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Abstract

Risk-sharing rules have been applied to mortality pooling products to ensure these products are actuarially fair and self-sustaining. However, most of the existing studies on the risk-sharing rules of mortality pooling products assume deterministic mortality rates, whereas the literature on mortality models provides empirical evidence suggesting that mortality rates are stochastic and correlated between cohorts. In this paper, we extend existing risk-sharing rules and introduce a new risk-sharing rule, named the joint expectation rule, to ensure the actuarial fairness of mortality pooling products while accounting for stochastic and correlated mortality rates. Moreover, we perform a systematic study of how the choice of risk-sharing rule, the volatility and correlation of mortality rates, pool size, account balance, and age affect the distribution of mortality credits. Then, we explore a dynamic pool that accommodates heterogeneous members and allows new entrants, and we track the income payments for different members over time. Furthermore, we compare different risk-sharing rules under the scenario of a systematic shock in mortality rates. We find that the account balance affects the distribution of mortality credits for the regression rule, while it has no effect under the proportional, joint expectation, and alive-only rules. We also find that a larger pool size increases the sensitivity to the deviation in total mortality credits for cohorts with mortality rates that are volatile and highly correlated with those of other cohorts, under the stochastic regression rule. Finally, we find that risk-sharing rules significantly influence the effect of mortality shocks on fund balances since, under different risk-sharing rules, fund balances have different sensitivities to deviations in mortality credits.

Keywords: Risk-sharing rule, mortality pooling product, stochastic mortality rate, correlated mortality rate

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1 Introduction

Mortality pooling products are useful tools for reducing the idiosyncratic mortality risks of their participants. These products involve retirees in a risk-sharing pool in which surviving members benefit from the mortality credits accrued by members who have passed away. Compared with conventional life annuity products, mortality pooling products have the advantage of requiring less capital because they are self-sustaining and do not guarantee lifetime income. As such, potential participants can purchase these products at a lower price and providers can reduce their financial and longevity risk exposure with less capital. At the same time, they also provide retirees with retirement income streams. Mortality pooling products can be broadly categorised into pooled annuities (Piggott et al., 2005; Qiao and Sherris, 2013; Bernhardt and Donnelly, 2021), tontines (Milevsky and Salisbury, 2015; Milevsky and Salisbury, 2016; Chen and Rach, 2019; Chen et al., 2019; Chen et al., 2020; Weinert and Gründl, 2021), and risk-sharing products (Sabin, 2010; Donnelly et al., 2014; Donnelly and Young, 2017; Denuit, 2019; Fullmer and Sabin, 2018) with an additional decumulation plan. These products share the advantage of requiring zero or almost zero capital, while the ways of distributing the mortality credits and determining income payments are different.

Within the three broad categories of mortality pooling products, risk-sharing products apply a specific risk-sharing rule in the step of distributing mortality credits. The risk-sharing rule is a concept of sharing insurance losses, which can be applied to sharing the loss of individuals in the case of death for mortality pooling products. An extensive study is conducted in Denuit et al. (2022a) on the properties of risk-sharing rules. Due to the application of a risk-sharing rule with desirable properties, risk-sharing products allow heterogeneity of fund members, allow new entrants, and are actuarially fair. These ideal properties can attract potential buyers so that the fund can benefit from a large pool size to reduce the volatility of payments. Therefore, risk-sharing products become a natural choice for this study.

Previous studies on risk-sharing products include Sabin (2010), Donnelly et al. (2014), Forman and Sabin (2015), Donnelly (2017), Donnelly and Young (2017), Fullmer and Sabin (2018), Denuit (2019), Weinert and Gründl (2021) and Denuit et al. (2022b). Sabin (2010) focuses on actuarial fairness and proposes fair tontine and fair tontine annuity designs that allow heterogeneity in the pool. Despite the product being named tontine, the risk-transfer plan essentially uses a risk-sharing rule to distribute the total mortality credits to individual accounts of members alive. Then, an income payment is paid from the account balance of the individual after risk sharing following a decumulation plan. Forman and Sabin (2015) state that the two main advantages of these products are the low probability of underfunding and the issuer will not need to bear all the mortality risk and investment risk. Weinert and Gründl (2021) build on the framework of Sabin (2010) for tontines by introducing a model that evolves over time and allows new members to join. However, the fair transfer plan involves solving a system of linear equations that increases in size with the number of members, making the transfer plan computationally difficult to implement and hard to explain to members and potential buyers. Donnelly and Young (2017) propose a risk-sharing rule for mortality pooling products which is actuarially fair for individuals of all ages at any point in time and thus gives members the freedom to join and leave the pool. The distribution of mortality credits in Donnelly and Young (2017) is determined by the risk exposure of individuals. The proportional risk-sharing rule in Donnelly and Young (2017) is the discrete-time version of the annuity overlay fund in Donnelly et al. (2014). Denuit (2019) applies the conditional-mean risk-sharing rule proposed in Denuit and Dhaene (2012) on sharing mortality credits and compares it with the formalised discrete-time risk-sharing rule in Donnelly (2017). Denuit et al. (2022b) then study the effect of pool size on conditional mean mortality risk sharing and find that the individual risk can be fully diversified if the pool size tends to infinity. However, the focus of Donnelly and Young (2017) and Denuit (2019) is mainly on the fair game of the fund balance but does not specify a decumulation plan. Individuals can decide to spend part of the fund balance and invest the remaining back into the pool, or they can take the money and leave the pool forever. As such, there is a risk that individuals overspend their fund balance due to a preference for early spending or short-sighted behaviour, leaving an insufficient amount of income to sustain them for their remaining lifetimes. Fullmer and Sabin (2018) propose a risk-sharing rule that is slightly different from the one in Donnelly and Young (2017) and only distributes mortality credits to the members alive, and they are also one of the earliest to study two decumulation strategies: a 10-year lump sum or annuity-like payments. However, their risk-sharing rule is almost fair but not strictly fair mathematically. Therefore, the literature lacks focus on both the actuarial fairness and factors that affect the decumulation of a risk-sharing product at the same time. This research is thus motivated to study risk-sharing rules with these properties along with a decumulation plan. The income payments are investigated over a period of time since the initial establishment of the product.

Moreover, despite multiple risk-sharing rules proposed to distribute mortality credits, most existing studies on mortality risk-sharing products assume deterministic mortality rates, whereas the developments in the literature on mortality models suggest that mortality rates are stochastic and correlated (Jevtić et al., 2013; Xu et al., 2020; Zhou et al., 2023). Therefore, this paper is motivated to extend current risk-sharing rules to a stochastic setting and explore new risk-sharing rules that are fair and self-sustaining under the stochastic setting. Furthermore, few studies have been conducted so far on the comparison of different risk-sharing rules and the evolution of a dynamic pool allowing heterogeneous members. This motivates us to study the income payments and balances over time and how they are affected by the choice of risk-sharing rules, member profiles, and mortality shock.

This paper contributes to the literature in the following aspects. Firstly, we contribute to the literature on risk-sharing rules in mortality pooling products by considering the fact that mortality rates are stochastic and correlated random variables in risk sharing. Most of the existing papers on mortality risk sharing assume deterministic mortality rates (Sabin, 2010; Donnelly, 2017; Denuit and Robert, 2021; Fullmer and Sabin, 2018). We analyse existing risk-sharing rules and extend them to the case in which mortality rates are stochastic and correlated. Another contribution is that a new risk-sharing rule called the joint expectation rule is proposed and tested, which takes stochastic mortality rates and correlations between mortality rates of different cohorts into account. We show that the joint expectation risk-sharing rule is actuarially fair and sustainable with heterogeneous cohorts when mortality rates are stochastic and when a death benefit is included.

Moreover, we contribute to the literature by investigating the effect of fund balance, age, pool size, and volatilities and correlations of mortality rates on fund balances and benefit payments for different risk-sharing rules. Our findings show that for regression rules, in which the risk-sharing weights take into account the covariance between the individual fund balance at risk and the total mortality credits, the higher the initial balance, the higher the weighting in the difference between the empirical and expected total mortality credits. Meanwhile, for regression, joint expectation, and alive-only rules, individual account balance does not affect the weight of the deviation in the total mortality credits. The weight of the deviation in the total mortality credits measures the impact on a risk-sharing rule when there is a mortality shock. For example, when there is a systematic reduction in mortality rates, then there will be a negative deviation from the expected total mortality credits, which will cause more reduction in the balance and income payments in the risk-sharing rules that have a higher weight in this deviation. Furthermore, we find that pool size plays an important role when stochasticity and correlation of mortality rates are included in risk sharing. For the stochastic regression rule, with the assumed correlation matrix between the mortality rates of cohorts, when the pool size increases, the weight in the deviation of total mortality credits increases for middle-aged retirees at age 80 who have more volatile mortality rates and are more correlated with other cohorts. Meanwhile, for young and very old retirees at ages 60 and 100, the weight in the deviation decreases.

Finally, we study how a mortality shock will affect income payments and fund balances for different risk-sharing rules when mortality rates are assumed to be either deterministic or stochastic and correlated. A dynamic pool with new and heterogeneous members joining is investigated over time. It is found that with a 5-year systematic reduction in mortality rates, the stochastic and deterministic regression rules give lower account balances at the end of the period for younger retirees and middle-aged retirees with high balances, compared with proportional and joint expectation rules. Meanwhile, for older retirees and middle-aged retirees with low balances, the stochastic and deterministic regression rules give higher account balances than proportional and joint expectation rules. The alive-only rule always gives the highest account balances in old age.

The rest of the paper is structured as follows. The operation of the risk-sharing product with a decumulation rule along with the two important properties, namely actuarial fairness and self-sustainability, are introduced in Section 2. We extend risk-sharing rules to the setting with stochastic and correlated mortality rates in Section 3 and prove their fairness and self-sustainability. In particular, Subsection 3.4 introduces a new risk-sharing rule, named the joint expectation rule, which is actuarially fair and self-sustaining with stochastic and correlated mortality rates. Meanwhile, the joint expectation rule reduces to the proportional risk-sharing rule when mortality rates are deterministic. We examine the risk-sharing rules by considering an open pool with heterogeneous members joining every year. Section 4 presents the data, assumptions, and results of how the deviation in expected mortality credits, different fund balances, pool sizes, and mortality shocks will affect different risk-sharing rules respectively. Section 5 concludes the paper.

2 Fund Operation

The fund operation allows each member i to have their own account. When the pooling product commences, each member i contributes an initial amount $F_i(0)$ as the initial account balance. At the end of the first period, the initial investment has an accumulated value denoted by $s_i(1) = F_i(0)(1 + ROR_i(0))$, where $ROR_i(0)$ is the rate of return realised on the investment of the i^{th} member over the period $[0, 1]$. A risk-sharing rule is then applied to obtain the fund balance $V_i(1)$ after distributing the total mortality credits from the members who have passed away over the period $[0, 1]$, and the benefit $B_i(1)$ based on $V_i(1)$ is paid to the individual. The remaining balance $F_i(1) = V_i(1) - B_i(1)$ is then reinvested. New members can join the fund at time 1, and the process is repeated for all members.

We now describe the fund mechanics at an arbitrary point in time t. The fund operates as the following steps.

Step 1: Accumulation
Assume that the fund value of member i at time t after all the payments to be made is $F_i(t)$. Then, at time $t + 1$, the fund value is accumulated to:

$$s_i(t+1) = F_i(t)(1 + ROR_i(t)), \tag{1}$$

where $ROR_i(t)$ is the return rate of individual i between time t and $t + 1$.

Step 2: Risk sharing
The set of members who have passed away over the period $[t, t + 1]$ is denoted by $D(t + 1)$. The accumulated fund values of members in the set $D(t+1)$ who have passed away over the period $[t, t+1]$ are added up to form the total mortality credits $S(t + 1)$ at time $t + 1$, that is:

$$S(t+1) = \sum_{j=1}^{N(t)} 1_{j \in D(t+1)} s_j(t+1),$$

where $N(t)$ is the total number of members in the pool at time t. The total mortality credits $S(t + 1)$ will be distributed at time $t + 1$ to either every member alive at time t, or only those who survive at the end of the period time $t + 1$, depending on the choice of the risk-sharing rule.

The distribution of the total mortality credit $S(t + 1)$ to each individual account with a risk-sharing rule is represented in Equation (2). The fund value $V_i(t + 1)$ of individual i at time $t + 1$ after risk sharing and before paying the benefit will be:

$$V_i(t+1) = \begin{cases} g_A(s_i(t+1), S(t+1)) & \text{if individual } i \text{ survives this period,} \\ g_D(s_i(t+1), S(t+1)) & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{2}$$

where $g_A(\cdot)$ and $g_D(\cdot)$ are functions of fund balance $s_i(t+1)$ of individual i and total mortality credits

$S(t+1)$ at time $t+1$ in the case of being alive or dead at time $t+1$ respectively, representing the risk-sharing rule. One common setting of the risk-sharing rule as an example is:

$$V_i(t+1) = \begin{cases} s_i(t+1) + w_i^A(t+1)S(t+1) & \text{if individual } i \text{ survives this period,} \\ w_i^D(t+1)S(t+1) & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{3}$$

where $w_i^j(t+1)$ is the weighting of individual i on the total mortality credits $S(t+1)$ at time $t+1$ for $j = A$ or D representing alive or dead. The weighting functions often depend on the one-year probability of death $q_i(t)$ for individual i at time t. We can see that under this setting, members who are alive will be better off because $s_i(t+1) \leq s_i(t+1) + w_i^A(t+1)S(t+1)$ when the weighting $w_i^A(t+1)$ and the total mortality credits $S(t+1)$ are non-negative. Meanwhile, members who have died will lose their accumulated fund value $s_i(t+1)$.

Step 3: Benefit Payment
After risk sharing, the benefit payment as the retirement income to every individual at time $t+1$ is determined by:

$$B_i(t+1) = \begin{cases} \frac{V_i(t+1)}{\ddot{a}_{x_{i,t+1}}} & \text{if individual } i \text{ survives this period,} \\ V_i(t+1) & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{4}$$

where $\ddot{a}_{x_{i,t+1}}$ is the actuarial notation of an annuity due for individual i who is aged x_i at time $t+1$. The fund recalculates the annuity-like payment values at each point in time, similar to the idea of a group self-annuitisation (GSA). The annuity due factor is calculated as:

$$\ddot{a}_{x_{i,t+1}} = 1 + \sum_{s=1}^{\infty} \frac{E[{}_s p_{x_i}(t+1)]}{(1+ROR_i)^s} \geq 1,$$

where ${}_s p_{x_i}(t+1)$ is the s-year survival probability for individual i aged x at time $t+1$. The annuity due factor $\ddot{a}_{x_{i,t+1}} \geq 1$ ensures that the benefit payment is always smaller or equal to the fund value after risk sharing $B_i(t+1) \leq V_i(t+1)$. If member i dies, their balance after risk sharing will be paid out to them so that the remaining balance will be 0 and their account will be closed. The remaining fund balance for member i is represented as:

$$F_i(t+1) = \begin{cases} V_i(t+1) - B_i(t+1) & \text{if individual } i \text{ survives this period,} \\ 0 & \begin{array}{l}\text{if individual } i \text{ dies during this period and} \\ \text{thus leaves the pool at the end of the period.}\end{array} \end{cases} \tag{5}$$

If member i survives, the fund value of member i after risk sharing will often be higher than the fund value before risk sharing: $V_i(t+1) > s_i(t+1)$ if the return of member i from the distribution of total mortality credits is positive. Hence, if they live longer than expected, their sum of income payments discounted to time zero will be higher than what they initially invested, similar to life annuities. This coincides with the idea in the monograph Milevsky (2022) which states that the major purpose of mortality risk sharing is pooling with people who are willing to share that risk and benefit from the mortality credits.

Step 4: Accumulation in the Next Period
The fund value after risk sharing and benefit payment $F_i(t+1)$ becomes the initial value for the next period $[t+1, t+2]$. New members can join the fund at time $t+1$ with their initial contributions. The fund value of member i is accumulated to $s_i(t+2) = F_i(t+1)(1 + ROR_i(t+1))$ following Equation (1).

2.1 Fairness and Self-sustainability

Fairness and self-sustainability are two important properties of a risk-sharing rule in Step 2, as discussed in Denuit et al. (2022a) and Hieber and Lucas (2022).

Definition 1 (*Fairness*)**.** A risk-sharing rule is said to be fair if for each member i at time $t+1$ for $t = 0, 1, 2, 3, ...$, the expected fund value after risk sharing is equal to its value before risk sharing:

$$\mathrm{E}\left[V_i(t+1)\right] = s_i(t+1). \tag{6}$$

Condition (6) promises that members are not taking advantage or disadvantage of other members by joining risk sharing.

Definition 2 (*Self-sustainability*)**.** A risk-sharing rule is said to be self-sustaining if at any time $t = 0, 1, 2, 3, ...$, the sum of member fund balances before and after risk sharing are equal to each other:

$$\sum_{j=1}^{N(t)} V_j(t+1) = \sum_{j=1}^{N(t)} s_j(t+1). \tag{7}$$

When a risk-sharing rule is self-sustaining, it will not pay higher than the total fund balance. Thus, insurance companies do not need to worry about the huge loss in the case of systematic mortality improvement. Therefore, it benefits from a lower capital requirement and thus a lower loading compared with life annuities.

3 Risk-sharing Rules and Extensions to Stochastic Mortality Rates

We study and extend three risk-sharing rules in the literature, which are the proportional rule in Donnelly and Young (2017), regression rule in Denuit and Robert (2021), and alive-only rule in Fullmer and Sabin (2018). We explain each of the risk-sharing rules in a consistent framework and extend them to the setting with stochastic and correlated mortality rates.

3.1 Proportional Rule

Firstly, we define the proportional rule that pays the total mortality credits in proportion to the capital at risk when the mortality rates $q_i(t)$ are deterministic.

Definition 3 (*Deterministic Proportional Rule*)**.** Consider a risk-sharing rule that the weights of the total mortality credits in Equation (3) are determined in proportion to the capital at risk $s_i(t+1)q_i(t)$ which is the product of accumulated fund balance and the one-year probability of death. At the end of a period, the fund value after risk sharing of an individual i initially alive is:

$$V_i(t+1) = \begin{cases} s_i(t+1) + \frac{s_i(t+1)q_i(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j(t)} S(t+1) & \text{if individual } i \text{ survives this period,} \\ \frac{s_i(t+1)q_i(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j(t)} S(t+1) & \text{if individual } i \text{ dies during this period.} \end{cases} \tag{8}$$

We call the risk-sharing rule in Equation (8) a deterministic proportional rule.

Proposition 1. *The deterministic proportional rule in Equation (8) is fair and self-sustaining under deterministic mortality rates.*

Proof. See Denuit (2019). □

We now extend the proportional risk-sharing rule to the case where mortality rates are stochastic and correlated. That is, we seek a risk-sharing rule of the form

$$V_i(t+1) = \begin{cases} s_i(t+1) + w_i(t+1)S(t+1) & \text{if individual } i \text{ survives this period,} \\ w_i(t+1)S(t+1) & \text{if individual } i \text{ dies during this period,} \end{cases}$$

where the weights $w_i(t+1)$ are to be determined such that they are proportional to the total risk exposure and the rule is actuarially fair.

Proposition 2. *When mortality rates are stochastic and correlated random variables, the risk-sharing rule:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]} S(t+1) & \text{individual } i \text{ survives this period,} \\ \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]} S(t+1) & \text{individual } i \text{ dies during this period,} \end{cases} \tag{9}$$

is fair and self-sustaining.

Proof. When the mortality rates are stochastic, we denote $Q(t)$ as the set of mortality rates $Q(t) = \{q_1(t), q_2(t), ..., q_i(t)\}$, where $q_i(t)$ are random variables of the one-year mortality rates of individual i during the period.

Assume the following payout function holds:

$$V_i(t+1) = \begin{cases} s_i(t+1) + w_i(t+1)S(t+1) & \text{if individual } i \text{ survives this period,} \\ w_i(t+1)S(t+1) & \text{if individual } i \text{ dies during this period.} \end{cases}$$

Using the law of conditional expectation, we obtain:

$$\begin{aligned} E[V_i(t+1)] =& E[E[V_i(t+1)|Q(t)]] \\ =& E\left[s_i(t+1)(1-q_i(t)) + w_i(t+1)\sum_{j=1}^{N(t)} s_j(t+1)q_j(t) \right] \\ =& s_i(t+1)(1-E[q_i(t)]) + E[w_i(t+1)]\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)] = s_i(t+1), \end{aligned}$$

which yields

$$E[w_i(t+1)] = \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]}.$$

Since the weight $w_i(t+1)$ is deterministic, we can write $w_i(t+1) = E[w_i(t+1)] = \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]}$. Therefore, the risk-sharing rule in Equation (9) is a fair risk-sharing rule.

The risk-sharing rule in Equation (9) is self-sustaining because:

$$\begin{aligned} \sum_{j=1}^{N(t)} V_i(t+1) &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]} S(t+1) \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + S(t+1) \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1) \\ &= \sum_{j=1}^{N(t)} s_j(t+1), \end{aligned}$$

where $A(t+1)$ is the set of people alive at time $t+1$ given alive at time t. □

The risk-sharing rule satisfying Equation (9) is called the stochastic proportional rule. One thing to notice is that $E[q_i(t)]$ is assumed to be equal to the $q_i(t)$ used in the numerical illustration of the deterministic case. Effectively, this is saying that the results on weighting are not affected when we move from deterministic to stochastic mortality rates.

Lemma 1. *The stochastic proportional rule in Equation (9) can be rewritten as:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + s_i(t+1)E[q_i(t)] \\ \quad + w_i^{\text{Proportional}}(t+1)(S(t+1) - E[S(t+1)]) & \text{if individual } i \text{ survives this period,} \\ \\ s_i(t+1)E[q_i(t)] \\ \quad + w_i^{\text{Proportional}}(t+1)(S(t+1) - E[S(t+1)]) & \text{if individual } i \text{ dies during this period,} \end{cases}$$

where $w_i^{\text{Proportional}}(t+1) = \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]}$ stands for the weighting for individual i at time $t+1$ with the stochastic proportional rule.

Proof. We can rewrite the share of mortality credits as:

$$\begin{aligned} & s_i(t+1)E[q_i(t)] + \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]}(S(t+1) - E[S(t+1)]) \\ =& s_i(t+1)E[q_i(t)] + \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]}S(t+1) - s_i(t+1)E[q_i(t)] \\ =& \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]}S(t+1), \end{aligned}$$

which is equal to the form in Equation (9). □

3.2 Regression Rule

Denuit and Robert (2021) consider a regression risk-sharing rule, which develops from linear regression and takes the volatility of the risky event into account. The regression risk-sharing rule is also referred to as the covariance risk-sharing rule in Jiao et al. (2022). However, this risk-sharing rule has not been discussed in detail with the setting of mortality-sharing products, nor under stochastic mortality rates.

Definition 4 (*Regression Rule*)**.** Consider a risk-sharing rule that the distribution of the total mortality credits in Equation (2) is defined as the following:

$$V_i(t+1) = \begin{cases} s_i(t+1) + E[X_i(t+1)] \\ \quad + \dfrac{Cov(X_i(t+1), S(t+1))}{Var(S(t+1))}(S(t+1) - E[S(t+1)]) & \text{if individual } i \text{ survives this period,} \\ \\ E[X_i(t+1)] \\ \quad + \dfrac{Cov(X_i(t+1), S(t+1))}{Var(S(t+1))}(S(t+1) - E[S(t+1)]) & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{10}$$

where $X_i(t+1) = 1_{i \in D(t+1)} s_i(t+1)$ and $S(t+1) = \sum_{j=1}^{N(t)} X_j(t+1)$. The risk-sharing rule that satisfies Equation (10) is called the regression rule.

Proposition 3. *The regression risk-sharing rule in Equation (10) is actuarially fair and self-sustaining.*

Proof. This risk-sharing rule in Equation (10) is fair because:

$$\begin{aligned}E[V_i(t+1)] =& s_i(t+1)p_i(t) + E[X_i(t+1)] \\ &+ \frac{Cov(X_i(t+1), S(t+1))}{Var(S(t+1))}(E[S(t+1)] - E[S(t+1)]) \\ =& s_i(t+1)p_i(t) + s_i(t+1)q_i(t) \\ =& s_i(t+1).\end{aligned}$$

This risk-sharing rule in Equation (10) is self-sustaining because:

$$\begin{aligned}\sum_{j=1}^{N(t)} V_i(t+1) =& \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} E[X_j(t+1)] \\ &+ (S(t+1) - E[S(t+1)]) \sum_{j=1}^{N(t)} \frac{Cov(X_j(t+1), S(t+1))}{Var(S(t+1))} \\ =& \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + E[S(t+1)] + (S(t+1) - E[S(t+1)]) \\ =& \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1) \\ =& \sum_{j=1}^{N(t)} s_j(t+1).\end{aligned}$$

$\square$

Proposition 4. *When mortality rates are deterministic, the regression risk-sharing rule in Equation (10) is:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + s_i(t+1)q_i(t) & \text{if individual } i \\ \quad + w_i^{\mathrm{RD}}(t+1)(S(t+1) - \sum_{j=1}^{N(t)} s_j(t+1)q_j(t)) & \text{survives this period,} \\ & \\ s_i(t+1)q_i(t) & \text{if individual } i \text{ dies} \\ \quad + w_i^{\mathrm{RD}}(t+1)(S(t+1) - \sum_{j=1}^{N(t)} s_j(t+1)q_j(t)) & \text{during this period,} \end{cases} \tag{11}$$

where $w_i^{\mathrm{RD}}(t+1) = \frac{s_i(t+1)^2 q_i(t)(1-q_i(t))}{\sum_{j=1}^{N(t)} s_j(t+1)^2 q_j(t)(1-q_j(t))}$ *stands for the weighting for individual* i *at time* $t+1$ *with the regression deterministic (RD) risk-sharing rule.*

Proof. By using $S(t+1) = \sum_{j=1}^{N(t)} X_j(t+1)$, we have:

$$\begin{aligned}Cov(X_i(t+1), S(t+1)) =& Var(X_i(t+1)) \\ =& s_i(t+1)^2 q_i(t)(1 - q_i(t)),\end{aligned} \tag{12}$$

and

$$
\begin{aligned}
Var(S(t+1)) &= \sum_{j=1}^{N(t)} Var(X_j(t+1)) \\
&= \sum_{j=1}^{N(t)} s_j(t+1)^2 q_j(t)(1-q_j(t)).
\end{aligned} \tag{13}
$$

Substituting Equations (12) and (13) into Equation (10) completes the proof. □

The regression risk-sharing rule with deterministic mortality rates in Equation (11) is referred to as the deterministic regression rule in this paper. Since the mortality rate $q_i(t)$ in $w_i^{\mathrm{RD}}(t+1)$ of Equation (11) is deterministic, the source of randomness comes from the uncertainty of survival for given deterministic mortality rates.

Proposition 5. *Assuming stochastic mortality rates, the fair regression risk-sharing rule in Equation (10) becomes:*

$$
V_i(t+1) = \begin{cases} s_i(t+1) + s_i(t+1)E[q_i(t)] & \\ \quad + w_i^{\mathrm{RS}}(t+1)(S(t+1) - \sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]) & \text{if individual } i \text{ survives this period,} \\ & \\ s_i(t+1)E[q_i(t)] & \\ \quad + w_i^{\mathrm{RS}}(t+1)(S(t+1) - \sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]) & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{14}
$$

where $w_i^{\mathrm{RS}}(t+1) = \frac{s_i(t+1)^2 E\left[q_i(t)(1-q_i(t))\right] + s_i(t+1)\sum_{j=1}^{N(t)} s_j(t+1)Cov(q_i(t),q_j(t))}{\sum_{j=1}^{N(t)} s_j(t+1)^2 E[q_j(t)(1-q_j(t))] + \sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)Cov(q_j(t),q_k(t))}$ *stands for the weighting for individual* i *at time* $t+1$ *with the regression stochastic (RS) risk-sharing rule, and* $E[q_i(t)(1-q_i(t))] = E[q_i(t)] - E[q_i(t)^2] = E[q_i(t)] - Var[q_i(t)] - E[q_i(t)]^2$.

Proof. Using the law of conditional expectation, we have:

$$
\begin{aligned}
E[X_i(t+1)] =& E[E[X_i(t+1)|Q(t)]] \\
=& E[E[s_i(t+1)1_{i\in D(t+1)}|Q(t)]] \\
=& E[s_i(t+1)q_i(t)] \\
=& s_i(t+1)E[q_i(t)],
\end{aligned}
$$

and

$$
\begin{aligned}
E[S(t+1)] =& E[E[S(t+1)|Q(t)]] \\
=& E[E[\sum_{j=1}^{N(t)} s_j(t+1)1_{j\in D(t+1)}|Q(t)]] \\
=& E[\sum_{j=1}^{N(t)} s_j(t+1)q_j(t)] \\
=& \sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)].
\end{aligned}
$$

By the law of total covariance, we have:

$$\begin{aligned}
&Cov(X_i(t+1), S(t+1)) \\
=&E[Cov(X_i(t+1), S(t+1)|Q(t))] + Cov(E[X_i(t+1)|Q(t)], E[S(t+1)|Q(t)]) \\
=&E\left[Cov\left(s_i(t+1)1_{D_i}, \sum_{j=1}^{N(t)} s_j(t+1)1_{D_j}|Q(t)\right)\right] \\
&+ Cov\left(E\left[s_i(t+1)1_{D_i}|Q(t)\right], E\left[\sum_{j=1}^{N(t)} s_j(t+1)1_{D_j}|Q(t)\right]\right) \\
=&E\left[Var(s_i(t+1)1_{D_i}|Q(t))\right] + Cov\left(s_i(t+1)q_i(t), \sum_{j=1}^{N(t)} s_j(t+1)q_j(t)\right) \\
=&s_i(t+1)^2 E\left[q_i(t)(1-q_i(t))\right] + s_i(t+1)\sum_{j=1}^{N(t)} s_j(t+1)Cov(q_i(t), q_j(t)).
\end{aligned} \tag{15}$$

Similarly, we have:

$$\begin{aligned}
&Var(S(t+1)) \\
=&E[Var(S(t+1)|Q(t))] + Var(E[S(t+1)|Q(t)]) \\
=&E\left[\sum_{j=1}^{N(t)} s_j(t+1)^2 q_j(t)(1-q_j(t))\right] + Var\left(\sum_{j=1}^{N(t)} s_j(t+1)q_j(t)\right) \\
=&\sum_{j=1}^{N(t)} s_j(t+1)^2 E[q_j(t)(1-q_j(t))] + \sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)Cov(q_j(t), q_k(t)).
\end{aligned} \tag{16}$$

Substituting Equations (15) and (16) into Equation (10) completes the proof. □

The regression risk-sharing rule with stochastic mortality rates in Equation (14) is referred to as the stochastic regression rule in this paper.

3.3 Alive-only Rule

The alive-only rule is a risk-sharing rule proposed in Fullmer and Sabin (2018) that only distributes the total mortality credits to members alive at the end of each period.

Definition 5 (Alive-only Rule)**.** Consider a risk-sharing rule that only the members alive at the end of the period share the total mortality credits as the following:

$$V_i(t+1) = \begin{cases} s_i(t+1) + \frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)} S(t+1) & \text{if individual } i \text{ survives this period,} \\ 0 & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{17}$$

where $r_i(t) = \frac{q_i(t)}{1-q_i(t)}$, and the major difference is that the dead individual loses everything. The risk-sharing rule in Equation (17) is called the alive-only rule.

The weight $\frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)}$ is not predetermined, but it depends on the realised survivorship at the end of the period. As the name implies, only members alive receive part of the total mortality credits to compensate, while members who die lose everything.

Proposition 6. *The alive-only risk-sharing rule in Equation (17) is self-sustaining and almost fair, but not exactly fair.*

Proof. The expected value of $V_i(t+1)$ is:

$$\begin{aligned}
\mathrm{E}\left[V_i(t+1)\right] =& E[E[V_i(t+1)|\mathcal{F}_i(t+1)]] \\
=& P(i \in A(t+1))E[V_i(t+1)|i \in A(t+1)] + P(i \notin A(t+1))E[V_i(t+1)|i \notin A(t+1)] \\
=& \left(1 - q_i(t)\right)\left[s_i(t+1) + E\left[\frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)} S(t+1)|i \in A(t+1)\right]\right],
\end{aligned}$$

where $\mathcal{F}_i(t+1)$ is the filtration representing the survival status of individual i up to time $t+1$. Fullmer and Sabin (2018) mention that this is not strictly fair. This is because of the approximation of $E\left[\frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)} S(t+1)|i \in A(t+1)\right] = s_i(t+1)r_i(t)$, which gives:

$$\begin{aligned}
\mathrm{E}\left[V_i(t+1)\right] &= \left(1 - q_i(t)\right) s_i(t+1) + (1 - q_i(t))s_i(t+1)r_i(t) \\
&= \left(1 - q_i(t)\right) s_i(t+1) + (1 - q_i(t))s_i(t+1)\frac{q_i(t)}{1 - q_i(t)} \\
&= s_i(t+1).
\end{aligned}$$

This is an approximation because:

$$\begin{aligned}
E\left[\frac{1}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)}|i \in A(t+1)\right] =& E\left[\frac{1}{\sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1)r_j(t)}|i \in A(t+1)\right] \\
\neq& \frac{1}{\sum_{j=1}^{N(t)} (1 - q_j(t))s_j(t+1)r_j(t)}
\end{aligned}$$

since we cannot move expectation into the summation in the denominator.

Due to the approximation, the alive-only rule in Equation (17) is an almost fair risk-sharing rule. However, the alive-only rule in Equation (17) is still self-sustaining because:

$$\begin{aligned}
\sum_{j=1}^{N(t)} V_i(t+1) &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} \frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)} S(t+1) \\
&= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1) \\
&= \sum_{j=1}^{N(t)} s_j(t+1).
\end{aligned}$$

$\square$

Proposition 7. *The risk-sharing rule below in Equation (18) is a self-sustaining and almost fair alive-only rule under stochastic mortality rates:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + w_i^{\text{Alive}}(t+1)S(t+1) & \text{if individual } i \text{ survives this period,} \\ 0 & \text{if individual } i \text{ dies during this period,} \end{cases} \tag{18}$$

where $w_i^{\text{Alive}}(t+1) = \frac{s_i(t+1)\frac{E[q_i(t)]}{1-E[q_i(t)]}}{\sum_{j\in A(t+1)} s_j(t+1)\frac{E[q_j(t)]}{1-E[q_j(t)]}}$ stands for the weighting for individual i at time $t+1$ with the alive-only rule under stochastic mortality rates.

Proof. By the law of total expectation, we have:

$$\begin{aligned}
E[V_i(t+1)] =& E\left[E\left[E\left[V_i(t+1)|Q(t)\right]|\mathcal{F}_i(t+1)\right]\right] \\
=& E\left[(1-q_i(t))(s_i(t+1) + \frac{s_i(t+1)\frac{q_i(t)}{1-q_i(t)}}{\sum_{j\in A(t+1)} s_j(t+1)\frac{q_j(t)}{1-q_j(t)}}S(t+1))|i \in A(t+1)\right] \\
=& s_i(t+1)(1-E[q_i(t)]) + E\left[\frac{s_i(t+1)q_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)\frac{q_j(t)}{1-q_j(t)}}S(t+1)|i \in A(t+1)\right].
\end{aligned}$$

When we make the approximation $E\left[\frac{s_i(t+1)q_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)\frac{q_j(t)}{1-q_j(t)}}S(t+1)|i \in A(t+1)\right] = s_i(t+1)E[q_i(t)]$, it gives us:

$$E[V_i(t+1)] \approx s_i(t+1),$$

which means the risk-sharing rule in Equation (18) is almost fair. The risk-sharing rule in Equation (18) is self-sustaining because:

$$\begin{aligned}
\sum_{j=1}^{N(t)} V_j(t+1) =& \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_i(t+1) + \sum_{j\in A(t+1)} \frac{s_i(t+1)\frac{E[q_i(t)]}{1-E[q_i(t)]}}{\sum_{j\in A(t+1)} s_j(t+1)\frac{E[q_j(t)]}{1-E[q_j(t)]}}S(t+1) \\
=& \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_i(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_i(t+1) \\
=& \sum_{j=1}^{N(t)} s_j(t+1).
\end{aligned} \tag{19}$$

□

3.4 A New Risk-sharing Rule: Joint Expectation Rule

In this subsection, a new fair and self-sustaining risk-sharing rule is proposed. The weight in the total mortality credits of an individual is higher with a higher fund balance, the mean of the mortality rate, the variance of the mortality rate, and the covariance between the mortality rates of other fund members are higher. The risk-sharing rule is named the joint expectation (JE) rule, and it will reduce to the proportional rule when the mortality rates are no longer stochastic and correlated random variables but deterministic values.

Proposition 8. *The following risk-sharing rule is a fair and self-sustaining rule incorporating correlations between stochastic mortality rates:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + s_i(t+1)E[q_i(t)] & \text{if individual } i \\ \ + w_i^{\text{JE}}(t+1)(S(t+1) - E[S(t+1)]) & \text{survives this period,} \\ & \\ s_i(t+1)E[q_i(t)] & \text{if individual } i \text{ dies} \\ \ + w_i^{\text{JE}}(t+1)(S(t+1) - E[S(t+1)]) & \text{during this period,} \end{cases} \tag{20}$$

where $w_i^{\text{JE}}(t+1) = \frac{s_i(t+1)\sum_{k=1}^{N(t)} s_k(t+1)E[q_i(t)q_k(t)]}{\sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)E[q_j(t)q_k(t)]}$, and $S(t+1) = \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1)$.

Proof. The risk-sharing rule in Equation (20) is fair because:

$$\begin{aligned} E[V_i(t+1)] =& E[E[V_i(t+1)|Q(t)]] \\ =& s_i(t+1)(1-E[q_i(t)]) + s_i(t+1)E[q_i(t)] + w_i(t+1)(E[S(t+1)] - E[S(t+1)]) \\ =& s_i(t+1). \end{aligned}$$

The risk-sharing rule in Equation (20) is self-sustaining because:

$$\begin{aligned} \sum_{j=1}^{N(t)} V_j(t+1) &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)] + S(t+1) - E[S(t+1)] \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1) \\ &= \sum_{j=1}^{N(t)} s_j(t+1). \end{aligned}$$

□

We name the risk-sharing rule in Equation (20) as the joint expectation rule because the term $E[q_i(t)q_k(t)]$ in the numerator is the joint expectation of the one-year mortality rates $q_i(t)$ and $q_k(t)$ of individual i and individual k at time t.

The joint expectation $E[q_i(t)q_k(t)]$ in the weight captures not only the expected values of mortality rates but also the volatility of mortality rates and the correlation between mortality rates of different individuals because:

$$\begin{aligned} E[q_i(t)q_k(t)] =& Cov(q_i(t), q_k(t)) + E[q_i(t)]E[q_k(t)] \\ =& \rho(q_i(t), q_k(t))\sigma(q_i(t))\sigma(q_k(t)) + E[q_i(t)]E[q_k(t)], \end{aligned} \tag{21}$$

where $Cov(q_i(t), q_k(t))$ is the covariance between the one-year mortality rates $q_i(t)$ and $q_k(t)$ of individuals i and k at time t, $\rho(q_i(t), q_k(t))$ is the correlation between $q_i(t)$ and $q_k(t)$, and $\sigma(q_i(t))$ is the standard deviation of $q_i(t)$.

Lemma 2. *When the mortality rates are deterministic (that is $E[(q_j(t))^2] = E[q_j(t)]^2$ so $Var(q_j(t)) = E[(q_j(t))^2] - E[q_j(t)]^2 = 0$) and the mortality rates for different cohorts are independent (that is $E[q_j(t)q_k(t)] = E[q_j(t)]E[q_k(t)]$), then the risk-sharing rule proposed in Equation (20) reduces to the fair proportional rule in Equation (9):*

$$V_i(t+1) = \begin{cases} s_i(t+1) + \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]} S(t+1) & \text{if individual } i \text{ survives this period,} \\ \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]} S(t+1) & \text{if individual } i \text{ dies during this period.} \end{cases}$$

Proof. When $E[(q_j(t))^2] = E[q_j(t)]^2$ and $E[q_j(t)q_k(t)] = E[q_j(t)]E[q_k(t)]$, the weighting becomes:

$$\begin{aligned} w_i^{\text{JE}}(t+1) =& \frac{s_i(t+1)\sum_{k=1}^{N(t)} s_k(t+1)E[q_i(t)q_k(t)]}{\sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)E[q_j(t)q_k(t)]} \\ =& \frac{s_i(t+1)\sum_{k=1}^{N(t)} s_k(t+1)E[q_i(t)]E[q_k(t)]}{\sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)E[q_j(t)]E[q_k(t)]} \\ =& \frac{s_i(t+1)E[q_i(t)]\sum_{k=1}^{N(t)} s_k(t+1)E[q_k(t)]}{(\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)])(\sum_{k=1}^{N(t)} s_k(t+1)E[q_k(t)])} \\ =& \frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_j(t)]}. \end{aligned}$$

□

The proposed risk-sharing rule can be extended to include death benefit so that when a member dies they do not lose all of the accumulated fund balance but get $d_i(t+1)$ as the death benefit in the case of death. The capital at risk thus becomes $s_i(t+1) - d_i(t+1)$.

Proposition 9. *With death benefit protection included, the risk-sharing rule in Equation (20) can be extended to:*

$$V_i(t+1) = \begin{cases} s_i(t+1) + (s_i(t+1) - d_i(t+1))E[q_i(t)] & \text{if individual } i \\ \quad + w_i(t+1)(S(t+1) - E[S(t+1)]) & \text{survives this period,} \\ & \\ d_i(t+1) + (s_i(t+1) - d_i(t+1))E[q_i(t)] & \text{if individual } i \text{ dies} \\ \quad + w_i(t+1)(S(t+1) - E[S(t+1)]) & \text{during this period,} \end{cases} \tag{22}$$

where $w_i^{\text{JE}}(t+1) = \frac{(s_i(t+1)-d_i(t+1))\sum_{k=1}^{N(t)}(s_k(t+1)-d_k)E[q_i(t)q_k(t)]}{\sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)}(s_j(t+1)-d_j(t+1))(s_k(t+1)-d_k)E[q_j(t)q_k(t)]}$, and $S(t+1) = \sum_{j=1}^{N(t)} 1_{j\in D(t+1)}(s_j(t+1) - d_j(t+1))$. In this case, the risk-sharing rule is still fair and self-sustaining.

Proof. The risk-sharing rule in Equation (22) is fair because:

$$\begin{aligned} E[V_i(t+1)] = E[E[V_i(t+1)|Q(t)]] =& s_i(t+1)(1 - E[q_i(t)]) + d_i(t+1)E[q_i(t)] \\ &+ (s_i(t+1) - d_i(t+1))E[q_i(t)] \\ &+ w_i(t+1)(E[S(t+1)] - E[S(t+1)]) \\ =& s_i(t+1). \end{aligned}$$

The risk-sharing rule in Equation (22) is self-sustaining because:

$$\begin{aligned}\sum_{j=1}^{N(t)} V_j(t+1) &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} d_j(t+1) + \sum_{j=1}^{N(t)} (s_j(t+1) - d_j(t+1))E[q_j(t)] \\ &\quad + S(t+1) - E[S(t+1)] \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} d_j(t+1) + S(t+1) \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} d_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} (s_j(t+1) - d_j(t+1)) \\ &= \sum_{j=1}^{N(t)} 1_{j\in A(t+1)} s_j(t+1) + \sum_{j=1}^{N(t)} 1_{j\in D(t+1)} s_j(t+1) \\ &= \sum_{j=1}^{N(t)} s_j(t+1).\end{aligned}$$

□

3.5 Summary of Risk-sharing Rules

We summarise the properties of different risk-sharing rules in Tables 1 and 2. Table 1 compares the weightings in the total mortality credits $S(t+1)$ between different deterministic and stochastic risk-sharing rules. We can see that when extended to stochastic risk-sharing rules, only the joint expectation rule and the stochastic regression rules take the correlation between mortality rates into consideration. Table 2 further compares how the increment in one statistic (mean, variance, or correlation of mortality

Table 1: Comparison of weighting in $S(t+1)$ of member i between deterministic and stochastic versions of risk-sharing rules.

	Deterministic	Stochastic
Proportional	$\frac{s_i(t+1)q_i(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j(t)}$	$\frac{s_i(t+1)E[q_i(t)]}{\sum_{j=1}^{N(t)} s_j(t+1)E[q_i(t)]}$
Joint Expectation	Reduces to the proportional rule	$\frac{s_i(t+1)\sum_{k=1}^{N(t)} s_k(t+1)E[q_i(t)q_k(t)]}{\sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)E[q_j(t)q_k(t)]}$
Regression	$\frac{s_i(t+1)^2 q_i(t)(1-q_i(t))}{\sum_{j=1}^{N(t)} s_j(t+1)^2 q_j(t)(1-q_j(t))}$	$\frac{s_i(t+1)^2 E\left[q_i(t)(1-q_i(t))\right] + s_i(t+1)\sum_{j=1}^{N(t)} s_j(t+1)Cov(q_i(t),q_j(t))}{\sum_{j=1}^{N(t)} s_j(t+1)^2 E[q_j(t)(1-q_j(t))] + \sum_{j=1}^{N(t)}\sum_{k=1}^{N(t)} s_j(t+1)s_k(t+1)Cov(q_j(t),q_k(t))}$
Alive-only	$\frac{s_i(t+1)r_i(t)}{\sum_{j\in A(t+1)} s_j(t+1)r_j(t)} 1_{i\in A(t+1)}$	$\frac{s_i(t+1)\frac{E[q_i(t)]}{1-E[q_i(t)]}}{\sum_{j\in A(t+1)} s_j(t+1)\frac{E[q_j(t)]}{1-E[q_j(t)]}} 1_{i\in A(t+1)}$

rates) affects the weighting in the total mortality credits between different risk-sharing rules, holding the other two statistics the same. We can see that the stochastic regression risk-sharing rule gives higher weight when the variance and correlation of mortality rates are higher, while the effect of the mean is not monotonic. The proposed joint expectation risk-sharing rule will distribute a higher proportion of total mortality when the mean, variance, or correlation of mortality rates is higher.

4 Numerical Analysis

This section outlines the methodology used in this research, including data and assumptions on mortality rates, the definition of the rate of return on mortality credits, and an overview of experiments to

Table 2: Change in weighting $w_i(t+1)$ when mean, variance of mortality rates, or correlation to mortality rates of other cohorts increase for different risk-sharing rules.

	Mean	Variance	Correlation
Proportional Det	Increase	No Change	No Change
Regression Det	Not Monotonic	No Change	No Change
Alive-only Det	Increase	No Change	No Change
Proportional Sto	Increase	No Change	No Change
Regression Sto	Not Monotonic	Increase	Increase
Alive-only Sto	Increase	No Change	No Change
Joint Expectation	Increase	Increase	Increase

be conducted.

4.1 Data and Assumptions

We establish a risk-sharing pool that allows mixed-age cohorts with different initial balances and new members to join. The assumptions on the risk-sharing pool are the following:

- A total of 586 members in the initial pool at time zero as presented in Table 3.
- Age range: 60 to 100.
- Age distribution refers to Australian population exposure in 2020.
- For each age, half of the members have a high balance and the other half have a low balance, where the high balance is 1.5 times the low balance.
- The balance decreases with age to reflect the consumption of retirement balance.
- New members joining every year have the same size of 586 members and the same age and balance distributions as in Table 3.

The pool size of 586 is chosen to demonstrate the pooling effect while maintaining computation efficiency, as we study 50 times the pool size and perform 30 years of analysis with new members of the same size joining each year.

The assumptions on mortality rates are:

- Mortality rates follow a multivariate log-normal distribution since they are non-negative.
- The covariance matrix of the multi-variate log-normal distribution is estimated by using an 11-year bracket for each cohort at every point in time and calculating the covariance.

Examples of the means, standard deviations, and correlation matrix of mortality rates for a selection of cohorts at ages $60, 70, 80, 90, 100$ in the calendar year 2020 are displayed in Table 4 and Table 5. We can see that the means of mortality rates increase with age. The standard deviations of mortality rates also increase with age, except for age 100. From Table 5, we can see that the correlation is generally higher when the age difference is lower. Age 100 is also an exception in this case because an improvement in the mortality at younger ages, for example, age 90 often leads to a worsening mortality at age 100 because people will die in the end.

Table 3: Assumptions on the pool and members.

Age	Size	Balance High	Balance Low
60	30	720,000	480,000
61	30	708,000	472,000
62	30	696,000	464,000
63	28	684,000	456,000
64	28	672,000	448,000
65	26	660,000	440,000
66	26	648,000	432,000
67	26	636,000	424,000
68	26	624,000	416,000
69	24	612,000	408,000
70	24	600,000	400,000
71	24	588,000	392,000
72	24	576,000	384,000
73	22	564,000	376,000
74	20	552,000	368,000
75	18	540,000	360,000
76	18	528,000	352,000
77	16	516,000	344,000
78	14	504,000	336,000
79	14	492,000	328,000
80	12	480,000	320,000
81	12	468,000	312,000
82	12	456,000	304,000
83	10	444,000	296,000
84	10	432,000	288,000
85	8	420,000	280,000
86	8	408,000	272,000
87	6	396,000	264,000
88	6	384,000	256,000
89	6	372,000	248,000
90	4	360,000	240,000
91	4	348,000	232,000
92	4	336,000	224,000
93	2	324,000	216,000
94	2	312,000	208,000
95	2	300,000	200,000
96	2	288,000	192,000
97	2	276,000	184,000
98	2	264,000	176,000
99	2	252,000	168,000
100	2	240,000	160,000
Total	586	NA	NA

Table 4: Mean and standard deviation of mortality rates at different ages in 2020.

q_x	60	70	80	90	100
μ	0.00638	0.01370	0.04159	0.14561	0.38578
σ	0.00017	0.00076	0.00307	0.00525	0.00277

Table 5: Correlation of mortality rates at different ages in 2020.

corr	60	70	80	90	100
60	1	0.88435	0.81642	0.84444	-0.51903
70	0.88435	1	0.87124	0.92396	-0.43913
80	0.81642	0.87124	1	0.96875	-0.67331
90	0.84444	0.92396	0.96875	1	-0.66694
100	-0.51903	-0.43913	-0.67331	-0.66694	1

4.2 Overview of Experiments

We compare risk-sharing rules by the rate of return from the distribution of mortality credits in the case of survival: $ROR_i^{mc}(t) = (V_i(t+1) - s_i(t+1))/F_i(t)$. We do not directly compare the weightings because they are heavily affected by the pool size and the fund value. The $ROR_i^{mc}(t)$ for different risk-sharing rules in the case of survival are shown below:

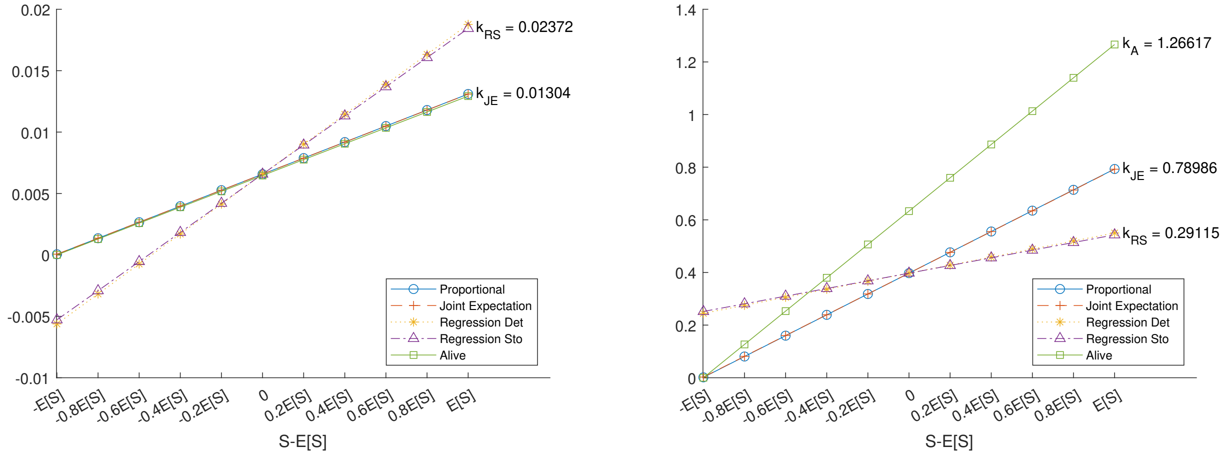
- Proportional: $\frac{s_i(t+1)E[q_i(t)]+w_i^{\text{Proportional}}(t+1)(S(t+1)-E[S(t+1)])}{F_i(t)}$
- Joint Expectation: $\frac{s_i(t+1)E[q_i(t)]+w_i^{\text{JE}}(t+1)(S(t+1)-E[S(t+1)])}{F_i(t)}$
- Regression Det: $\frac{s_i(t+1)q_i(t)+w_i^{\text{RD}}(t+1)(S(t+1)-E[S(t+1)])}{F_i(t)}$
- Regression Sto: $\frac{s_i(t+1)E[q_i(t)]+w_i^{\text{RS}}(t+1)(S(t+1)-E[S(t+1)])}{F_i(t)}$
- Alive: $\frac{w_i^{\text{Alive}}(t+1)S(t+1)}{F_i(t)}$

We omit the deterministic proportional and alive-only rules because we assume that the mortality rates $q_i(t)$ used in the deterministic case are equal to $E[q_i(t)]$ used in the stochastic case. We can see that the major difference in risk-sharing rules is their weighting on the difference between the empirical and expected mortality credits $S(t+1) - E[S(t+1)]$, except for the alive-only rule which is on $S(t+1)$. This means different risk-sharing rules have different sensitivities when there is a deviation from the expected mortality credits. We measure this difference in the sensitivity by plotting $ROR^{mc}(t)$ versus the deviation in total mortality credits $S(t+1) - E[S(t+1)]$ for different (1) risk-sharing rules, (2) ages and thus mortality rates of members, (3) fund balances of members, and (4) pool sizes.

Then, we will study the performance of the fund over time with new members joining. We will assess benefit payments of different cohorts for the next 30 years since the initial establishment of the pool assuming: (1) No systematic mortality risk, and (2) 20% reduction of mortality rates for the first 5 years.

4.3 Comparison of Rates of Return from Mortality Credits against Deviation in Total Mortality Credits

We first compare the rates of return from mortality credits $ROR^{mc}(t)$ of different risk-sharing rules against the deviation in total mortality credits for time $t = 0$. As illustrated in Figure 1, $S(t+1) - E[S(t+1)] = 0$ implies zero deviation in mortality credits and this is the point of the expected rate of return from mortality credits $E[ROR^{mc}(t)]$ where all risk-sharing rules except for the alive-only rule pay the same.



(a) Age 60 with high balance $= 720,000$.

(b) Age 100 with high balance $= 240,000$.

Figure 1: Comparison of $ROR^{mc}(t)$ between risk-sharing rules.

Figure 1(a) shows the comparison of risk-sharing rules for the cohort aged 60 with a high balance $720,000$ in year 1. The slope of the plot implies the sensitivity of $ROR^{mc}(t)$ to the deviation in $S(t+1)$. A higher slope implies a higher sensitivity, and the value of the slope is shown in the figure as k, calculated as the vertical change in $ROR^{mc}(t)$ between the two endpoints. This adjustment controls variations in the scale of $S(t+1) - E[S(t+1)]$ under different settings in this paper. We can see from Figure 1(a) that the deterministic regression rule and the stochastic regression rule give higher slopes at age 60 than the other three risk-sharing rules, while the deterministic regression rule has a slightly higher slope than the stochastic regression rule. The differences between the regression rules and other risk-sharing rules are due to the different weightings in mortality credits, as illustrated in Table 1. The weighting of the regression rules have the quadratic of the accumulated fund balance in the numerator, which results in higher slopes than the other risk-sharing rules for cohorts with high fund balances in the pool. In addition to the quadratic accumulated fund balance, this can also be explained by the $E\left[q_i(t)(1-q_i(t))\right]$ and $q_i(t)(1-q_i(t))$ terms in the numerators of regression rules, whose relative rates to $E[q_i(t)]$ and $q_i(t)$ decrease as age increases and thus expected mortality rate increases. Meanwhile, the proportional rule, joint expectation rule, and alive-only rule have minor differences at age 60.

Figure 1(b) shows the comparison at age 100 with high balance $240,000$. In contrast to age 60, the alive-only rule gives a higher slope than the other four risk-sharing rules with a significant difference. This is because the $\frac{E[q_i(t)]}{1-E[q_i(t)]}$ term for the alive-only rule in Table 1 is much higher than the expected mortality rates $E[q_i(t)]$ at older ages, since the $(1-E[q_i(t)])$ term is much smaller than 1 at older ages with higher expected mortality rates. Meanwhile, the $\frac{E[q_i(t)]}{1-E[q_i(t)]}$ term at younger ages is close to $E[q_i(t)]$ because their expected mortality rates are relatively lower, which explains the minor difference between the alive-only rule with the joint expectation rule and proportional rule at younger ages. The deterministic and stochastic regression rules now have lower slopes than proportional and joint expectation rules. The expected rate of return from mortality credits at age 100 is much higher compared with age 60 because of the much higher expected mortality rate at age 100.

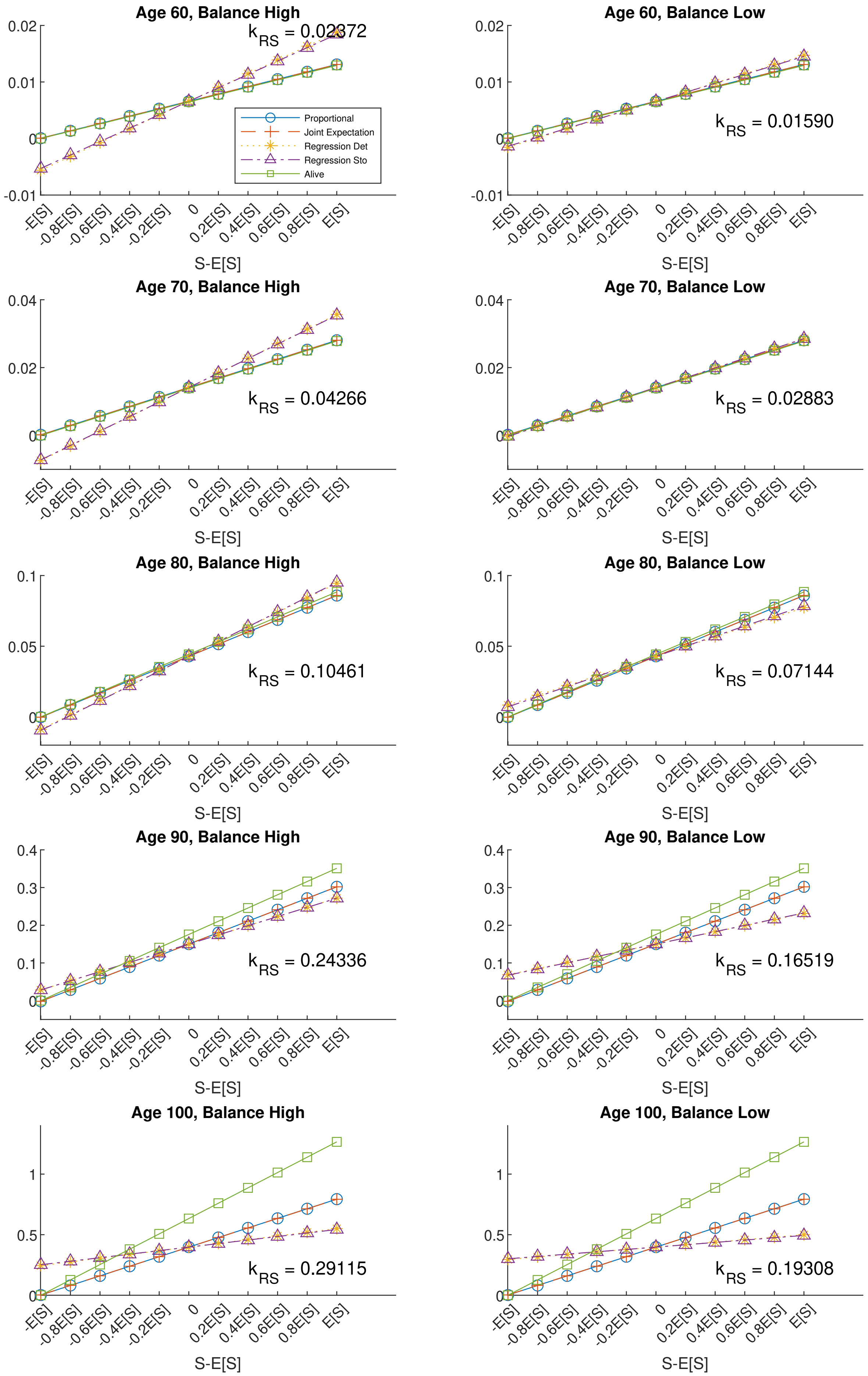


Figure 2: Comparison of $ROR^{mc}(t)$ with different ages, balances, and rules.

Figure 2 compares 5 cohorts aged 60, 70, 80, 90, 100, and with high or low balance. We can see that as age increases, the expected mortality rate $E[q_i(t)]$ increases, which leads to higher $E[ROR^{mc}(t)]$ at the point of $S(t+1) - E[S(t+1)] = 0$. Meanwhile, as age increases, the slopes of the regression rules increase, but not as fast as the proportional rule and joint expectation rule, leading them to be relatively flatter at older ages. This is due to the quadratic accumulated fund balance and the $E\left[q_i(t)(1-q_i(t))\right]$ term in the regression rules explained earlier. The slope of the alive-only rule increases at the fastest rate with age, and it starts to dominate the other rules for old cohorts. We also find that the slopes of the regression rules are higher with high balance, keeping age the same.

4.4 Effect of Balance

To further study the effect of balance on the slope, we divide the slope of high balance by the slope of low balance at every age and present the results in Table 6. From Table 6, we find that the ratios for the proportional rule, joint expectation rule, and alive-only rule are 1, indicating that balance does not affect the sensitivity to deviation in total mortality credits. However, we can see from Table 6 that the ratio for the deterministic regression rule is 1.5, which is exactly the ratio of high balance over low balance, indicating that the slope increases proportionally to balance for the deterministic regression rule. Moreover, for the stochastic regression rule, it is above 1 but not equal to 1.5 exactly, indicating that the slope still increases with balance, but the ratio is affected also by the mean, variance, and correlation of mortality rates of all fund members.

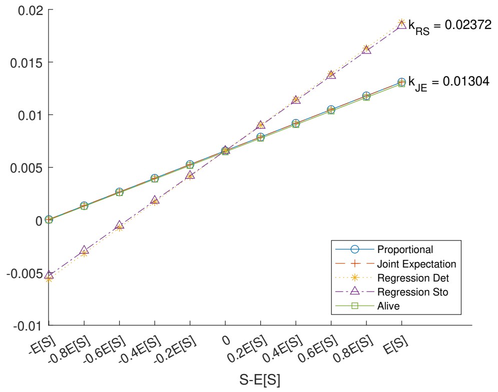
Table 6: Slope high balance over slope low balance at different ages.

	Risk-sharing Rules				
Age	Proportional	Joint Expectation	Regression Det	Regression Sto	Alive
60	1	1	1.5	1.49239	1
70	1	1	1.5	1.47962	1
80	1	1	1.5	1.46428	1
90	1	1	1.5	1.47324	1
100	1	1	1.5	1.50793	1

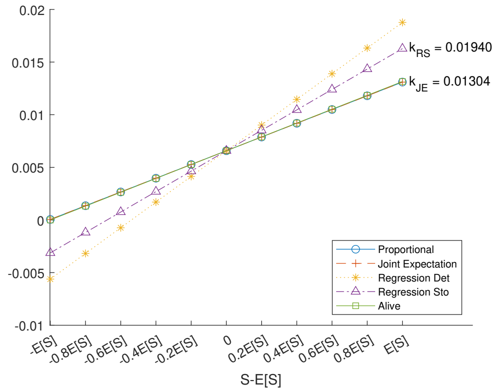
4.5 Effect of Pool Size

Figures 3 to 5 show the slopes of different risk-sharing rules for different pool sizes and for ages 60, 80, and 100 respectively. The expected rate of return from mortality credits $E[ROR^{mc}(t)]$ at the point of $S(t+1) - E[S(t+1)] = 0$ does not change with the pool size. We observe from Figure 3 that the slopes of the proportional rule, joint expectation rule, alive-only rule, and deterministic regression rule are relatively stable when we increase pool size to 10 times and 50 times the original size. However, we find that the slope for the stochastic regression rule decreases for member age 60. The slope values of the stochastic regression rule and the joint expectation rule are shown in Table 7.

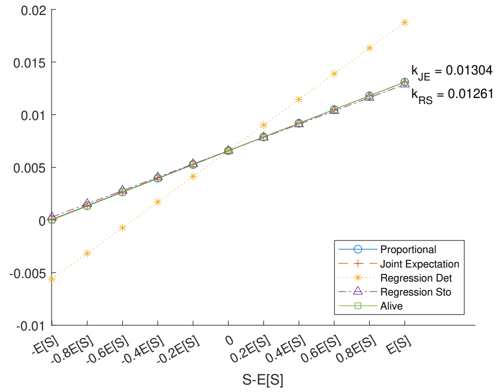
From Figure 4 and Table 7, we can see that the slope of the stochastic regression rule increases with pool size at age 80. Meanwhile, from Figure 5 and Table 7, a decrease in the slope with pool size is observed at age 100. The decrease in slope at ages 60 and 100 can be explained by the low volatility in the mortality rates we assume, and being correlated to fewer members in the pool. Especially at age 100, the mortality rates are negatively correlated with the mortality rates of most members at other ages. Meanwhile, the slopes of the joint expectation rule remain the same when the pool size increases. The comparisons of all cohorts with 10 and 50 times the original pool size are displayed in Figures 9 and 10 respectively in Appendix 1.



(a) Original size.

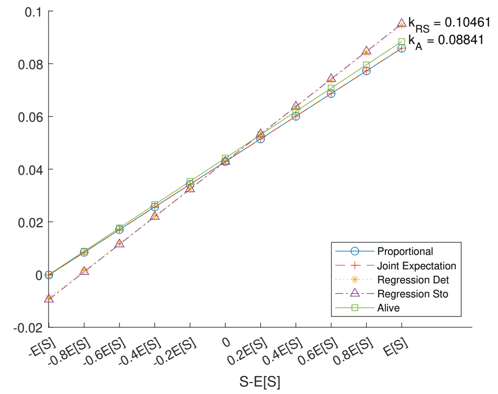


(b) 10 × original size.

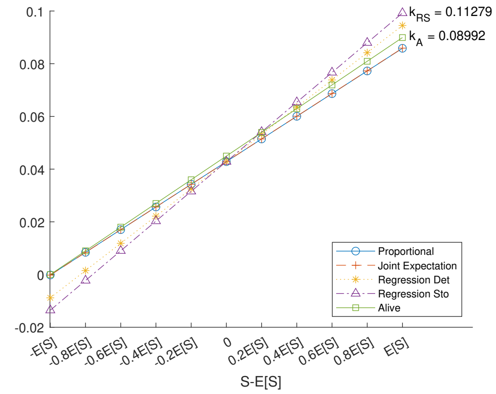


(c) 50 × original size.

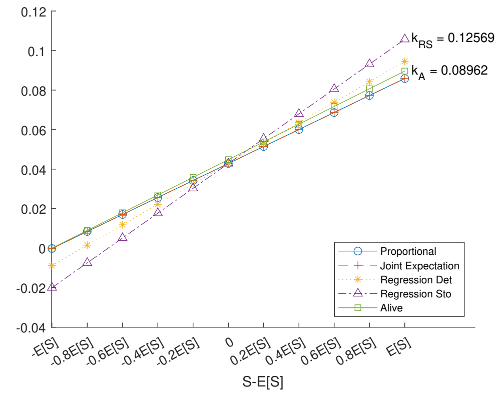
Figure 3: Comparison of $ROR^{mc}(t)$ with increasing pool size: Age 60 with high balance $= 720,000$.



(a) Original size.

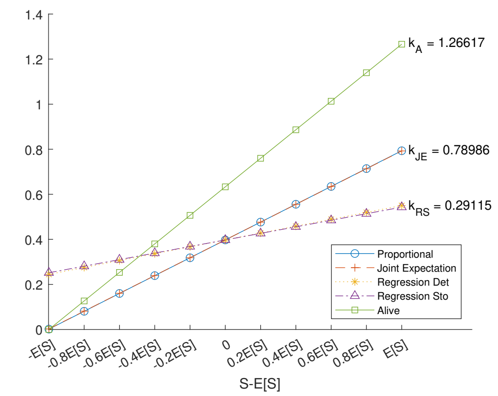


(b) 10 × original size.

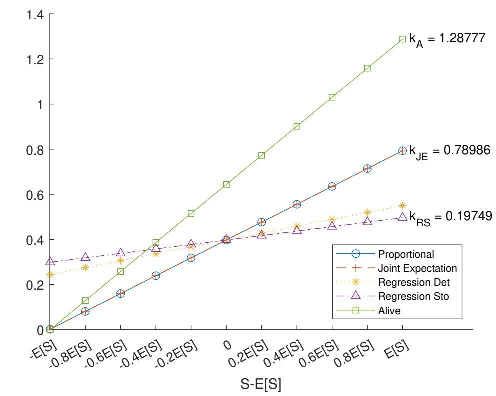


(c) 50 × original size.

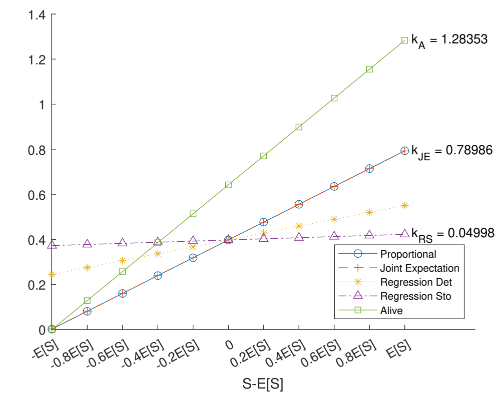
Figure 4: Comparison of $ROR^{mc}(t)$ with increasing pool size: Age 80 with high balance $= 480,000$.



(a) Original size.



(b) 10 × original size.



(c) 50 × original size.

Figure 5: Comparison of $ROR^{mc}(t)$ with increasing pool size: Age 100 with high balance $= 240,000$.

Table 7: Slopes of $ROR^{mc}(t)$ for stochastic regression rule and joint expectation rule and for high-balance individuals at ages 60, 80, and 100 with different pool sizes.

	Origianl Size	10× size	50× size
k_{RS}^{60}	0.02372	0.01940	0.01261
k_{RS}^{80}	0.10461	0.11279	0.12569
k_{RS}^{100}	0.29115	0.19749	0.04998
k_{JE}^{60}	0.01304	0.01304	0.01304
k_{JE}^{80}	0.08614	0.08614	0.08614
k_{JE}^{100}	0.78986	0.78986	0.78986

4.6 Benefit Payments over Time

The discussions so far are between time 0 and time 1. We now allow new heterogeneous members as shown in Table 3 to enter the pool at the beginning of each year. The ages of existing members increase by one every year, and their balances after benefit payments become the initial balances at the beginning of the next year. Under the dynamic setting, we track the performance of the pool over the next 30 years since the initial establishment. The means, standard deviations, and correlations of mortality rates at different ages in Table 4 and Table 5 are updated over time to reflect the time evolution of mortality rates.

Figure 6 shows the simulated income payments for different cohorts over the next 30 years. A sudden drop in income payments to zero represents the death of the member. We can see that for younger cohorts, all risk-sharing rules pay a relatively stable income over the next 30 years. The alive-only rule results in higher income payments at older ages, as illustrated in the previous subsections from their higher slope. The income for people who joined at older ages, for example, age 90 is slightly decreasing because of the very small annuity factor at that age, leading to high income payments relative to initial contributions and the balance to be consumed very quickly. However, the income keeps increasing for the alive-only rule because of the very high slope at old ages. At age 100, the income for risk-sharing rules except for the alive-only rule is decreasing fast, while it is still increasing for the alive-only rule.

Figure 7 shows the simulated income payments when there is a systematic mortality shock of 20% reduction in all cohorts for the first 5 years. We can see that for younger cohorts, their income payments are still relatively stable over their lifetime. However, for older cohorts, the payments for all risk-sharing rules experience a decrease compared with no mortality shock, mainly because the benefits from mortality credits decrease when mortality improves. We can also observe differences between different risk-sharing rules because as illustrated in previous sections, different risk-sharing rules allocate different weightings to the deviation in total mortality credits.

Figure 8 further compares the difference in fund balance in year 5 when there is a mortality shock for the first 5 years. The difference in the way of distributing this deviation leads to the difference in the fund balances, and a lower slope leads to a higher balance. We can see that there is a significant difference when we move from the proportional or joint expectation risk-sharing rule to the deterministic or stochastic regression rule, or to the alive-only rule. The two regression rules give lower fund balances at younger ages than the other three rules, but they give higher fund balances at older ages than the proportional rule and joint expectation rule.

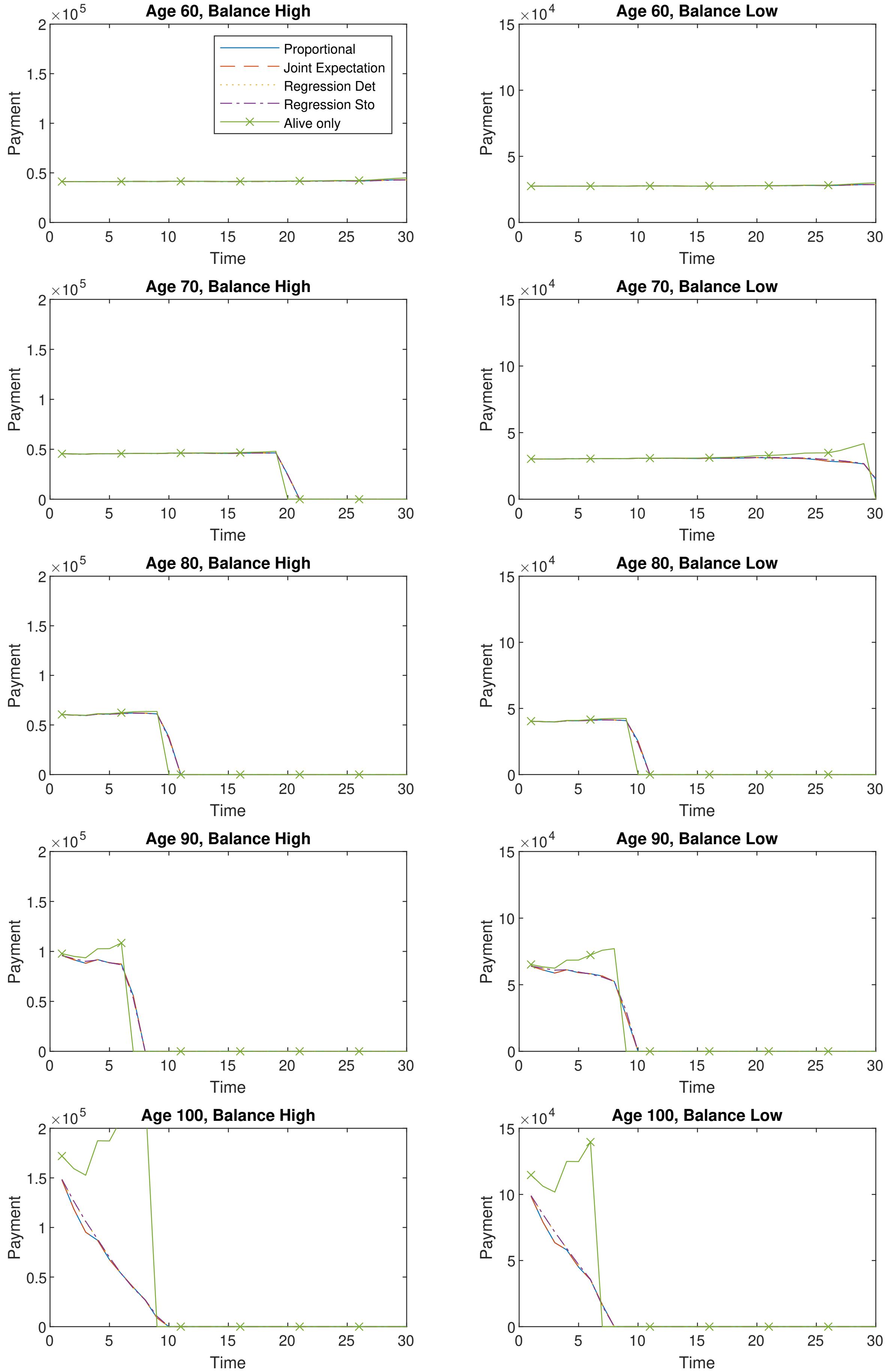


Figure 6: Benefit payments over time in dynamic pool allowing new members joining.

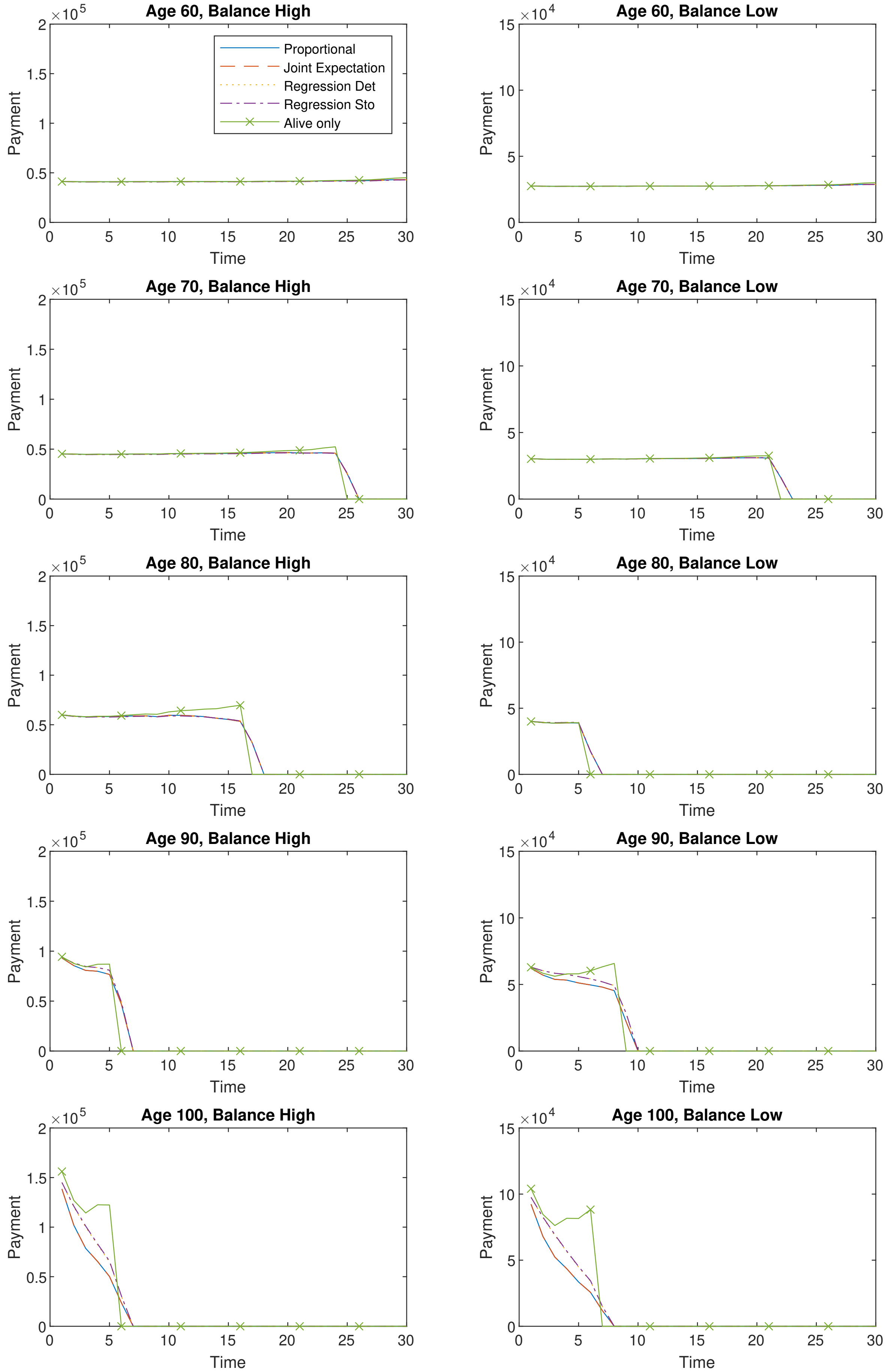


Figure 7: Benefit payments over time in dynamic pool allowing new members joining, under 5-year systematic mortality shock.

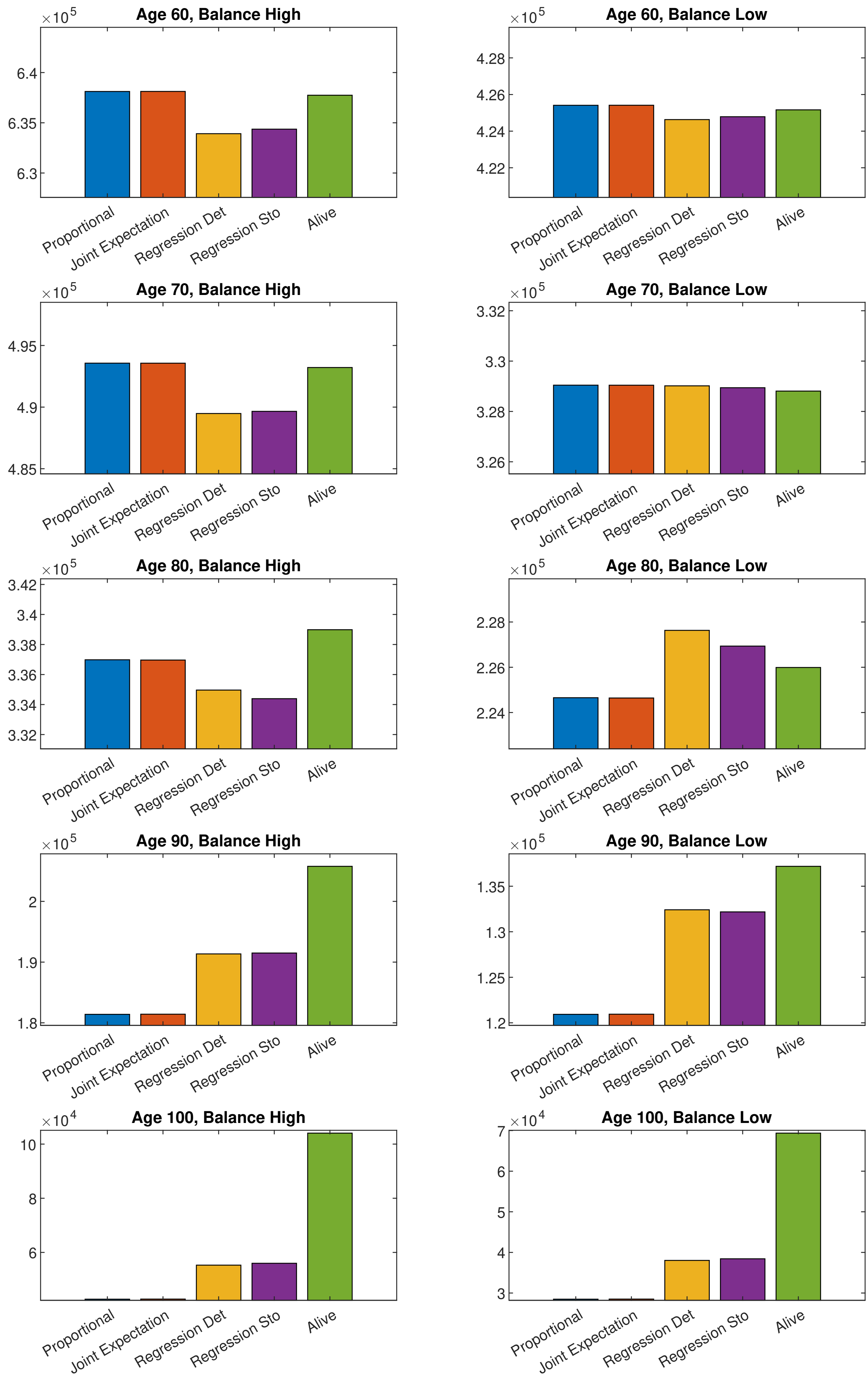


Figure 8: Balance at time 5, under 5-year systematic mortality shock.

From Figure 8, we can see that the difference in balance is several thousand dollars for ages 60, 70, and 80, and can exceed $10,000$ for ages 90 and 100 with the initial balances we set. An obvious difference is also observed between deterministic and stochastic regression rules. The stochastic regression rule gives higher balances than the deterministic regression rule at age 60, 70 with high balance, 90 with high balance, and 100. In the other cases, they will give lower balances than the deterministic regression rule. This is because when a systematic mortality shock of 20% reduction in mortality rates happens, empirical total mortality credits will tend to be lower than expected. The difference between the joint expectation rule and the proportional rule is small, because when we include variance and correlation to all cohorts, the numerator of every fund member increases, leading to a smaller change in the weighting. Therefore, there is a dilution effect when we move from the proportional rule to the joint expectation rule.

Finally, the difference between the alive-only rule to the other risk-sharing rules is also obvious. We can see from Figure 8 that the difference between the alive-only rule and the joint expectation rule ranges from a few hundred dollars at age 60, to a few thousand dollars at age 80, up until around $50,000$ dollars at age 100. We can see that a higher difference between risk-sharing rules normally happens at older ages despite the lower initial contribution at older ages. Therefore, this paper provides a more accurate calculation of risk sharing with different rules under stochastic and correlated mortality rates and can help issuers decide which risk-sharing rule to choose when they establish the product according to their needs. If the issuers do not prefer the distribution of mortality credits to be affected by individual account balances, then they should not choose stochastic or deterministic regression rules. Meanwhile, if the issuers prefer the account balance of younger retirees to be higher than older retirees when a systematic reduction in mortality rates happens, then the proportional, joint expectation, and alive-only rules are preferred. Moreover, if issuers prefer to pay more to surviving members, then they would prefer the alive-only rule. While the difference between different risk-sharing rules exists, we need to emphasise that all risk-sharing rules are still actuarially fair, or almost fair for the alive-only rule.

5 Conclusions

In conclusion, this paper studies mortality pooling products that use risk-sharing rules to distribute mortality credits first and then decumulate and calculate the benefit payments with the pooled-annuity strategy. Existing studies on the risk sharing of mortality pooling products mostly use deterministic mortality rates in the distribution of mortality credits. However, mortality rates are stochastic and correlated random variables between cohorts. This paper extends the existing risk-sharing rules to the case of stochastic mortality rates and proposes a new risk-sharing rule named the joint expectation rule which is the general form of the proportional rule when mortality rates are stochastic. The joint expectation rule with death benefit is also proposed and proven to be fair and self-sustaining.

Moreover, the risk-sharing pool in this paper contains heterogeneous members with different ages and balances. In addition, the pool is dynamic so new heterogeneous members are joining every year. The pool of mixed members is observed for 30 years since the initial establishment. The effect of age, balance, pool size, and choice of the risk-sharing rule on the distribution of mortality credits, and thus on the income payments and remaining balances of different members are studied in this paper.

Our results show that age mainly affects the distribution of mortality credits by the higher mortality rates at older ages. People at higher ages have higher mean mortality rates, and thus a larger proportion of the mortality credits, controlling balance the same. Meanwhile, with the annuity-like decumulation strategy, a larger proportion of their remaining balance is paid out every year due to the smaller annuity due factor at older ages. Therefore, this framework of risk sharing and decumulation can consistently provide stable income payments to the majority of members. Moreover, we find that the fund balance does not affect mortality risk sharing in the proportional rule, alive-only rule, and the proposed joint expectation rule. However, for the deterministic and stochastic regression rules, a higher balance results in a higher share of the total mortality credits.

Furthermore, with the assumption that the ages of people joining the pool range from 60 to 100, we study how the volatility and correlation of mortality rates and the pool size affect the distribution of total mortality credits. The results show that for the stochastic regression rule which takes into account the volatility and correlation of mortality rates, a larger pool size results in a smaller sensitivity to the deviation in total mortality credits for the younger (age 60) cohorts who have less volatile mortality rates and the older (age 100) cohorts who are less correlated with other cohorts. Meanwhile, for the middle-aged 80 cohorts who have relatively volatile mortality rates and are highly correlated with other cohorts, a larger pool size results in a higher sensitivity to the deviation in total mortality credits using the stochastic regression rule.

Finally, the effect of a mortality shock is compared between different risk-sharing rules. Our results show that under a 5-year mortality shock, the younger cohorts with deterministic and stochastic regression rules have lower fund balances, compared with the proportional, joint expectation, and alive-only rules. Meanwhile, the older cohorts have higher fund balances using the deterministic and stochastic regression rules, compared with proportional and joint expectation rules, while the alive-only rule always results in the highest account balance at older ages.

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Competing Interests There are no competing interests to declare.

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Appendix 1: Comparison of Rates of Returns from Mortality Credits for Cohorts with Different Ages and Fund Balances, and for Pools with Different Sizes

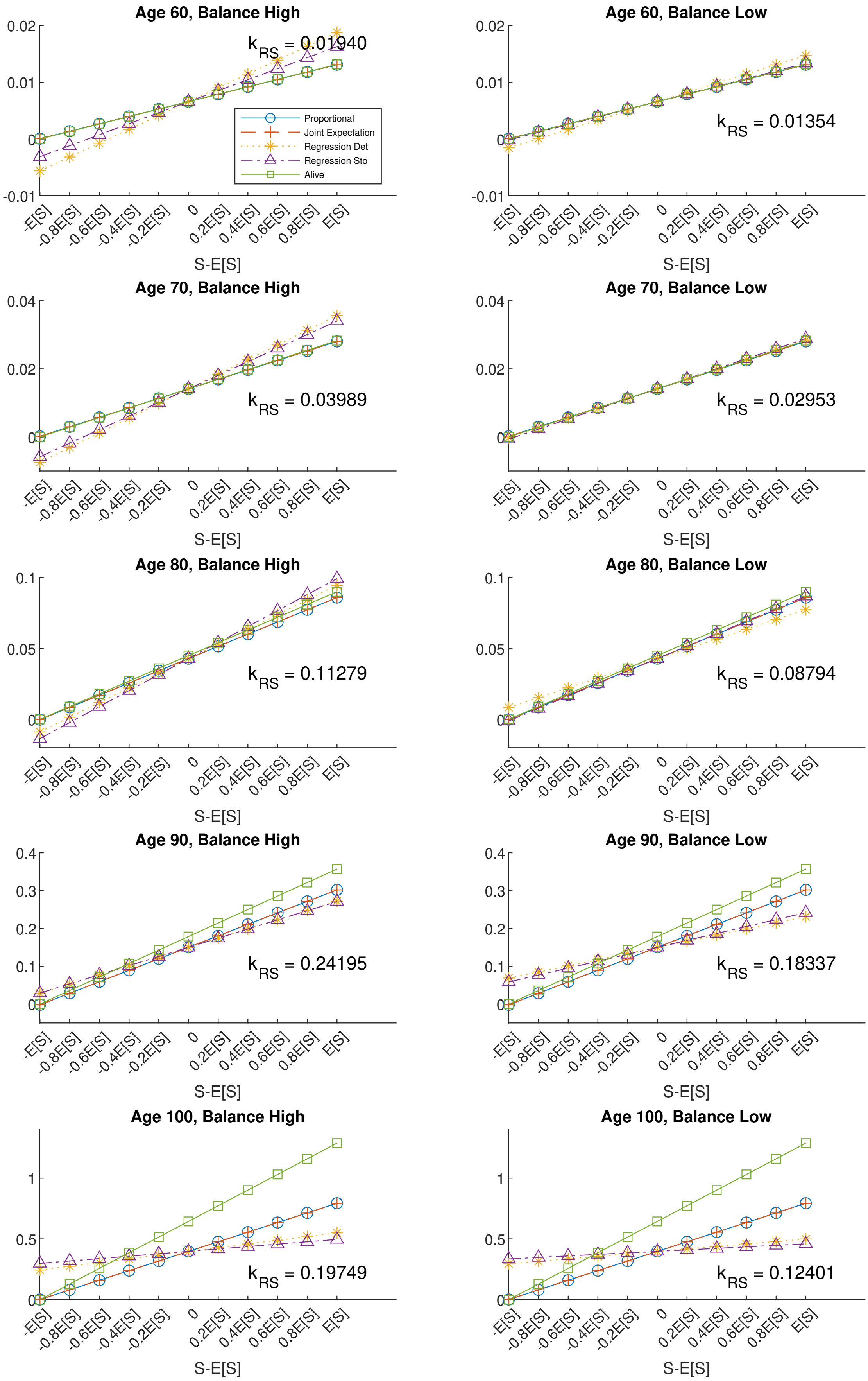

Figure 9: Comparison of $ROR^{mc}(t)$: All cohorts with 10 times the pool size.

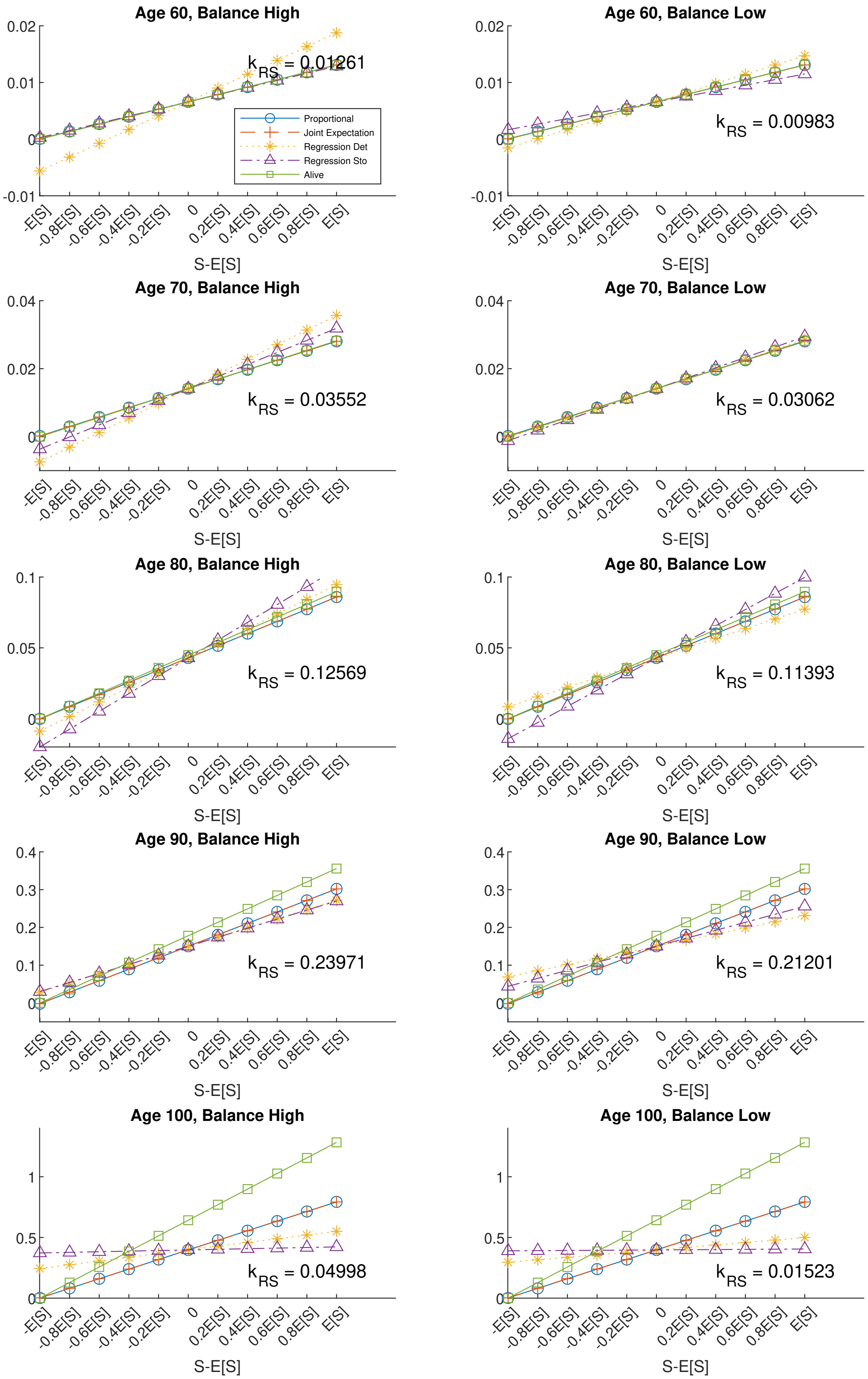

Figure 10: Comparison of $ROR^{mc}(t)$: All cohorts with 50 times the pool size.