Define ${}_{\tau'}\mathbf{Y}_j = ({}_{\tau'}Y_{1,j}, \ldots, {}_{\tau'}Y_{N,j})'$. For the first raw sample moment, we obtain

$$a_1({}_{\tau}\mathbf{T}_j) = Y_{0,j} + \frac{1}{N}\sum_{i=1}^{N} {}_{\tau'}Y_{i,j} = Y_{0,j} + a_1({}_{\tau'}\mathbf{Y}_j).$$

For the second and third central sample moments, we obtain

$$\widetilde{m}_2({}_{\tau}\mathbf{T}_j) = \widetilde{m}_2({}_{\tau'}\mathbf{Y}_j) \quad \text{and} \quad \widetilde{m}_3({}_{\tau}\mathbf{T}_j) = \widetilde{m}_3({}_{\tau'}\mathbf{Y}_j).$$

Note that $Y_{0,j}$ is present in the truncation point τ'. Hence, unlike in the untruncated case, we cannot solely use the second and third central moments to estimate θ_j and λ_j.

Now suppose that $Y_{0,j}$ is given. Then ${}_{\tau'}Y_{1,j}, \ldots, {}_{\tau'}Y_{N,j}$ are independent and identically distributed. Consequently, the first raw sample moment is an unbiased estimator of $\alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j})$. Moreover,

$$a_1({}_{\tau}\mathbf{T}_j|Y_{0,j}) \xrightarrow{P} Y_{0,j} + \alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j})$$

and the (adjusted) second and third central sample moments are unbiased and consistent estimators of $\mu_2({}_{\tau'}Y_{1,j}|Y_{0,j})$ and $\mu_3({}_{\tau'}Y_{1,j}|Y_{0,j})$, respectively. We take conditional expectations of the sample moments, with respect to $Y_{0,j}$, and using Proposition 1 obtain

$$\begin{aligned}
E[a_1({}_{\tau}\mathbf{T}_j)|Y_{0,j}] &= Y_{0,j} + E[a_1({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] \\
&= Y_{0,j} + \alpha_1({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= Y_{0,j} + \lambda_j\kappa'(\theta_j) + g_1(\tau'), && (4)\\
E[\widetilde{m}_2({}_{\tau}\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_2({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] = \mu_2({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \lambda_j\kappa''(\theta_j) + g_2(\tau') - g_1(\tau')^2, && (5)\\
E[\widetilde{m}_3({}_{\tau}\mathbf{T}_j)|Y_{0,j}] &= E[\widetilde{m}_3({}_{\tau'}\mathbf{Y}_j)|Y_{0,j}] = \mu_3({}_{\tau'}Y_{1,j}|Y_{0,j}) \\
&= \lambda_j\kappa'''(\theta_j) + g_3(\tau') - 3g_1(\tau')g_2(\tau') + 2g_1(\tau')^3, && (6)
\end{aligned}$$

where $g_k(\tau') = g_k(\tau'; \theta_j, \lambda_j), k = 1, 2, 3$, as defined in Proposition 1.

It is evident that equations (4)-(6) are generalizations of equations (1)-(3). The latter set of equations are obtained when $\overline{F}(\tau', \theta_j, \lambda_j)$ takes value one, thus setting the values of the g_k to zero, which only occurs when there is no truncation present.

3.2.2 The Simplified Case

In addition to the assumption that $\tau_j = \tau$, for all j, we further assume that $\theta_j = \theta$, and $\lambda_j = \lambda$ for all j. This additional assumption is equivalent to assuming that lives in every pool have similar risk profiles. The level of dependence within pools, however, still varies since this depends on the value $Y_{0,j}$. The consequence of the simplified case is that the model should only be applied to pools of a similar nature. In the case of heterogeneous pools, the problem may have to be compartmentalized and studied separately for each homogeneous sub-group.

In this simplified case, we begin our estimation procedure by combining all pools. Define ${}_\tau\mathbf{T} = ({}_\tau T_{1,1}, \ldots, {}_\tau T_{N,M})'$. Due to our simplifying assumptions, the components of ${}_\tau\mathbf{T}$ are identically distributed, although not independent. Recall that $T_{i,j} \sim Tw_p(\theta, \tilde{\lambda} = \lambda_0 + \lambda)$. Then ${}_\tau T_{i,j}$ has a truncated $Tw_p(\theta, \tilde{\lambda})$ distribution. Utilizing the first raw and second central moments, we obtain the following from the Proposition 1:

$$
\begin{aligned}
E[a_1({}_\tau\mathbf{T})] &= \alpha_1({}_\tau T_{1,1}) = \tilde{\lambda}\kappa'(\theta) + g_1(\tau; \theta, \tilde{\lambda}), &(7)\\
E[\widetilde{m}_2({}_\tau\mathbf{T})] &\approx \mu_2({}_\tau T_{1,1}) = \tilde{\lambda}\kappa''(\theta) + g_2(\tau; \theta, \tilde{\lambda}) - g_1(\tau; \theta, \tilde{\lambda})^2. &(8)
\end{aligned}
$$

Equation (7) arises from the fact that, for the simplified case, the raw sample moments of ${}_\tau\mathbf{T}$ are unbiased estimators of the raw moments of ${}_\tau T_{1,1}$. This does not hold for central moments, but the approximation given by (8) yields very good estimation results, as will be seen below. Notice that we no longer condition on a single $Y_{0,j}$. This is due to the fact that ${}_\tau\mathbf{T}$ contains M different realizations from the $Tw_p(\theta, \lambda_0)$ distribution, rather than one. It is, therefore, a viable option to take expectations with respect to the $Y_{0,j}$.

Equations (7) and (8) provides a two by two system of equations, but due to the presence of the g's, requires the development of a computational algorithm to provide solutions. To apply an iteration algorithm we first notice that

$$
\frac{\kappa'(\theta)}{\kappa''(\theta)} = \begin{cases} 1, & p = 1, \\ \frac{\theta}{\alpha - 1}, & p \neq 1. \end{cases}
$$

Then system of equations (7) and (8) for $p \neq 1$ can be reduced to the following system:

$$
\begin{aligned}
\theta &\approx \frac{(\alpha - 1)(\alpha_1({}_\tau T_{1,1}) - g_1(\tau; \theta, \tilde{\lambda}))}{\mu_2({}_\tau T_{1,1}) - g_2(\tau; \theta, \tilde{\lambda}) + g_1(\tau; \theta, \tilde{\lambda})^2}, &(9)\\
\tilde{\lambda} &= \frac{\alpha_1({}_\tau T_{1,1}) - g_1(\tau; \theta, \tilde{\lambda})}{\kappa'(\theta)}. &(10)
\end{aligned}
$$

We apply an iterative algorithm that is found to perform exceptionally well.

Algorithm 1

1. *Assume starting values for* θ *and* $\tilde{\lambda}$*, denote them* $\theta(1)$ *and* $\tilde{\lambda}(1)$*.*

2. *Substitute* $\theta(r)$ *and* $\tilde{\lambda}(r)$ *into equations* (9) *and* (10) *to obtain parameter estimators* $\theta(r+1)$ *and* $\tilde{\lambda}(r+1)$ *as follows:*

$$\begin{aligned}\theta(r+1) &= \frac{(\alpha-1)(a_1({}_\tau\mathbf{T}) - g_1(\tau;\theta(r),\tilde{\lambda}(r)))}{\widetilde{m}_2({}_\tau\mathbf{T}) - g_2(\tau;\theta(r),\tilde{\lambda}(r)) + g_1(\tau;\theta(r),\tilde{\lambda}(r))^2},\\ \tilde{\lambda}(r+1) &= \frac{a_1({}_\tau\mathbf{T}) - g_1(\tau;\theta(r),\tilde{\lambda}(r))}{\kappa'(\theta(r+1))},\end{aligned}$$

   where the sample moments of ${}_\tau\mathbf{T}$ *are used to estimate the theoretical moments.*

3. *Return to Step 2 with* $r = r+1$ *until parameter estimates are stable.*

From Algorithm 1, we obtain parameter estimate $\widehat{\theta}$. With this estimate in hand, we return our consideration to individual pool j. We reconsider equations (4) and (5), this time, utilizing $\widehat{\theta}$.

$$\begin{aligned}E[a_1({}_\tau\mathbf{T}_j)|Y_{0,j}] &\approx Y_{0,j} + \lambda\kappa'(\widehat{\theta}) + g_1(\tau';\widehat{\theta},\lambda), & (11)\\ E[\widetilde{m}_2({}_\tau\mathbf{T}_j)|Y_{0,j}] &\approx \lambda\kappa''(\widehat{\theta}) + g_2(\tau';\widehat{\theta},\lambda) - g_1(\tau';\hat{\theta},\lambda)^2. & (12)\end{aligned}$$

Again, we are presented with a non-linear system of equations. We apply the following iterative algorithm.

Algorithm 2

1. *Assume starting values for* $Y_{0,j}$ *and* λ*, denote them* $Y_{0,j}(1)$ *and* $\lambda(1)$*.*

2. *Substitute* $Y_{0,j}(r)$ *and* $\lambda(r)$ *into equation* (12) *to obtain* $\lambda(r+1)$*,*

$$\lambda(r+1) = \frac{\widetilde{m}_2({}_\tau\mathbf{T}_j) - g_2(\tau'(r);\widehat{\theta},\lambda(r)) + g_1(\tau'(r);\widehat{\theta},\lambda(r))^2}{\kappa''(\widehat{\theta})},$$

   where $\tau'(r) = \tau - Y_{0,j}(r)$*.*

3. *Substitute* $\lambda(r+1)$ *into equation* (11) *to obtain* $Y_{0,j}(r+1)$*,*

$$Y_{0,j}(r+1) = a_1({}_\tau\mathbf{T}_j) - \lambda(r+1)\kappa'(\widehat{\theta}) - g_1(\tau'(r);\widehat{\theta},\lambda(r+1)),$$

   where $\tau'(r) = \tau - Y_{0,j}(r)$*.*

4. Return to Step 2 with $r = r + 1$ until parameter estimates are stable.

To complete the estimation procedure, we set

$$\widehat{\lambda} = \frac{1}{M} \sum_{j=1}^{M} \widehat{\lambda}^{(j)}, \quad \text{and} \quad \widehat{\lambda}_0 = \frac{1}{M} \sum_{j=1}^{M} \frac{\widehat{Y}_{0,j}}{\kappa'(\widehat{\theta})},$$

where $\widehat{\lambda}^{(j)}$ and $\widehat{Y}_{0,j}$ are the estimate of λ and predicted value of $Y_{0,j}$, respectively, obtained using Algorithm 2 on pool j.

4 Application to Specific Distributions

In the above, the parameter estimation has been outlined for general members of the Tweedie family. The function g_k was introduced to facilitate theory. Recall,

$$g_k(\tau) = g_k(\tau; \theta, \lambda) = \frac{1}{\overline{F}(\tau, \theta, \lambda)} \frac{\partial^k \overline{F}(\tau, \theta, \lambda)}{\partial \theta^k}, \quad k = 1, 2, 3.$$

It is evident that to calculate these functions is a non-trivial exercise. In this section we focus on the simplified scenario of Section 3.2.2, and determine the necessary g_k for some important distributions belonging to the Tweedie family. Namely, the normal, gamma, inverse Gaussian, and compound Poisson distributions, for which it is progressively more difficult to obtain the g_k.

4.1 Truncated Normal Lifetimes

Suppose $Y \sim Tw_p(\theta, \lambda)$ with $p = 0$. Equivalently, Y may be represented using the normal distribution, that is, $Y \sim N(\mu = \theta\lambda, \sigma^2 = \lambda)$.

Using this equivalence, we have that

$$\overline{F}_Y(\tau, \theta, \lambda) = \bar{\Phi}((\tau - \theta\lambda)/\sqrt{\lambda}),$$

where $\bar{\Phi}(x)$ is the standard normal survival function.

It is a trivial exercise to obtain the functions g_1 and g_2 for normally distributed variables. That is, for $Tw_{p=0}(\theta, \lambda)$, we have

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \sqrt{\lambda} \, \frac{\varphi((\tau - \lambda\theta)/\sqrt{\lambda})}{\bar{\Phi}((\tau - \lambda\theta)/\sqrt{\lambda})}, \\ g_2(\tau; \theta, \lambda) &= -\lambda \, \frac{\varphi'((\tau - \lambda\theta)/\sqrt{\lambda})}{\bar{\Phi}((\tau - \lambda\theta)/\sqrt{\lambda})}, \end{aligned}$$

where $\Phi(x)$ and $\varphi(x)$ are the standard normal cumulative and density distributions.

Therefore, suppose that $Y_{i,j} \sim Tw_p(\theta, \lambda)$ and $Y_{0,j} \sim Tw_p(\theta, \lambda_0)$, with $p = 0$. Equivalently, we have that $Y_{i,j} \sim N(\theta\lambda, \lambda)$ and $Y_{0,j} \sim N(\theta\lambda_0, \lambda_0)$. This consequently implies that $T_{i,j} \sim Tw_{p=0}(\theta, \tilde{\lambda} = \lambda + \lambda_0) \equiv N(\theta\tilde{\lambda}, \tilde{\lambda})$. Note that $\kappa_{p=0}(\theta) = \theta^2/2$, $\kappa'_{p=0}(\theta) = \theta$, $\kappa''_{p=0}(\theta) = 1$, and $\alpha = 2$; this information together with g_1 and g_2 yields Algorithms 1 and 2 easily executed for truncated multivariate normal lifetimes.

Numerical Results

We have simulated truncated multivariate normal lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=0}(\theta = 0.2, \lambda = 375) \equiv N(\theta\lambda = 75, \lambda = 375), \\ Y_{0,j} &\sim Tw_{p=0}(\theta = 0.2, \lambda_0 = 25) \equiv N(\theta\lambda_0 = 5, \lambda_0 = 25). \end{aligned}$$

Consequently, we have that each individual lifetime is normally distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=0}(\theta = 0.2, \tilde{\lambda} = \lambda + \lambda_0 = 400) \equiv N(\theta\tilde{\lambda} = 80, \tilde{\lambda} = 400).$$

In Table 1 we investigate the performance of Algorithm 1. Recall that the principal concern of Algorithm 1 is to provide an estimate of θ. Each column of Table 1 represents a scenario with various numbers of pools and individuals, we find that θ is well estimated in each case. Furthermore, the algorithm is robust to required initial estimates and converges quickly. In

N	1,000	100,000	10,000	1,000	1,000	10,000
M	1	1	50	1,000	10,000	1,000
N*M	1,000	100,000	500,000	1,000,000	10,000,000	10,000,000
τ	60	60	60	60	60	60
$\tilde{\lambda}$	400	400	400	400	400	400
$\widehat{\tilde{\lambda}}$	417	372	393	403	400	401
θ	0.200	0.200	0.200	0.200	0.200	0.200
$\widehat{\theta}$	0.188	0.207	0.202	0.197	0.199	0.199

Table 1: Simulation results to test Algorithm 1 using the normal distribution.

Table 2 we investigate the performance of Algorithm 2. Recall that Algorithm 2 requires θ known (estimated, practically speaking), and produces $\widehat{\lambda}$ and $\widehat{Y}_0$ for one pool. Therefore, in our simulation, we focus on one pool of various sizes, stipulate $Y_0 = 5$, which is its expected value, and use the true θ. The reason we use the true value of θ is that otherwise, the results would reflect the accuracy of $\widehat{\theta}$; this would not be testing Algorithm 2, it would rather be a reflection of the performance of Algorithm 1. As can be seen in Table 2, Algorithm 2 performs well. Furthermore, it is robust to required initial estimates and converges quickly.

N	100	1,000	10,000	100,000	1,000,000
τ	60	60	60	60	60
θ	0.2	0.2	0.2	0.2	0.2
Y_0	5.000	5.000	5.000	5.000	5.000
$\widehat{Y}_0$	3.480	10.411	6.639	4.047	4.964
λ	375.000	375.000	375.000	375.000	375.000
$\widehat{\lambda}$	378.291	348.675	369.063	379.008	375.114

Table 2: Simulation results to test Algorithm 2 using the normal distribution.

4.2 Truncated Gamma Lifetimes

Suppose $Y \sim Tw_p(\theta, \lambda)$ with $p = 2$. Equivalently, Y may be represented using the gamma distribution, that is, $Y \sim \Gamma(\lambda, \beta = -\theta)$, where λ, β, are the shape and rate parameters, respectively.

Using this equivalence, we have that

$$\overline{F}_Y(\tau, \theta, \lambda) = \overline{G}(\tau, \lambda, -\theta)$$

where

$$\overline{G}(\tau, \lambda, \beta) = \frac{\beta^\lambda}{\Gamma(\lambda)} \int_\tau^\infty x^{\lambda-1} e^{-\beta x} dx$$

is the survival function of gamma random variable with shape parameter λ and rate parameter β.

We wish to find the functions g_1 and g_2. We differentiate the gamma survival function with respect to θ and obtain

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \frac{\lambda}{\theta}\Big(1 - K_1(\tau; \theta, \lambda)\Big), \\ g_2(\tau; \theta, \lambda) &= \frac{\lambda}{\theta^2}\Big((\lambda - 1) - 2\lambda K_1(\tau; \theta, \lambda) + (\lambda + 1) K_2(\tau; \theta, \lambda)\Big), \end{aligned}$$

where

$$K_k(\tau; \theta, \lambda) = \frac{\overline{G}(\tau, \lambda + k, -\theta)}{\overline{G}(\tau; \lambda, -\theta)}, \quad k = 1, 2.$$

Therefore, suppose that $Y_{i,j} \sim Tw_p(\theta, \lambda)$ and $Y_{0,j} \sim Tw_p(\theta, \lambda_0)$, with $p = 2$. Equivalently, we have that $Y_{i,j} \sim \Gamma(\lambda, -\theta)$ and $Y_{0,j} \sim \Gamma(\lambda_0, -\theta)$. This consequently implies that $T_{i,j} \sim Tw_{p=2}(\theta, \tilde{\lambda} = \lambda + \lambda_0) \equiv \Gamma(\tilde{\lambda}, -\theta)$. Note that $\kappa_{p=2}(\theta) = -\ln(-\theta)$, $\kappa'_{p=2}(\theta) = -1/\theta$, $\kappa''_{p=2}(\theta) = 1/\theta^2$, and $\alpha = 0$; this information together with g_1 and g_2 yields Algorithms 1 and 2 easily executed for truncated multivariate gamma lifetimes.

Numerical Results

We have simulated truncated multivariate gamma lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=2}(\theta = -0.2, \lambda = 15) \equiv \Gamma(\lambda = 15, \beta = 0.2), \\ Y_{0,j} &\sim Tw_{p=2}(\theta = -0.2, \lambda_0 = 1) \equiv \Gamma(\lambda_0 = 1, \beta = 0.2). \end{aligned}$$

Consequently, we have that each individual lifetime is gamma distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=2}(\theta = -0.2, \tilde{\lambda} = \lambda + \lambda_0 = 16) \equiv \Gamma(\tilde{\lambda} = 16, \beta = 0.2).$$

In Table 3 we investigate the performance of Algorithm 1 using the gamma distribution. We find that θ is well estimated. Again, the algorithm is robust to required initial estimates and converges quickly. In Table 4 we investigate

N	1,000	100,000	10,000	1,000	1,000	10,000
M	1	1	50	1,000	10,000	1,000
N*M	1,000	100,000	500,000	1,000,000	10,000,000	10,000,000
τ	60	60	60	60	60	60
$\tilde{\lambda}$	16.00	16.00	16.00	16.00	16.00	16.00
$\widehat{\tilde{\lambda}}$	17.26	17.49	16.16	15.97	15.96	15.97
θ	-0.200	-0.200	-0.200	-0.200	-0.200	-0.200
$\widehat{\theta}$	-0.226	-0.214	-0.204	-0.201	-0.200	-0.201

Table 3: Simulation results to test Algorithm 1 using the gamma distribution.

the performance of Algorithm 2 using the gamma distribution. As before, we focus on one pool of various sizes, stipulate $Y_0 = 5$, and use the true θ. As can be seen in Table 4, Algorithm 2 performs well. Furthermore, it is robust to required initial estimates and converges quickly.

N	100	1,000	10,000	100,000	1,000,000
τ	60	60	60	60	60
θ	-0.2	-0.2	-0.2	-0.2	-0.2
Y_0	5.000	5.000	5.000	5.000	5.000
$\widehat{Y_0}$	22.634	7.639	3.546	5.461	5.357
λ	15.000	15.000	15.000	15.000	15.000
$\widehat{\lambda}$	12.030	14.575	15.311	14.949	14.933

Table 4: Simulation results to test Algorithm 2 using the gamma distribution.

4.3 Truncated Inverse Gaussian Lifetimes

Suppose $Y \sim Tw_p(\theta, \lambda)$ with $p = 3$. Equivalently, Y may be represented using the inverse Gaussian distribution, that is,

$$Y \sim Tw_{p=3}(\theta, \lambda) \equiv IG(\mu = \lambda/\sqrt{-2\theta}, \phi = \lambda^2),$$

where μ and ϕ are the mean and the shape parameter, respectively, of the inverse Gaussian distribution. Using this equivalence, we have that

$$\overline{F}_Y(\tau, \theta, \lambda) = 1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)),$$

where $\Phi(x)$ is the standard normal distribution function and

$$\begin{aligned} z_1(\tau) &= z_1(\tau; \theta, \lambda) = \sqrt{-2\tau\theta} - \frac{\lambda}{\sqrt{\tau}}, \\ z_2(\tau) &= z_2(\tau, \theta, \lambda) = -\sqrt{-2\tau\theta} - \frac{\lambda}{\sqrt{\tau}}; \end{aligned}$$

see, for example, p. 137 of Jørgensen (1997), or Klugman *et al.* (1998).

We obtain the functions g_1 and g_2 by differentiating the survival function with respect to θ. That is, for $Tw_{p=3}(\theta, \lambda)$, we have

$$\begin{aligned} g_1(\tau; \theta, \lambda) &= \frac{\sqrt{\tau}\varphi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Big(\sqrt{\tau}\varphi(z_2(\tau)) - 2\lambda\Phi(z_2(\tau)\Big)}{\sqrt{-2\theta}(1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)))}, \\ g_2(\tau; \theta, \lambda) &= \frac{\tau\varphi'(z_1(\tau)) - \sqrt{\frac{\tau}{-2\theta}}\varphi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}h(\tau; \theta, \lambda)}{2\theta(1 - \Phi(z_1(\tau)) - e^{2\lambda\sqrt{-2\theta}}\Phi(z_2(\tau)))}, \end{aligned}$$

where,

$$h(\tau; \theta, \lambda) = \Big(\frac{2\lambda}{\sqrt{-2\theta}} - 4\lambda^2\Big)\Phi(z_2(\tau)) - \Big(\frac{1}{\sqrt{-2\theta}} - 4\lambda\Big)\sqrt{\tau}\varphi(z_2(\tau)) - \tau\varphi'(z_2(\tau)),$$

and where $\Phi(x)$ and $\varphi(x)$ are the standard normal cumulative and density distributions.

Therefore, suppose that $Y_{i,j} \sim Tw_p(\theta, \lambda)$ and $Y_{0,j} \sim Tw_p(\theta, \lambda_0)$, with $p = 3$. Equivalently, we have that

$$\begin{aligned} Y_{i,j} &\sim IG(\mu = \lambda/\sqrt{-2\theta}, \phi = \lambda^2), \\ Y_{0,j} &\sim IG(\mu = \lambda_0/\sqrt{-2\theta}, \phi = \lambda_0^2). \end{aligned}$$

This consequently implies that

$$T_{i,j} \sim Tw_{p=3}(\theta, \tilde{\lambda} = \lambda + \lambda_0) \equiv IG(\mu = \tilde{\lambda}/\sqrt{-2\theta}, \phi = \tilde{\lambda}^2).$$

Note that $\kappa_{p=3}(\theta) = -(-2\theta)^{1/2}$, $\kappa'_{p=3}(\theta) = (-2\theta)^{-1/2}$, $\kappa''_{p=3}(\theta) = (-2\theta)^{-3/2}$, and $\alpha = 1/2$; this information together with g_1 and g_2 yields Algorithms 1 and 2 easily executed for truncated multivariate inverse Gaussian lifetimes.

Numerical Results

We have simulated truncated multivariate inverse Gaussian lifetimes where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=3}(\theta = -0.1, \lambda = \sqrt{1125}) \equiv IG(\mu = 75, \phi = 1125), \\ Y_{0,j} &\sim Tw_{p=3}(\theta = -0.1, \lambda_0 = \sqrt{5}) \equiv IG(\mu = 5, \phi = 5). \end{aligned}$$

Consequently, we have that each individual lifetime is inverse Gaussian distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=2}(\theta = -0.1, \tilde{\lambda} = \lambda + \lambda_0 = \sqrt{1280}) \equiv IG(\mu = 80, \phi = 1280).$$

In Tables 5 and 6 we investigate the performance of Algorithm 1 and 2, respectively, using the inverse Gaussian distribution. The algorithms are robust to required initial estimates and converge quickly.

4.4 Truncated Compound Poisson Lifetimes

Suppose $Y \sim Tw_p(\theta, \lambda)$ with $p \in (1, 2)$. Equivalently, Y may be represented using the compound Poisson distribution, that is, Y may be written as a random sum of independent and identically distributed gamma random variables,

$$Y = S_1 + S_2 + \ldots + S_N,$$

resulting in the equivalence,

$$Y \sim Tw_{p\in(1,2)}(\theta, \lambda) \equiv CP(N \sim P(\lambda\kappa(\theta)), S_1 \sim \Gamma(-\alpha, -\theta)),$$

N	1,000	100,000	10,000	1,000	1,000	10,000
M	1	1	50	1,000	10,000	1,000
N*M	1,000	100,000	500,000	1,000,000	10,000,000	10,000,000
τ	60	60	60	60	60	60
$\tilde{\lambda}$	35.78	35.78	35.78	35.78	35.78	35.78
$\widehat{\tilde{\lambda}}$	42.64	40.83	36.04	35.65	35.69	35.47
θ	-0.100	-0.100	-0.100	-0.100	-0.100	-0.100
$\widehat{\theta}$	-0.120	-0.111	-0.102	-0.100	-0.100	-0.099

Table 5: Simulation results to test Algorithm 1 using the inverse Gaussian distribution.

N	100	1,000	10,000	100,000	1,000,000
τ	60	60	60	60	60
θ	-0.1	-0.1	-0.1	-0.1	-0.1
Y_0	5.000	5.000	5.000	5.000	5.000
$\widehat{Y}_0$	16.966	16.451	1.505	5.514	4.792
λ	34.641	34.641	34.641	34.641	34.641
$\widehat{\lambda}$	29.419	28.776	35.008	33.318	33.611

Table 6: Simulation results to test Algorithm 2 using the inverse Gaussian distribution.

where $\lambda\kappa(\theta)$, $-\alpha$, and $-\theta$ are the mean of the Poisson distribution and the shape and rate parameters of the gamma distribution, respectively; see, for example, Jørgensen and De Souza (1994). Alternatively, we may refer to this distribution as a Poisson-gamma mixture,

$$Y \sim Tw_{p\in(1,2)}(\theta,\lambda) \equiv \Gamma(-\alpha N, -\theta), \text{ where } N \sim P(\lambda\kappa(\theta)),$$

with the convention that $\Gamma(0,-\theta)$ is degenerate with value zero, and therefore $P_\theta[S_0 \le s] = 1$ for $s \ge 0$. Using the compound Poisson (or Poisson-gamma) equivalence, we have that

$$\overline{F}_Y(\tau,\theta,\lambda) = 1 - \sum_{n=0}^{\infty} P_\theta[S_n \le \tau] P_\theta[N = n],$$

where $S_n \sim \Gamma(-\alpha n, -\theta)$. We differentiate the survival function with respect to θ; we provide some details.

$$\frac{\partial \overline{F}_Y(\tau)}{\partial\theta} = -\sum_{n=0}^{\infty}\left\{\frac{\partial P_\theta[S_n \le \tau]}{\partial\theta} P_\theta[N=n] + P_\theta[S_n \le \tau]\frac{\partial P_\theta[N=n]}{\partial\theta}\right\}.$$

Let us focus on the partial derivative of the gamma random variable S_n. We have that for $n \in \mathbb{Z}_+$,

$$\begin{aligned}
\frac{\partial P_\theta[S_n \le \tau]}{\partial\theta} &= \frac{\partial}{\partial\theta}\int_0^\tau \frac{(-\theta)^{-n\alpha}}{\Gamma(-n\alpha)} e^{x\theta} x^{-n\alpha-1} dx \\
&= -\frac{n\alpha}{\theta}\int_0^\tau \left\{\frac{(-\theta)^{-n\alpha}}{\Gamma(-n\alpha)} e^{x\theta} x^{-n\alpha-1} - \frac{(-\theta)^{-n\alpha+1}}{\Gamma(-n\alpha+1)} e^{x\theta} x^{-n\alpha} dx\right\} \\
&= -\frac{n\alpha}{\theta}\left\{P_\theta[S_n \le \tau] - P_\theta[S_{n'} \le \tau]\right\},
\end{aligned}$$

where $S_{n'} \sim \Gamma(-\alpha n+1, -\theta)$. Applying the infinite sum and coefficient yields,

$$\sum_{n=0}^{\infty}\frac{\partial P_\theta[S_n \le \tau]}{\partial\theta} P_\theta[N=n] = -\sum_{n=0}^{\infty}\frac{n\alpha}{\theta}\left\{P_\theta[S_n \le \tau] - P_\theta[S_{n'} \le \tau]\right\} P_\theta[N=n].$$

Similarly, the partial derivate of the Poisson random variable N is given by

$$\begin{aligned}
\frac{\partial P_\theta[N=n]}{\partial\theta} &= \frac{\partial}{\partial\theta}\frac{e^{-\lambda\kappa(\theta)}(\lambda\kappa(\theta))^n}{n!} \\
&= -\lambda\kappa'(\theta)\left\{\frac{e^{-\lambda\kappa(\theta)}(\lambda\kappa(\theta))^n}{n!} - \frac{e^{-\lambda\kappa(\theta)}(\lambda\kappa(\theta))^{n-1}}{(n-1)!}\right\} \\
&= -\lambda\kappa'(\theta)\left\{P_\theta[N=n] - P_\theta[N=n-1]\right\}.
\end{aligned}$$

Applying the infinite sum and coefficient yields,

$$\sum_{n=0}^{\infty} P_\theta[S_n \le \tau] \frac{\partial P_\theta[N=n]}{\partial \theta} = -\lambda\kappa'(\theta) \sum_{n=0}^{\infty} \Big\{ P_\theta[S_n \le \tau] - P_\theta[S_{n+1} \le \tau] \Big\} P_\theta[N=n].$$

Putting the above together yields,

$$\begin{aligned}\frac{\partial \overline{F}_Y(\tau)}{\partial \theta} &= \sum_{n=0}^{\infty} \Big(\lambda\kappa'(\theta) \Big\{ P_\theta[S_n \le \tau] - P_\theta[S_{n+1} \le \tau] \Big\} \\ &\qquad + \frac{n\alpha}{\theta} \Big\{ P_\theta[S_n \le \tau] - P_\theta[S_{n'} \le \tau] \Big\} \Big) P_\theta[N=n].\end{aligned}$$

The function $g_1(\tau)$ is obtained by dividing the above by the survival function $\overline{F}_Y(\tau)$. The second derivate follows similarly. The function $g_2(\tau)$ is obtained by dividing the equation below by the survival function $\overline{F}_Y(\tau)$.

$$\begin{aligned}\frac{\partial^2 \overline{F}_Y(\tau)}{\partial \theta^2} &= \sum_{n=0}^{\infty} \Big(\lambda\kappa''(\theta) \Big\{ P_\theta[S_n \le \tau] - P_\theta[S_{n+1} \le \tau] \Big\} \\ &\qquad - (\lambda\kappa'(\theta))^2 \Big\{ P_\theta[S_n \le \tau] - 2P_\theta[S_{n+1} \le \tau] + P_\theta[S_{n+2} \le \tau] \Big\} \\ &- 2\lambda\kappa'(\theta)\frac{\alpha}{\theta} \Big\{ n(P_\theta[S_n \le \tau] - P_\theta[S_{n'} \le \tau]) - (n+1)(P_\theta[S_{n+1} \le \tau] - P_\theta[S_{n+1'} \le \tau]) \Big\} \\ &- \frac{n\alpha}{\theta^2} \Big\{ (n\alpha+1) P_\theta[S_n \le \tau] - 2n\alpha P_\theta[S_{n'} \le \tau] + (n\alpha - 1) P_\theta[S_{n''} \le \tau] \Big\} \Big) P_\theta[N=n],\end{aligned}$$

where $S_{n''} \sim \Gamma(-\alpha n + 2, -\theta)$ and $S_{n+1'} \sim \Gamma(-\alpha(n+1)+1, -\theta)$.

Numerical Results

We have simulated truncated multivariate compound Poisson lifetimes with $p = 1.5$ (equivalently, $\alpha = -1$), where

$$\begin{aligned} Y_{i,j} &\sim Tw_{p=1.5}(\theta = -0.4, \lambda = 3) \equiv CP(N \sim P(30), S_1 \sim \Gamma(1, 0.4)), \\ Y_{0,j} &\sim Tw_{p=1.5}(\theta = -0.4, \lambda_0 = 0.2) \equiv CP(N \sim P(2), S_1 \sim \Gamma(1, 0.4)).\end{aligned}$$

Consequently, we have that each individual lifetime is compound Poisson distributed with mean 80 and standard deviation 20,

$$T_{i,j} \sim Tw_{p=1.5}(\theta = -0.4, \tilde{\lambda} = 3.2) \equiv CP(N \sim P(32), S_1 \sim \Gamma(1, 0.4)).$$

For $p = 1.5$, we have

$$\kappa(\theta) = -4/\theta, \qquad \kappa'(\theta) = 4/\theta^2, \qquad \kappa''(\theta) = -8/\theta^3.$$

In Tables 7 and 8 we investigate the performance of Algorithm 1 and 2, respectively, using the compound Poisson distribution. The algorithms are robust to required initial estimates and converge quickly. Of note is that the infinite summation is approximated using n up to $1,000$; n up to $2,000$ yielded no visible difference in the results.

N	1,000	100,000	10,000	1,000	1,000	10,000
M	1	1	50	1,000	10,000	1,000
N*M	1,000	100,000	500,000	1,000,000	10,000,000	10,000,000
τ	60	60	60	60	60	60
$\tilde{\lambda}$	3.20	3.20	3.20	3.20	3.20	3.20
$\widehat{\tilde{\lambda}}$	2.82	3.00	3.26	3.18	3.19	3.15
θ	-0.400	-0.400	-0.400	-0.400	-0.400	-0.400
$\widehat{\theta}$	-0.388	-0.400	-0.403	-0.399	-0.400	-0.398

Table 7: Simulation results to test Algorithm 1 using the compound Poisson distribution.

N	100	1,000	10,000	100,000	1,000,000
τ	60	60	60	60	60
θ	-0.4	-0.4	-0.4	-0.4	-0.4
Y_0	5.000	5.000	5.000	5.000	5.000
$\widehat{Y_0}$	37.601	9.056	5.851	5.756	4.766
λ	3.000	3.000	3.000	3.000	3.000
$\widehat{\lambda}$	1.861	2.840	2.977	2.979	3.007

Table 8: Simulation results to test Algorithm 2 using the compound Poisson distribution.

5 Conclusion

We model lifetimes using a common stochastic component to address dependence within a portfolio of lives resulting from systematic mortality improvements. We develop parameter estimation in the presence of truncated observations for the Tweedie distribution, in general, and provide explicit solutions for important members of the Tweedie family, namely, the normal, gamma, inverse Gaussian, and compound Poisson distributions. Simulation is used to verify the performance of our estimation procedure.

The allowance for truncated observations is essential for the application of the model to pools of lives, especially when income insurance products are considered. However, the allowance for censored observations would further improve the applicability, and is something we consider for future research.

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