ARC Centre of Excellence in Population Ageing Research

Working Paper 2024/16

Multi-state Health-contingent Mortality Pooling: A Heterogeneous, Actuarially Fair, and Self-sustaining Product

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16th August 2024

Abstract

There is a growing need for higher retirement incomes to cover the higher long-term care (LTC) costs when retirees become functionally disabled or ill. However, most of the existing mortality pooling products in the literature do not consider the health status of members. Hence, they do not provide higher retirement incomes to members who have LTC needs due to deteriorated health conditions. To address this issue, we propose a health-contingent mortality pooling product that is actuarially fair and self-sustaining, featuring health-state-dependent income payments. The proposed framework allows free transitions between health states so that recovery from functional disability is allowed. The framework has the flexibility to allow any number of health states, while we use a five-state model with the health states constructed from two dimensions, which are functional disability and morbidity. Moreover, the product allows heterogeneity so members can have different ages, contributions, initial health states, joining times, and rates of investment returns. Allowing heterogeneous members to join helps increase the pool size and generate more stable income payments. We find that the proposed health-contingent pooling product consistently provides significantly higher retirement incomes to members with functional disability and morbidity, while the costs to healthy members are relatively low. We also find that the jump in income payments happens immediately when there is a transition to a less healthy state, allowing members to quickly obtain higher incomes to cover the higher costs incurred by being functionally disabled or ill. Meanwhile, if the member recovers from functional disability, the income payments will decrease to reflect the reduced LTC cost.

Keywords: Mortality pooling product, Long-term care, Retirement income, Health-contingent, Multi-state health model

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1 Introduction

Long-term care (LTC) costs have been a growing concern for retirees in recent years. The expenditure of the Australian government on aged care has increased from 13.63 billion AUD (0.73% GDP) in 2012 to 28.28 billion AUD (1.19% GDP) in 2022 (Productivity Commission, 2014; Productivity Commission, 2024; Australian Bureau of Statistics, 2023). A similar pattern has been observed in other OECD countries. For example, the total LTC cost relative to GDP has increased from 1.5% in 2004 to 2.5% in 2020 in Germany, and from 1.2% in 2014 to 1.5% in 2020 in the USA (OECD, 2020; OECD, 2024). On average, 1.5% of GDP across OECD countries has been spent on LTC cost in 2018, which is equal to around 760 USD per capita (OECD, 2020).

LTC risk refers to the risk that individuals incur higher costs due to the need for care services resulting from deteriorated health conditions. Products to hedge against LTC risk include LTC insurance, LTC annuities, and the recently developed LTC pooling products. LTC insurance is a traditional insurance product which provides a benefit, reimbursement, or LTC service to the individual in the event that the individual becomes functionally disabled. Meanwhile, LTC annuities combine LTC insurance with life annuities in different ways as in Murtaugh et al. (2001), Brown and Warshawsky (2013), Pitacco (2016) and Chen et al. (2022b). The benefit of LTC annuities is the lower cost of capital and lower tendency of adverse selection due to the natural hedging of longevity risk and LTC risk (Murtaugh et al., 2001). LTC annuities are also available to people with bad health who are often rejected from the underwriting of LTC insurance (Hieber and Lucas, 2022). However, it is still challenging to determine the reserve of LTC annuities. This has led to the development of LTC pooling products, also referred to as health-contingent mortality pooling products in this paper, which do not promise lifetime incomes and have individuals in a risk pool sharing the risk with each other. Thus, LTC pooling products have zero capital requirements and very low premium loadings.

Health-contingent mortality pooling products, or LTC pooling products, are developed from mortality pooling products, which involve individuals in a risk pool sharing the mortality risk by distributing the mortality credits of members who passed away to all members or only surviving members. Mortality pooling products are effective tools for retirees to hedge their longevity risk. At the same time, they also provide retirees with retirement income streams. Mortality pooling products can be broadly categorised into pooled annuities (Piggott et al., 2005; Qiao and Sherris, 2013; Bernhardt and Donnelly, 2021), tontines (Milevsky and Salisbury, 2015; Milevsky and Salisbury, 2016; Chen and Rach, 2019; Chen et al., 2019; Chen et al., 2020; Sabin, 2010; Weinert and Gründl, 2021), and risk-sharing products (Donnelly et al., 2014; Donnelly and Young, 2017; Denuit, 2019; Fullmer and Sabin, 2018) with an additional decumulation plan. These products share the advantage of requiring zero or almost zero capital, while the ways of distributing the mortality credits and determining income payments are different. In this paper, we extend the last category, which is risk-sharing products with a decumulation plan to further protect the LTC risk of policyholders. The risk-sharing products with a decumulation plan are chosen because they have the advantage over the first two categories in that they are actuarially fair, self-sustaining, and allow heterogeneity at the same time.

The key motivation for developing an LTC pooling product is that not all retirees have the same level of demand for retirement income. For example, functionally disabled people at old ages need higher retirement income to cover their higher LTC cost needs. Meanwhile, people being functionally disabled also have a higher probability of death compared with healthy individuals of the same age, thus it is not fair to distribute the same proportion of mortality credits to all members of the pool. People with a higher probability of death need more from the mortality credits to be compensated. This differs from mortality pooling products that do not consider the health status of individuals. Therefore, we aim to study the suitable risk-sharing rule mixing people in different health states and at different ages in a risk pool. We also expect people in the more disabled and ill states to receive higher payments to cover their higher LTC costs.

Previous studies on LTC pooling products include Hieber and Lucas (2022), Chen et al. (2022a) and Kabuche et al. (2024). However, Hieber and Lucas (2022) assume no recovery from the dependent state to the healthy state. Our framework allows the free transition between any health states which includes recovery from functional disability due to the use of forward iterations. Meanwhile, the main focus of Chen et al. (2022a) is the optimisation of individual lifetime utility, while our focus is to have properties like actuarial fairness and self-sustainability which are very important for mortality pooling products as discussed in Denuit (2019). Kabuche et al. (2024) extend the group self-annuitisation (GSA) in Piggott et al. (2005) and use a framework that allows recovery from a disability state. However, the product in Kabuche et al. (2024) does not allow heterogeneous members or new entrants in subsequent times. Members in Kabuche et al. (2024) need to be one cohort of the same age joining at time zero, and there requires at least one person in each health state. Therefore, there lacks an LTC pooling product that is actuarially fair and self-sustaining, allowing recovery from functional disability, heterogeneous members, and new members to join at the same time. The proposed framework for LTC pooling products in this paper satisfies all these features mentioned above.

To study the risk-sharing with multiple health states, a multi-state health model that can capture the transition probabilities between different health states is required. Recently, there has been growing literature on multi-state health models. Fong et al. (2015) use a generalized linear model for a three-state model, and they allow recovery from the disability state to the healthy state. Li et al. (2017) and Fu et al. (2021) further incorporate trend and uncertainty in disability rate in a three-state model. Sherris and Wei (2021) develop a five-state model that not only incorporates trend and uncertainty but also constructs the states from two dimensions: functional disability and health. We use the five-state model in Sherris and Wei (2021) for empirical illustration, while our framework allows any number of health states and any type of transition between health states.

This paper contributes to the literature by proposing a framework for health-contingent mortality pooling products that are actuarially fair at any point in time, self-sustaining thereby having no capital requirement, and allow different kinds of heterogeneity in members including age, gender, contribution, health state, mortality rate, rate of return, and joining time. The health-contingent design allows it to pay higher retirement incomes to members in less healthy states to cover their higher long-term care cost needs. The proposed framework is also flexible to allow health state models with any number of health states and free transition between health states. Thus, recovery from functional disability is allowed in this paper. Moreover, the benefit payments are determined by forward iterations, which are clear and straightforward. The use of forward iterations is also one reason that the framework allows recovery from functional disability. There is also flexibility in decumulation so that the product can generate annuity-like payments that use health-state-dependent annuity factors or level payments with predetermined drawdown rates.

We find that the proposed health-contingent mortality pooling product can provide higher retirement income to retirees who have morbidity or are functionally disabled than those who are healthy. The higher income payments of ill or functionally disabled retirees come from three mechanisms: a higher but fair health-state-dependent accumulation factor, a higher probability of death in a less healthy state and thus a higher proportion of mortality credits, and a higher drawdown rate in a less healthy state. We also find that retirement incomes are significantly higher for members who have functional disability and morbidity, with very little impact on the retirement incomes of healthy members. Moreover, we analyse the volatility of income payments and find that the payments are stable and the product is able to pay higher income in unhealthy states even in the worst-case scenario. Furthermore, we study the income payments when a transition to a less healthy state happens at a different time and find that the product can provide higher income payments for ill and functionally disabled members no matter when the transition happens, and the higher income payments persist if the individual stays in a less healthy state. Meanwhile, if the member recovers from functional disability, the income payments will reduce immediately to reflect the lower required long-term care cost. Finally, the level payments provide more stable income payments than annuity-like payments because the income payments remain the same as long as the balance does not run out.

The rest of the paper is structured as follows. Section 2 introduces the multi-state health model used in this paper. Section 3 explains in detail the framework and operation of the proposed health-contingent mortality pooling product. Section 4 discusses the assumptions of the numerical experiments and the results we get including no risk-sharing, risk-sharing with closed and open pools, and risk-sharing with health-state-dependent accumulation factors. We compare the income payments and balances of individuals with different ages, contributions, health states, and experience different transitions between health states for both annuity-like and predetermined level decumulation plans. Section 5 concludes the paper.

2 Multi-state Health Model

A multi-state health model is required for a health-contingent mortality pooling LTC product. Assume that there are N_{HS} health states, and an individual i belongs to one of the N_{HS} health states. States $h = 1, 2, ..., (N_{HS} - 1)$ represent the states that the individual is alive but with different health statuses, while State $h = N_{HS}$ represents the dead state. The one-year transition probability matrix of individual i from time t to time $t + 1$ between different health states is defined as $\mathbf{P}_i(t, t+1)$, which is represented as:

$$\mathbf{P}_i(t,t+1) = \begin{bmatrix} p_i^{1,1}(t,t+1) & p_i^{1,2}(t,t+1) & \cdots & p_i^{1,N_{HS}}(t,t+1) \\ p_i^{2,1}(t,t+1) & p_i^{2,2}(t,t+1) & \cdots & p_i^{2,N_{HS}}(t,t+1) \\ \vdots & \vdots & \ddots & \vdots \\ p_i^{N_{HS}-1,1}(t,t+1) & p_i^{N_{HS}-1,2}(t,t+1) & \cdots & p_i^{N_{HS}-1,N_{HS}}(t,t+1) \\ p_i^{N_{HS},1}(t,t+1) & p_i^{N_{HS},2}(t,t+1) & \cdots & p_i^{N_{HS},N_{HS}}(t,t+1) \end{bmatrix}$$
$$= \begin{bmatrix} p_i^{1,1}(t,t+1) & p_i^{1,2}(t,t+1) & \cdots & p_i^{1,N_{HS}}(t,t+1) \\ p_i^{2,1}(t,t+1) & p_i^{2,2}(t,t+1) & \cdots & p_i^{2,N_{HS}}(t,t+1) \\ \vdots & \vdots & \ddots & \vdots \\ p_i^{N_{HS}-1,1}(t,t+1) & p_i^{N_{HS}-1,2}(t,t+1) & \cdots & p_i^{N_{HS}-1,N_{HS}}(t,t+1) \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

where $p_i^{b,h}(t,t+1)$ is the one-year transition probability of individual i at time t from State b to State h between time t and $t+1$. Note that $p_i^{N_{HS},h\neq N_{HS}}(t,t+1) = 0$ and $p_i^{N_{HS},N_{HS}}(t,t+1) = 1$ because dead state is an absorbing state.

We use the multi-state model calibrated in Sherris and Wei (2021) with US Health and Retirement Study (HRS) data. The detailed multi-state health model is shown in Figure 1 below. People are defined to be functionally disabled if they have two or more difficulties in six activities of daily living (ADL) (Li et al., 2017). The six ADLs are dressing, walking, bathing, eating, transferring, and toileting as in Li et al. (2017). And people have ill health if they experience any one of the four non-recoverable illnesses: heart problems, diabetes, lung disease, or stroke (Brown and Warshawsky, 2013). Then, five states are constructed taking the interaction of disability and illness into consideration.

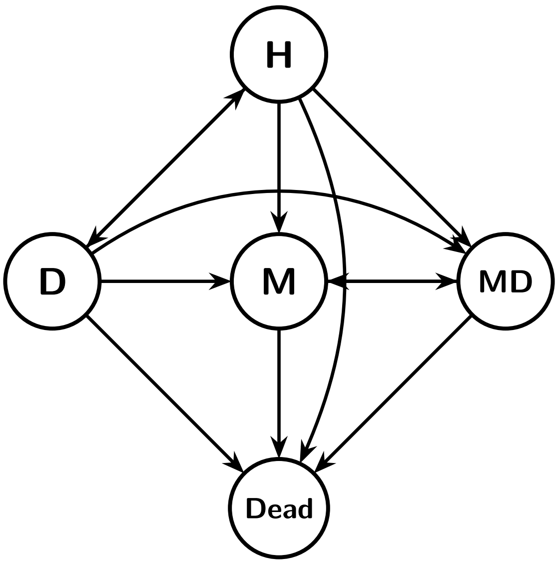


Figure 1: A five-state transition model in Sherris and Wei (2021).

1. H: No morbidity and not functionally disabled;
2. M: Morbidity and not functionally disabled;
3. D: No morbidity and functionally disabled;
4. MD: Morbidity and functionally disabled;
5. Dead.

For a given age x and time t, the estimated five-state model will give us a transition probability matrix as shown in the table below, which includes the transition probabilities between time t and $t+1$:

$$\mathbf{P}_i(t,t+1) = \begin{bmatrix} p_i^{1,1}(t,t+1) & p_i^{1,2}(t,t+1) & p_i^{1,3}(t,t+1) & p_i^{1,4}(t,t+1) & p_i^{1,5}(t,t+1) \\ 0 & p_i^{2,2}(t,t+1) & 0 & p_i^{2,4}(t,t+1) & p_i^{2,5}(t,t+1) \\ p_i^{3,1}(t,t+1) & p_i^{3,2}(t,t+1) & p_i^{3,3}(t,t+1) & p_i^{3,4}(t,t+1) & p_i^{3,5}(t,t+1) \\ 0 & p_i^{4,2}(t,t+1) & 0 & p_i^{4,4}(t,t+1) & p_i^{4,5}(t,t+1) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where some of the transition probabilities are 0 because recovery from morbidity to no morbidity is not allowed in this model, while recovery from functionally disabled to not functionally disabled is allowed. We can also write $q_i^b(t) = p_i^{b,5}(t,t+1)$ for $b = 1,2,3,4$ to represent the one-year probability of death from State b, and $p_i^b(t) = \sum_{j=1}^{4} p_i^{b,j}(t,t+1)$ for $b = 1,2,3,4$ to represent the one-year survival probability in State b. Note that we have the relationship that $p_i^b(t) + q_i^b(t) = 1$.

3 Multi-state Health-contingent Mortality Pooling

We allow retirees of different ages and in all health states to join the risk-sharing pool at any discrete point in time. Assume an individual i who joins the pool at time 0 has fund value $F_i(0)$ as the initial contribution. Then, at time 1 the health-contingent accumulated fund value becomes $s_i^h(1) = F_i(0)a_i^h(1)$, where $a_i^h(1)$ is the accumulation factor for individual i between time 0 and 1 and which depends on the health state $h_i(1)$ of individual i at time 1. A health-contingent risk-sharing rule is then applied to distribute the total mortality credits $S(1)$ coming from the accumulated fund balances of members who pass away between time 0 and 1 to the fund members. The health-contingent balance after risk-sharing is $V_i^h(1)$. The health-contingent benefit payment $B_i^h(1)$ is then paid from the balance after risk-sharing $V_i^h(1)$. The remaining balance $F_i(1) = V_i^h(1) - B_i^h(1)$ becomes the initial balance for the next time period $[1,2]$ and this iteration keeps going.

Now we look at the fund operation starting at an arbitrary point time t. The fund operation is summarised in Steps 1-4 and illustrated in Figure (2):

Step 1: Health-contingent accumulation of investment return in Equation (4) and Equation (5).

Step 2: Health-contingent risk-sharing and distributing the total mortality credits in Equation (12).

Step 3: Health-contingent benefit paying from the balance after risk-sharing. The benefit payment is determined as in Equation (13) or Equation (14).

Step 4: Health-contingent accumulation of investment return for the next period in Equation (4) and Equation (5).

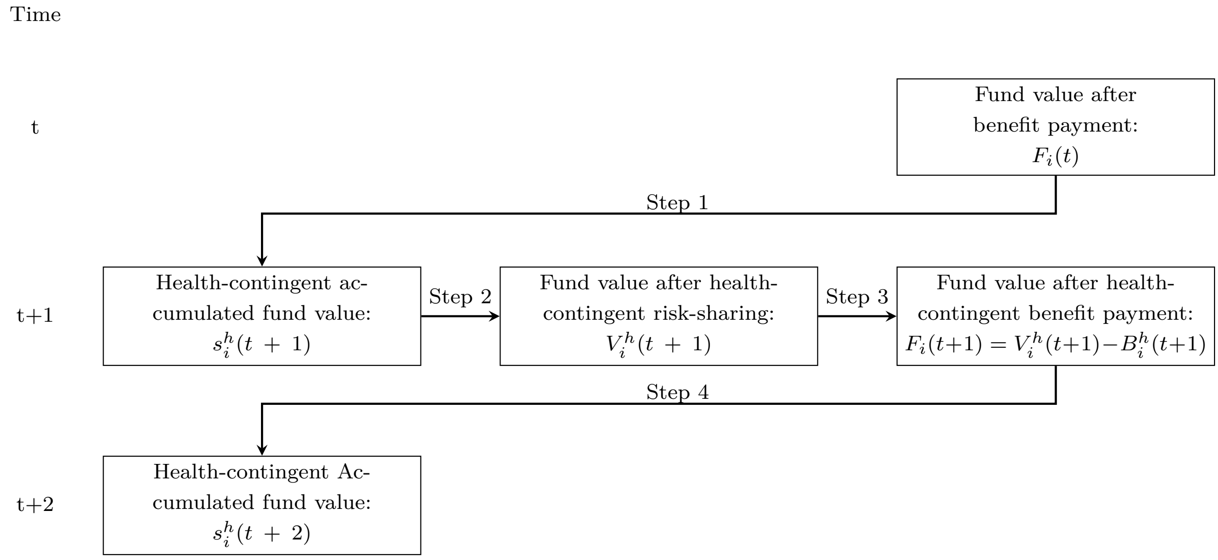


Figure 2: Health-contingent fund operation between time t and $t+2$.

3.1 Step 1: Health-contingent Accumulation

Denote $F_i(t)$ as the final fund balance of individual i at time t, and $a_i(t+1) = 1 + ROR_i(t)$ the accumulation factor of individual i between time t and $t+1$, where $ROR_i(t)$ is the rate of return of individual i between time t and $t+1$. Then, we propose the use of a health state-dependent accumulation factor $a_i^h(t+1)$ between time t and $t+1$ which differs for different health states $h_i(t+1)$ that the individual i is in at time $t+1$, i.e. the end of the period, represented as:

$$a_i^h(t+1) = \begin{cases} a_i^1(t+1) & \text{if } h_i(t+1) = 1, \\ a_i^2(t+1) & \text{if } h_i(t+1) = 2, \\ \quad\vdots & \qquad\vdots \\ a_i^{N_{HS}-1}(t+1) & \text{if } h_i(t+1) = N_{HS} - 1, \\ a_i^{N_{HS}}(t+1) = 1 + ROR_i(t) = a_i(t+1) & \text{if } h_i(t+1) = N_{HS}, \end{cases} \tag{1}$$

where $h_i(t+1) = 1, 2, ..., N_{HS} - 1, N_{HS}$ represents the state that individual i is in at time $t+1$. Generally speaking, if the LTC cost in State n is higher than that in State m, then the distribution factor in State n should be higher than that in State m for individuals to cover their higher LTC cost and health-related cost needs. Hence, it is reasonable to assume $a_i^n(t+1) > a_i^m(t+1)$.

We expect the following equation to hold:

$$E[a_i^h(t+1)] = a_i(t+1), \tag{2}$$

so that the state-dependent accumulation is actuarially fair. Denote $b_i(t) = 1, 2, \cdots, N_{HS} - 1$ the initial state of individual i at time t, i.e. the beginning of time period $[t, t+1]$. Thus, we expect:

$$\sum_{h=1}^{N_{HS}-1} a_i^h(t+1) p_i^{b_i,h}(t, t+1) = a_i(t+1) \sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t, t+1), \tag{3}$$

where $p_i^{b_i,h}(t, t+1)$ is the one-year transition probability of individual i over the period $[t, t+1]$, from the health state $b_i(t)$, which is known at time t for individual i, to a health state $h_i(t+1)$ at time $t+1$.

Equation (3) holds because we set $a_i^{N_{HS}}(t+1) = 1 + ROR_i(t) = a_i(t+1)$ in Equation (1). The accumulation factor in the dead state is set to be equal to the normal accumulation one $a_i^{N_{HS}}(t+1) = a_i(t+1)$ so that in the event of death, members will just lose the accumulated fund balance in their account and this will reduce complexity in risk-sharing and calculation of total mortality credits. We show in the next subsection how to find a set of $a_i^h(t+1)$ which suits this requirement.

Following the above considerations, the accumulated fund value at time $t+1$ of individual i initially alive at time t, and in State h at time $t+1$ is represented as $s_i^h(t+1)$:

$$s_i^h(t+1) = F_i(t) \times a_i^h(t+1), \tag{4}$$

which translates to

$$s_i^h(t+1) = \begin{cases} s_i^1(t+1) & \text{if } h_i(t+1) = 1, \\ s_i^2(t+1) & \text{if } h_i(t+1) = 2, \\ \quad\vdots & \qquad\vdots \\ s_i^{N_{HS}-1}(t+1) & \text{if } h_i(t+1) = N_{HS} - 1, \\ s_i^{N_{HS}}(t+1) = s_i(t+1) & \text{if } h_i(t+1) = N_{HS}, \end{cases} \tag{5}$$

where $h_i(t+1)$ represents the state of individual i at time $t+1$, and $s_i(t+1) = F_i(t)a_i(t+1)$.

Similar to Equation (2), we expect:

$$E[s_i^h(t+1)] = s_i(t+1), \tag{6}$$

which can be written as

$$\sum_{h=1}^{N_{HS}} s_i^h(t+1) p_i^{b_i,h}(t, t+1) = s_i(t+1),$$

and thus

$$\sum_{h=1}^{N_{HS}-1} s_i^h(t+1)p_i^{b_i,h}(t,t+1) = s_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1). \tag{7}$$

Equation (6) holds if Equation (2) holds because:

$$\begin{aligned} E[s_i^h(t+1)] =& E[a_i^h(t+1)]F_i(t+1) \\ =& a_i(t+1)F_i(t+1) \\ =& s_i(t+1). \end{aligned} \tag{8}$$

3.1.1 Determining the State-dependent Accumulation Factors

To determine the state-dependent accumulation factors $a_i^h(t+1)$, we assume that they are proportional to a set of predetermined factors $c_i^h(t+1)$ for $h = 1, 2, ..., N_{HS} - 1$.

Definition 1. We set:

$$\begin{cases} a_i^1(t+1) = c_i^1(t+1)x_i^c(t+1), \\ a_i^2(t+1) = c_i^2(t+1)x_i^c(t+1), \\ \vdots \\ a_i^{N_{HS}-1}(t+1) = c_i^{N_{HS}-1}(t+1)x_i^c(t+1), \\ a_i^{N_{HS}}(t+1) = a_i(t+1), \end{cases} \tag{9}$$

so that $a^n(t+1) > a^m(t+1)$ if $c_i^n(t+1) > c_i^m(t+1)$ and $x_i^c(t+1) > 0$, and $x_i^c(t+1)$ a constant of individual i for the time period $[t, t+1]$ to be solved to keep actuarial fairness.

Proposition 1. *The state-dependent distribution factors are:*

$$\begin{cases} a_i^1(t+1) = \frac{a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)}{\sum_{h=1}^{N_{HS}-1} c_i^h(t+1)p_i^{b_i,h}(t,t+1)} c_i^1(t+1), \\ a_i^2(t+1) = \frac{a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)}{\sum_{h=1}^{N_{HS}-1} c_i^h(t+1)p_i^{b_i,h}(t,t+1)} c_i^2(t+1), \\ \vdots \\ a_i^{N_{HS}-1}(t+1) = \frac{a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)}{\sum_{h=1}^{N_{HS}-1} c_i^h(t+1)p_i^{b_i,h}(t,t+1)} c_i^{N_{HS}-1}(t+1), \\ a_i^{N_{HS}}(t+1) = a_i(t+1), \end{cases} \tag{10}$$

if Equation (8) and Equation (9) hold.

Proof. To have actuarial fairness, we require:

$$E[a_i^h(t+1)] = a_i(t+1),$$

which gives us

$$\sum_{h=1}^{N_{HS}-1} a_i^h(t+1)p_i^{b_i,h}(t,t+1) = a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)$$
$$\sum_{h=1}^{N_{HS}-1} c_i^h(t+1)x_i^c(t+1)p_i^{b_i,h}(t,t+1) = a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1),$$

from which we obtain

$$x_i^c(t+1) = \frac{a_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)}{\sum_{h=1}^{N_{HS}-1} c_i^h(t+1)p_i^{b_i,h}(t,t+1)}. \tag{11}$$

Substituting Equation (11) back to Equation (9) completes the proof. □

3.2 Step 2: Health-contingent Risk-sharing

The accumulated fund balances of members who die between time t and $t+1$ sum up into the total mortality credits to be distributed. The total mortality credits $S(t+1)$ to be distributed at time $t+1$ is represented as:

$$\begin{aligned} S(t+1) &= \sum_{i=1}^{N(t)} (1 - I_i(t+1))s_i(t+1) \\ &= \sum_{i=1}^{N(t)} X_i(t+1), \end{aligned}$$

where $N(t)$ is the number of people initially alive at time t, i.e. the beginning of the period $[t, t+1]$,

$$I_i(t+1) = \begin{cases} 1 & \text{if individual } i \text{ is alive at time } t+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$X_i(t+1) = (1 - I_i(t+1))s_i(t+1).$$

Definition 2. The health-contingent fund value after risk-sharing $V_i^h(t)$ in the proposed framework is the following:

$$V_i^h(t+1) = \begin{cases} s_i^1(t+1) + \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = 1, \\ s_i^2(t+1) + \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = 2, \\ \vdots & \vdots \\ s_i^{N_{HS}-1}(t+1) + \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = N_{HS}-1, \\ \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = N_{HS}, \end{cases} \tag{12}$$

where $\delta(t+1) = \sum_{j \in A(t+1)} s_j^h(t+1) - s_j(t+1)$ is a deviation term equals to the sum of the differences between the empirical and expected state-dependent accumulated fund balance of members in the set $A(t+1)$ who are alive at time $t+1$ given alive at time t, and $q_i^{b_i}(t)$ is the one-year probability of death over the period $[t, t+1]$ for individual i initially in the health state $b_i(t)$ at time t. We have $E[\delta(t+1)] = 0$ since $E[s_i^h(t+1)] = s_i(t+1)$ for every individual i. This term is deducted from the total mortality credit to adjust the deviation from the empirical and expected sum of accumulated fund balances, which thus ensures the self-sustainability of the fund. The weighting in the total mortality credit minus the deviation is similar to the idea of the proportional risk-sharing rule in Donnelly and Young (2017).

Proposition 2. *The health-contingent risk-sharing rule in Equation (12) is actuarially fair for every individual i if $\sum_{h=1}^{N_{HS}-1} s_i^h(t+1)p_i^{b_i,h}(t,t+1) = s_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1)$ in Equation (7) holds.*

Proof. We calculate the expectation of the health-contingent fund value after risk-sharing:

$$\begin{aligned} E[V_i^h(t+1)] = &\sum_{h=1}^{N_{HS}-1} s_i^h(t+1)p_i^{b_i,h}(t,t+1) + \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)} E[S(t+1)], \\ =&s_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1) + \frac{s_i(t+1)q_i^{b_i}(t)}{\sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t)} \sum_{j=1}^{N(t)} s_j(t+1)q_j^{b_j}(t), \\ =&s_i(t+1)\sum_{h=1}^{N_{HS}-1} p_i^{b_i,h}(t,t+1) + s_i(t+1)q_i^{b_i}(t), \\ =&s_i(t+1), \end{aligned}$$

which completes the proof since the expectation of the health-contingent fund value after risk-sharing is equal to the accumulated fund value at risk before risk-sharing for every individual i. □

Proposition 3. *The health-contingent risk-sharing rule in Equation (12) is self-sustaining.*

Proof. We calculate the sum of the health-contingent fund values after risk-sharing across all individuals i:

$$\begin{aligned}\sum_i V_i^h(t+1) =& \sum_{i\in A(t+1)} s_i^h(t+1) + S(t+1) - \left(\sum_{i\in A(t+1)} s_i^h(t+1) - s_i(t+1)\right)\\ =& S(t+1) + \sum_{i\in A(t+1)} s_i(t+1)\\ =& \sum_{i\in D(t+1)} s_i(t+1) + \sum_{i\in A(t+1)} s_i(t+1)\\ =& \sum_i s_i(t+1),\end{aligned}$$

where $D(t+1)$ is the set of members who have died during the period $[t, t+1]$. This completes the proof since the sum of the health-contingent fund values after risk-sharing across all individuals i is equal to the sum of the accumulated fund values before risk-sharing across all individuals i. □

The proposed framework for health-contingent mortality pooling products has the flexibility to include other risk-sharing rules in Step 2. For example, one possible future extension can be the conditional mean risk-sharing introduced in Denuit and Dhaene (2012) and further investigated in Denuit (2019). The fund balance of member i after risk-sharing is:

$$V_i^h(t+1) = \begin{cases} s_i^1(t+1) + \frac{E[X_i(t+1)|S(t+1)]}{S(t+1)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = 1,\\ s_i^2(t+1) + \frac{E[X_i(t+1)|S(t+1)]}{S(t+1)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = 2,\\ \vdots & \vdots\\ s_i^{N_{HS}-1}(t+1) + \frac{E[X_i(t+1)|S(t+1)]}{S(t+1)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = N_{HS}-1,\\ \frac{E[X_i(t+1)|S(t+1)]}{S(t+1)}[S(t+1) - \delta(t+1)] & \text{if } h_i(t+1) = N_{HS}, \end{cases}$$

where the conditional mean $E[X_i(t+1)|S(t+1)]$ can be calculated following Denuit and Dhaene (2012) and Denuit (2019).

3.3 Step 3: Health-contingent Benefit Payments

After risk-sharing, the benefit payment is then determined from the balance after risk-sharing $V_i^h(t+1)$. The benefit paid to every individual at time $t+1$ has the flexibility to be either annuity-like payments or level payments. The annuity-like payments are represented as:

$$B_i^h(t+1) = \begin{cases} \frac{V_i^h(t+1)}{\ddot{a}^h_{x_{i,t+1}}} & \text{if individual } i \text{ survives},\\ V_i^h(t+1) & \text{if individual } i \text{ dies}, \end{cases} \tag{13}$$

where $\ddot{a}^h_{x_{i,t+1}}$ is the actuarial notation of an annuity due for individual i that is aged x and in State h at time $t+1$:

$$\ddot{a}^h_{x_{i,t+1}} = 1 + \sum_{k=1}^{\infty} p_i^h(t+1, t+1+k),$$

and $p_i^h(t+1, t+1+k)$ is the survival probability of individual i who is aged x and in health State h at time $t+1$ between time $t+1$ and $t+1+k$, no matter which health states the individual transits into during the period. We also have the relationship that:

$$p_i^h(t+1, t+1+k) = \sum_{j=1}^{N_{HS}-1} p_i^{h,j}(t+1, t+k+1),$$

which is because the survival probability is the sum of the transition probabilities to the health states except for the dead state. The fund recalculates the annuity-like payments at each point in time. This will naturally yield higher benefit payments for members in more disabled states because their fund balances will be higher and their annuity factor will be lower.

Meanwhile, the drawdown payments can be predetermined level payments which are higher in more disabled health states:

$$B_i^h(t+1) = \begin{cases} min(B_i^h(t+1)^{Predetermined}, V_i^h(t+1)) & \text{if individual } i \text{ survives,} \\ V_i^h(t+1) & \text{if individual } i \text{ dies.} \end{cases} \tag{14}$$

The minimum function $min(B_i^h(t+1)^{Predetermined}, V_i^h(t+1))$ is applied so that if the predetermined level payment $B_i^h(t+1)^{Predetermined}$ is higher than the fund value after risk-sharing $V_i^h(t+1)$, the fund value after risk-sharing will be paid and the remaining fund balance will become zero.

3.4 Step 4: Health-contingent Accumulation in the Next Period

The fund value of individual i after health-contingent risk-sharing and benefit payment $F_i(t+1) = V_i^h(t+1) - B_i^h(t+1)$ becomes the initial value for the next period $[t+1, t+2]$. The health state of individual i at time $t+1$ becomes the initial health state for the next period $[t+1, t+2]$, which determines the transition probabilities to the health states at time $t+2$. These transition probabilities along with the health state of individual i at time $t+2$ determine the health-contingent accumulation between time $t+1$ and $t+2$ in Step 4, followed by the health-contingent risk-sharing and benefit payments in subsequent steps.

3.5 Summary of fund operation

In summary, there are three mechanisms resulting in higher payments in less healthy states:

1. Health-contingent accumulation factor.
2. Higher probability of death in a less healthy state and thus a higher proportion of total

mortality credits.

3. Higher payout ratio in a less healthy state due to:
   - Lower annuity due values.
   - Higher level payout ratios.

4 Numerical Illustration: Income Payments and Balances under Different Settings

This section displays the results including income payments and balances of members with different settings of risk-sharing. The general assumptions we make for all experiments are:

- The interest rate is 3% per annum.
- The pool is established in calendar year 2022.
- Pool members are male whose transition rates and mortality rates are calculated from the Retirement Income Toolkit using the 5-state health model in Sherris and Wei (2021) and HRS US data.

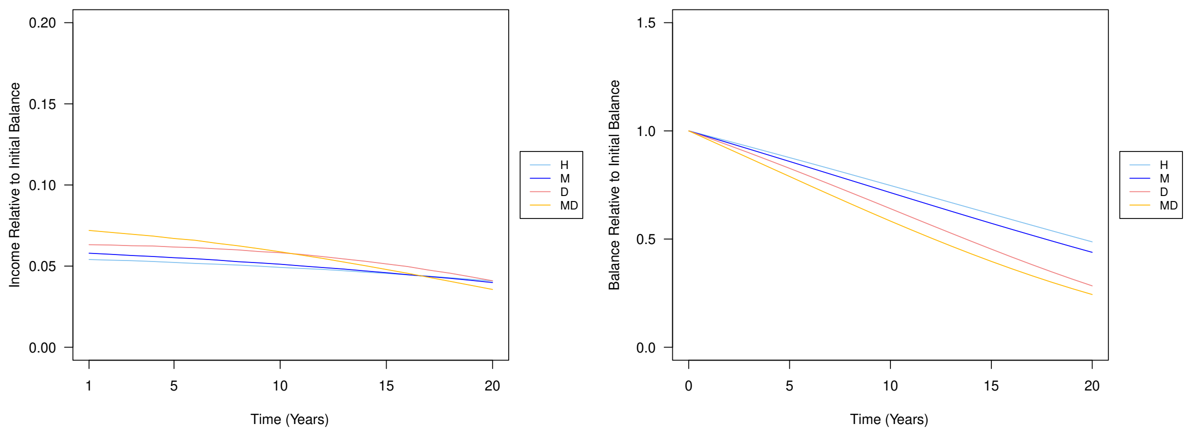
The transition probability matrices from ages 60 to 109 for the next 20 years are calculated with the Retirement Income Toolkit available at https://github.com/RI-Toolkit/rit. The parameters for the five-state health model used in the toolkit are calibrated in Sherris and Wei (2021) using the Health and Retirement Study (HRS) data from 1998 to 2014. The details of the model and the parameters can be found in the appendix. Meanwhile, for each age in each state in the next 20 years, the survival probabilities until age 109 are produced to construct the annuity factor $\ddot{a}_{x^j_{i,t+1}}$, which is used to calculate income payments as shown in Equation (13). We simulate the transition between states of members as well as deaths with the transition probability matrices, and we use the health-contingent risk-sharing rule in Equation (12).

4.1 No Risk-Sharing, Pool Size=1

We start with no risk-sharing, which means each individual withdraws the annuity-alike income payments according to Equation (13), but there is no distribution of the total mortality credits and individuals are managing their own longevity risks. Therefore, only Mechanism 3, the health-state-dependent payout ratio, leads to the difference in income payments between different health states.

We study an individual aged 60 in year 2022 which is time 0 with an initial balance of $600,000, and then transits into H, M, D, or, MD states at time 1 and stays in those states respectively for the next 20 years. The relative income payments and relative balances over the next 20 years are shown in Figure 3. From Figure 3(a), we can see the difference between different health states in income payments, which is mainly driven by the lower annuity factor $\ddot{a}_{x^j_{i,t+1}}$ in the less healthy and more disabled states, leading to a higher income payment according to

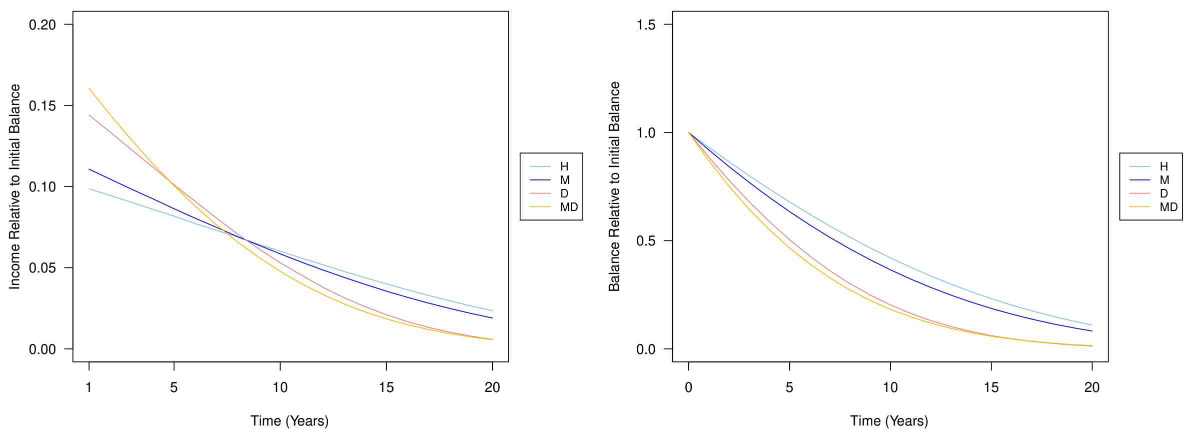
Equation (13). Meanwhile, we can see from Figure 3(b) that the balance is consumed faster in a less healthy or more disabled state. This leads to the income payments in the MD state after 11 years being lower than the income payments in the D state since the balance in the MD state is lower than the balance in the D state and not enough to generate a higher income payment than the D state. The income payments in all states are also decreasing over time because there is no risk-sharing to benefit from mortality credits under this setting.



(a) Relative income payments (b) Relative balances

Figure 3: Individuals joined at age 60 at time 0 initially healthy and transit into different health states (No risk-sharing over the next 20 years).

Similarly, Figure 4 displays the relative income payments and relative balances of individuals joined at age 80 at time 0, initially healthy with an initial balance of \$200,000, and transits into different health states.



(a) Relative income payments (b) Relative balances

Figure 4: Individuals joined at age 80 at time 0 initially healthy and transit into different health states (No risk-sharing over the next 20 years).

From Figure 4, we can see that both the income payments and balances reduce faster than the aged 60 cohort. Moreover, the relative income ratios start at a higher value compared with age 60, which is due to the lower annuity due factor at older ages used in Equation (13). For example, the relative income ratios start at around 10% in H and M states, and around 15% in D and MD states for the aged 80 cohort, compared with the 5% to 7.5% for the aged 60 cohort. Furthermore, since there is no risk-sharing and we assume these individuals are alive and stay in these health states for the next 20 years, the income payments and balances in Figures 3 and 4 are deterministic in this case.

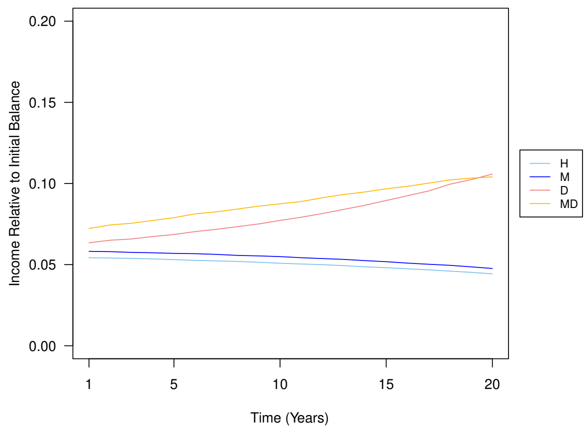
4.2 Risk-sharing, Closed Pool, and No State-dependent Index

We now include risk-sharing but with a closed pool and no state-dependent index, which means there are 100 or 1000 aged 60 members joining the pool at time 0 initially healthy with an initial balance of $\$600,000$, no new member is joining afterwards, and the state-dependent accumulation factors are proportional to $c_i^h = \{1, 1, 1, 1\}$ for $h = 1, 2, 3, 4$, every individual i, and every time period. Due to the participation of risk-sharing, in addition to Mechanism 3, Mechanism 2 that people in a less healthy state get a higher proportion of mortality credits also leads to the difference in income payments between different health states.

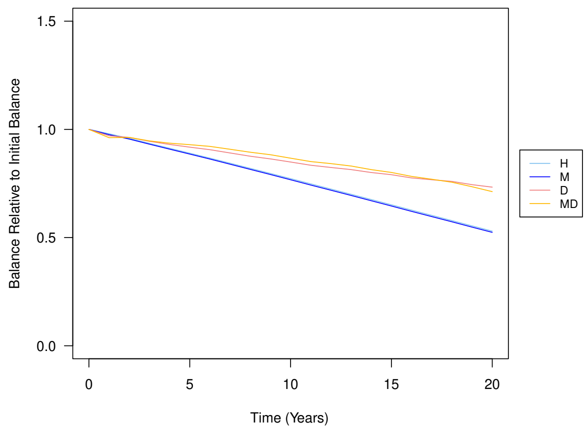
4.2.1 Age=60, Pool Size=100

With a pool size of 100, Figure 5(a) shows that the relative income payments still show the pattern that the income payments in a less healthy and more disabled state are higher. However, we can see that with a pool size of 100 instead of 1, the income payments in the MD state are higher than in the D state for a longer period of time. This is because from Equation (12), it can be seen that the probability of death in the MD state for the cohort aged 60 joining at time 0 is higher than the probability of death in the D state $q_i^4(t) > q_i^3(t)$ for the next 20 years $t \in [1, 20]$, which leads to a higher compensation from the total mortality credits in an MD state than in a D state. Meanwhile, the D state still benefits more than the H and M states since $q_i^3(t) > q_i^2(t) > q_i^1(t)$. We observe that the income payments in Figure 5(a) are higher than in Figure 3(a) and show a less decreasing trend, which is also because of the benefit from mortality credits for surviving members. We can also observe that from the balances in Figure 5(b) that the balance of the MD state is not decreasing as fast as in Figure 3(b) when there is no risk-sharing.

Then, we study how stable the relative income payments are. There are two sources of randomness, the volatility in the transition rates from the frailty model, and the empirical transition between states for given transition rates. Figure 6 displays the 95% confidence intervals (CIs) of income payments for individuals transit into H, M, D, and MD states from 100 simulated paths. It can be seen that for the relative income payments, the confidence intervals in the D and MD states are more volatile than in the H and M states.

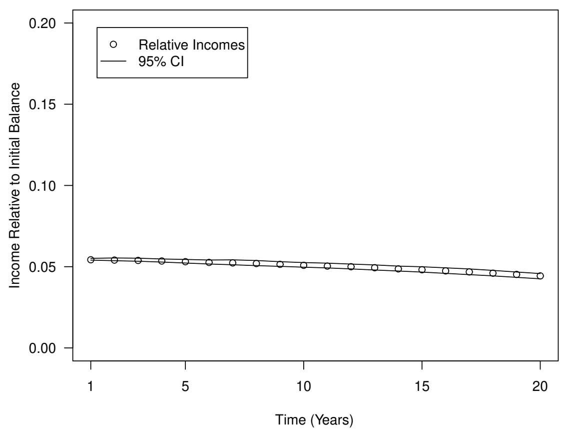


(a) Mean relative income payments

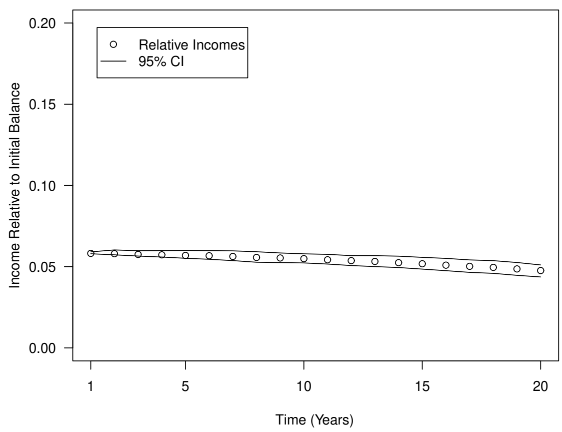


(b) Mean relative balances

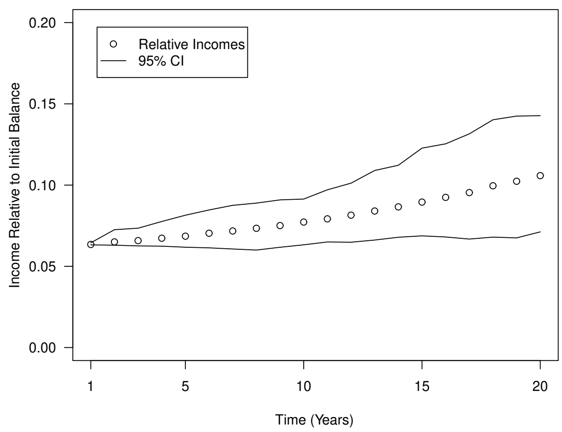
Figure 5: Individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with closed pool, size= 100).



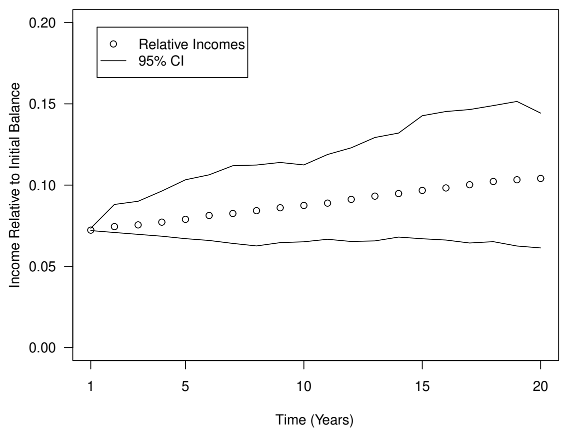
(a) H state



(b) M state



(c) D state

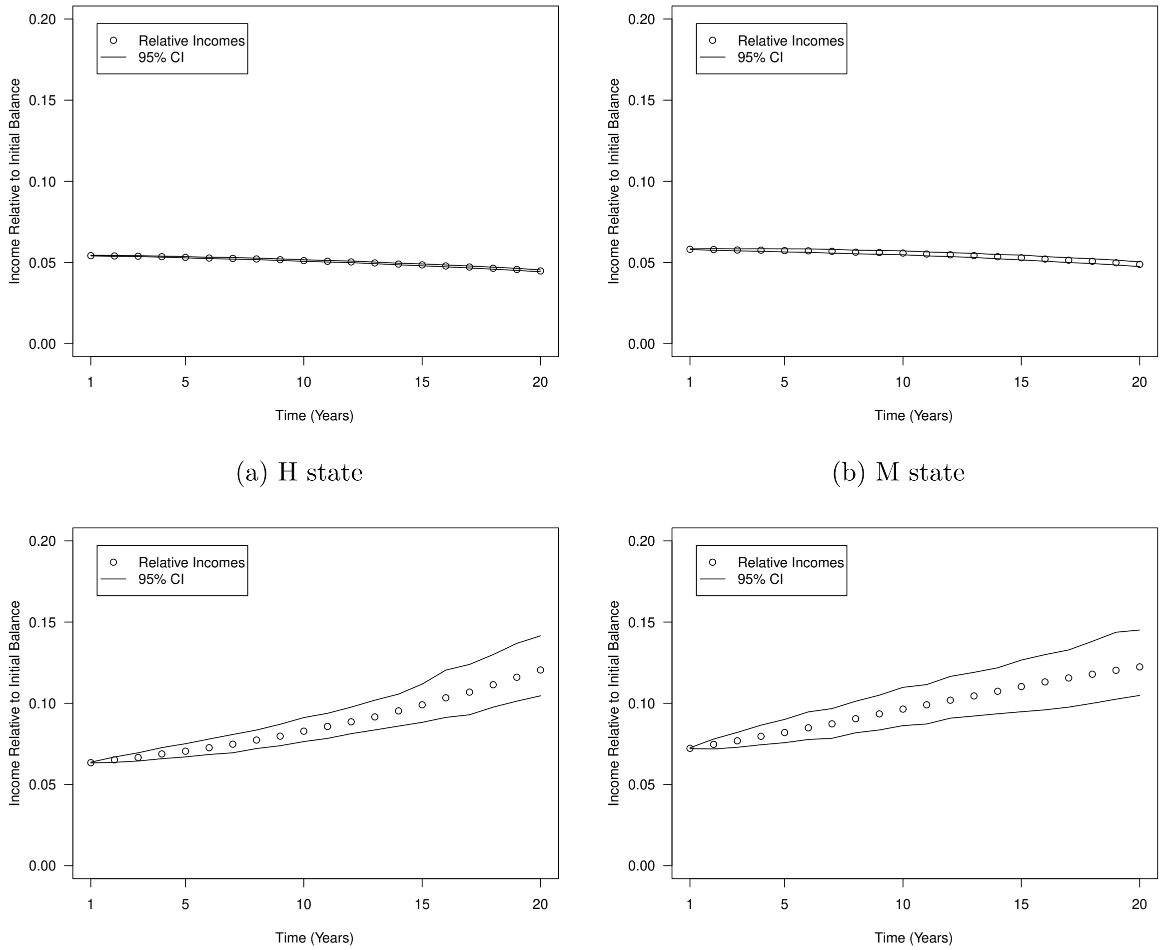


(d) MD state

Figure 6: Relative income payments: 95% confidence interval for individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with closed pool, size= 100).

4.2.2 Age=60, Pool Size=1000

We now increase the size of the closed pool from 100 to 1000 with the other assumptions remaining the same. Figure 7 displays the 95% confidence intervals for relative income payments when the size of the closed pool is 1000. Comparing Figure 7 with Figure 6, we can see that with a larger pool size, the relative income payments are significantly higher, with more trend of increase in D and MD states, and the 95% confidence intervals for relative income payments are much narrower.



(a) H state (b) M state

(c) D state (d) MD state

Figure 7: Relative income payments: 95% confidence interval for individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with closed pool, size= 1000).

4.3 Risk-sharing, Open Pool and Mixed Cohort, and No State-dependent Accumulation Factor

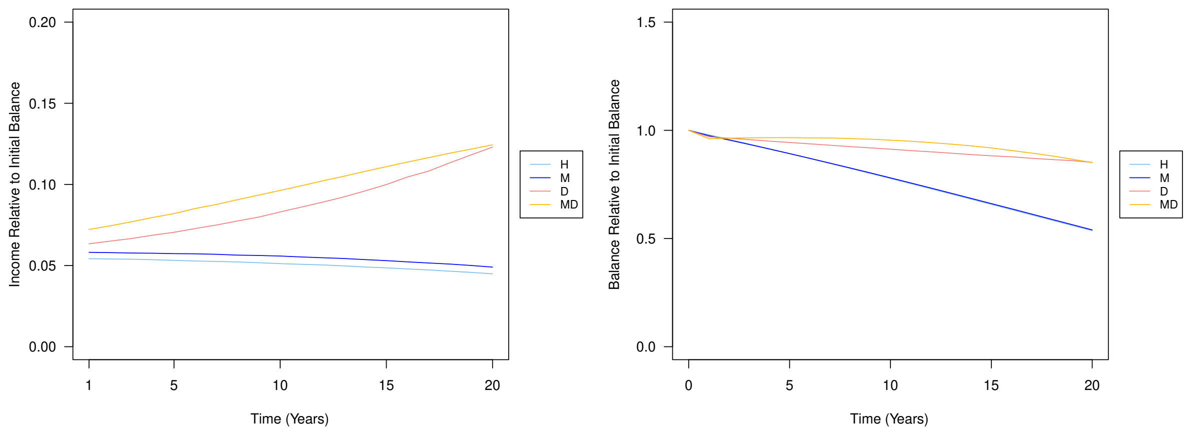
We now assume that the pool consists of members of different ages ranging from age 60 to 80, different initial health states, and initial age-dependent contributions. Table 1 displays the number of people at each age between 60 and 80, and initially in H, M, D, and MD states respectively. The distribution of people in different health states follows the health-state-dependent lifetable generated by the Retirement Income Toolkit. The number of people joining decreases with age, while the proportion of people in M, D, and MD states increases with age. Moreover, Table 1 also shows the age-dependent initial balance in the last column. The initial contribution at age 60 is $\$600,000$, and it decreases by $\$20,000$ when age increases by one to reflect the consumption of retirement savings. It is assumed to be an open pool so the combination of members in Table 1, that is a total of 1912 new members, will join the pool every year for the first 20 years. Moreover, we also assume that the accumulation factors are state-independent so they are proportional to $c_i^h = \{1, 1, 1, 1\}$ for $h = 1, 2, 3, 4$, every individual i, and every time period. Therefore, only Mechanisms 2 and 3 lead to health-state-dependent income payments.

Table 1: Number of people in different health states and ages with age-dependent balances joining the pool at the beginning of every year.

Age	Health States				Total	Balance
	H	M	D	MD		
60	100	0	0	0	100	600,000
61	94	5	0	0	99	580,000
62	88	10	1	0	99	560,000
63	83	15	1	0	99	540,000
64	77	20	1	1	99	520,000
65	72	24	1	1	98	500,000
66	67	28	1	1	97	480,000
67	62	32	1	2	97	460,000
68	57	35	1	2	95	440,000
69	52	38	1	2	93	420,000
70	48	41	1	3	93	400,000
71	44	44	1	3	92	380,000
72	40	46	1	4	91	360,000
73	36	48	1	4	89	340,000
74	32	50	1	4	87	320,000
75	29	51	1	5	86	300,000
76	26	52	1	5	84	280,000
77	23	52	1	6	82	260,000
78	20	53	1	6	80	240,000
79	17	53	1	6	77	220,000
80	15	52	1	7	75	200,000
Subtotal	1082	749	19	62	1912	NA

We assign the members who join at time 0 with ID numbers from 1 to 1912, ordered by age and within each age by the joining health state H, M, D, and MD. The ID numbers continue for members who join in the subsequent years. We assume that members 1, 2, 3, and 4 join the pool at time 0 aged 60, initially in the H state, and transit into the H, M, D, and MD states respectively and stay in their respective states for the next 20 years. It is the same for members 1838 to 1841 except that they are aged 80 initially in the H state when they join the pool at time 0. Only those 8 members stay in those states to make observation easier, while the other members move freely between states.

From Figure 8, we can see that the mean relative income payments and mean relative balances of the age 60 at time 0 cohort show similar patterns like increasing income payments in MD and D states and relatively stable payments in M and H states. Meanwhile, the pattern that $MD > D > M > H$ in terms of income payments persists.

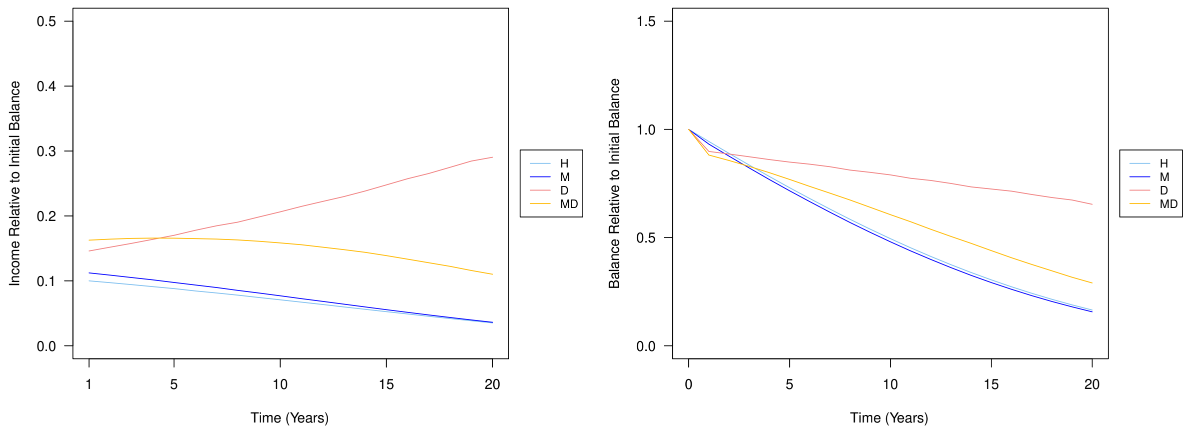


(a) Mean relative income payments

(b) Mean relative balances

Figure 8: Individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1, 1, 1\}$).

Figure 9 plots the mean relative income payments and mean relative balances of the age 80 cohort in different states. We can see that for the age 80 cohort, the income in the D state is higher than in the MD state because we observe that the probability of death in the D state is significantly higher than in the MD state after age 85, that is $q_i^3(t) > q_i^4(t)$ for $t > 5$. The very high income payments in the D state are because of the very high mortality rate $q_i^3(t) > 0.2$ after age 90 in the D state, which distributes very high mortality credits to that cohort and pays very high income from the remaining balance due to the small annuity factor. The expected future life at age 80 in the D state is 7.54 years, so the probability that an individual keeps getting income payments in the D state for the next 20 years is small.



(a) Mean relative income payments

(b) Mean relative balances

Figure 9: Individuals joined at age 80 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1, 1, 1\}$).

4.4 Risk-sharing, Open Pool, and State-dependent Accumulation Factors

We now wish to distribute more of the mortality credits to the members in a less healthy and more disabled state, so we assume that the state-dependent accumulation factors a_i^h are proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$ for $h = 1, 2, 3, 4$, every individual i, and every time period. Therefore, apart from Mechanisms 2 and 3, Mechanism 1 also starts to play a role in the health-state-dependent income payments. The rest of the assumptions are the same as in Subsection 4.3.

4.4.1 Numerical Illustration of Determining State-dependent Accumulation Factors

We perform a simple illustration of how the state-dependent accumulation factors are determined with a set of indexes that the accumulation factors are proportional to. We have the following known information:

- The interest rate is 3%, so $a_i = 1.03$.
- The accumulation factors are proportional to $\{1, 1.01, 1.02, 1.08\}$.

If we look at a male aged 60 in the H state in the year 2022, the one-year transition probabilities from H state to H, D, M, MD, and Dead states are $\{0.9409, 0.0519, 0.0030, 0.0008, 0.0034\}$.

Using Equation (10), the state-dependent accumulation factors are calculated to be:

$$\begin{cases} a_i^1(t+1) = 1.0293, \\ a_i^2(t+1) = 1.0396, \\ a_i^3(t+1) = 1.0499, \\ a_i^4(t+1) = 1.1117, \end{cases}$$

which are proportional to $\{1, 1.01, 1.02, 1.08\}$. This is fair because:

$$1.0293 \times 0.9409 + 1.0396 \times 0.0519 + 1.0499 \times 0.0030$$
$$+1.1117 \times 0.0008 + 1.03 \times 0.0034 = 1.03.$$

Similarly, if we look at a male aged 60 in the D state in the year 2022, the one-year transition probabilities from D state to H, D, M, MD, and Dead states are $\{0.1343, 0.0208, 0.7950, 0.0226, 0.0273\}$. Using Equation (10), the accumulation factors are calculated to be:

$$\begin{cases} a_i^1(t+1) = 1.0114, \\ a_i^2(t+1) = 1.0215, \\ a_i^3(t+1) = 1.0316, \\ a_i^4(t+1) = 1.0923, \end{cases}$$

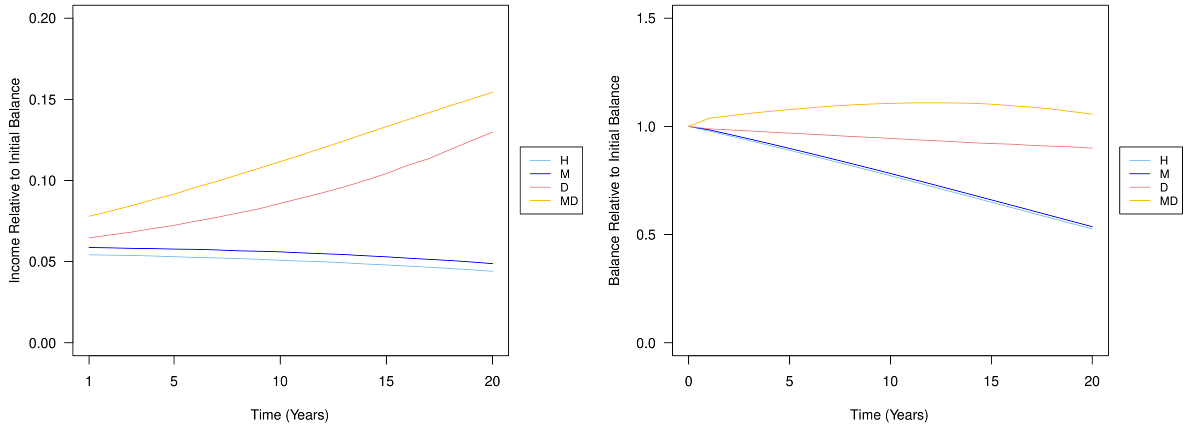
which are also proportional to $\{1, 1.01, 1.02, 1.08\}$. This is fair because:

$$1.0114 \times 0.1343 + 1.0215 \times 0.0208 + 1.0316 \times 0.7950$$
$$+1.0923 \times 0.0226 + 1.03 \times 0.0273 = 1.03.$$

From the above illustration, we can see that the state-dependent accumulation factor depends on the initial state at the beginning of the year and also the state at the end of the year. Following the same procedure, we thus determine the state-dependent accumulation factors for each member at different ages and in different states at the end of each period.

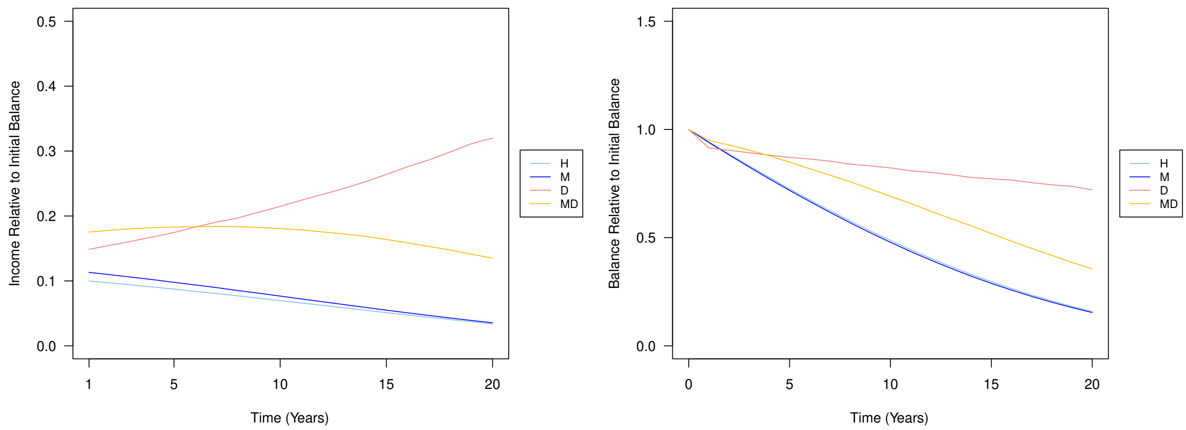
4.4.2 Comparison of Incomes and Balances in Different Health States

We wish to study how the mean relative income payments and mean relative balances will change when the accumulation factors change from state-independent to state-dependent. Figure 10 shows the mean relative income payments and mean relative balances of the aged 60 join at time 0 cohort in different states since time 1 with the accumulation factors proportional to $\{1, 1.01, 1.02, 1.08\}$. We can see that compared with when the accumulation factors are proportional to $\{1, 1, 1, 1\}$ as in Figure 8, the income payments in the MD and D states are higher in Figure 10(a), and the balances are consumed slower in Figure 10(b). The same observation can be found for the aged 80 cohort when we compare Figure 11 with Figure 9.



(a) Mean relative income payments (b) Mean relative balances

Figure 10: Individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$).



(a) Mean relative income payments (b) Mean relative balances

Figure 11: Individuals joined at age 80 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$).

Table 2 presents the ratio of the sum of the expected present value (EPV) of income payments between time 0 and 20 discounted back to time 0 over the initial contribution for different cohorts with state-independent accumulation factors or state-dependent accumulation factors proportional to $\{1, 1.01, 1.02, 1.08\}$. We can see that with the state-dependent accumulation factors, the sum of expected present income payments is lower in the H state, and higher in the D and MD states for both 60 and 80 cohorts joining at time 0. We can also see that the change in the H and M states are at a relatively low level at around 1%, while the change in the D state is around $4\% - 5\%$ and the change in the MD state is around $13\% - 16\%$, indicating that with the proposed health state-dependent risk-sharing rule, the income in the less healthy state can be increased by a significant amount at a relatively small cost of the healthy members.

Table 2: Sum of the expected present value of income payments relative to initial contribution for cohorts in different health states with state-independent accumulation factors or state-dependent accumulation factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$.

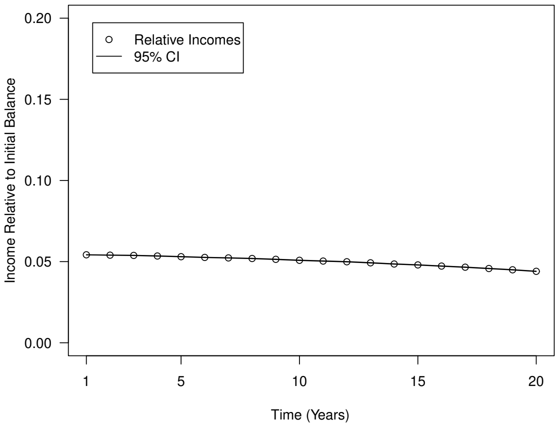
Age 60\Health States	H	M	D	MD
State-independent	0.7587	0.8223	1.2597	1.4169
State-dependent	0.7531	0.8248	1.3058	1.6464
Change in Percentage	-0.74%	0.30%	3.65%	16.20%
Age 80\Health States	H	M	D	MD
State-independent	1.0687	1.1688	3.0665	2.2599
State-dependent	1.0547	1.1673	3.2214	2.5644
Change in Percentage	-1.32%	-0.12%	5.05%	13.48%

We also want to emphasise that the ratios c_i^h to which the age-dependent accumulation factors are proportional can be specific to any individual i, although we currently set them to be the same for every pool member and the same across all time periods. For example, we can set the age $60 - 69$ cohort to have the accumulation factors proportional to $\{1, 1.005, 1.01, 1.04\}$, while the accumulation factors for the aged $70 - 80$ cohort to be proportional to $\{1, 1.01, 1.02, 1.08\}$. To be more specific, in theory, we can let every individual i have their own proportion index c_i^h, which can also vary for different time periods.

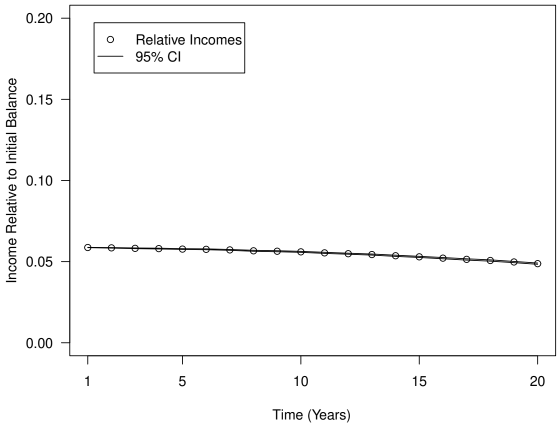
Another thing to notice is that an individual can withdraw any amount of the remaining account balance after risk sharing and benefit payment because the fund is actuarially fair at any point in time and at the individual level. Therefore, if one individual has an emergent liquidity requirement, or finds it too risky if they are in the MD state with a very high probability of death but also a high account balance which could be all lost if they die in the next period, they can partially or fully withdraw the remaining account balance.

4.4.3 Confidence Intervals

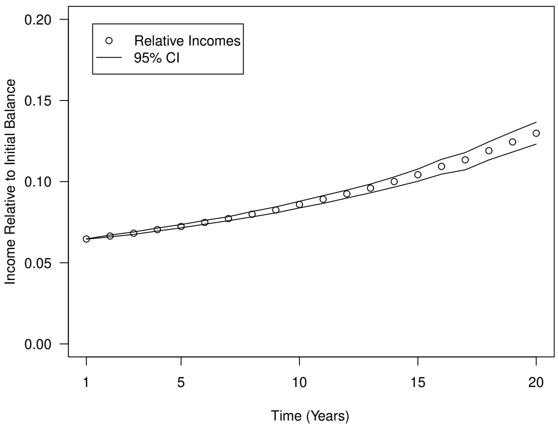
We now study how stable the income payments are with the assumptions in Subsection 4.4. We run 100 simulations and plot the 95% CIs for the income payments in H, M, D, and MD states for age 60 and 80 cohorts respectively in Figures 12 and 13. We can see that the CIs in the H and M states are quite narrow compared with those in the D and MD states. This is because of the lower benefit from the total mortality credits in the H and M states, which is where volatility mainly comes from. Moreover, it can be observed that the lower CI in the D and MD states can still provide higher income payments than in the H and M states, indicating that even in the low-income year either because of systematic change in transition rate or because of less people die than expected, the less healthy and more disabled members can still receive some level of protection for the higher income payments to cover their higher LTC needs.



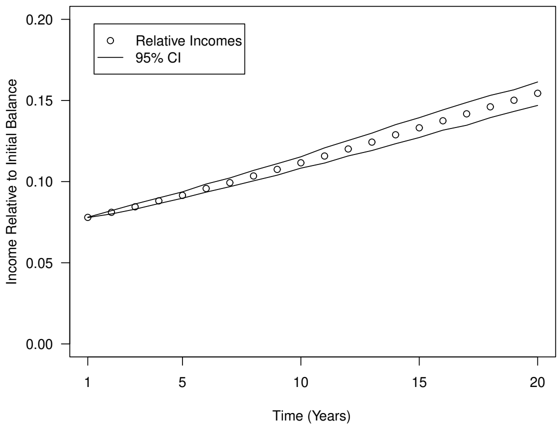
(a) H state



(b) M state

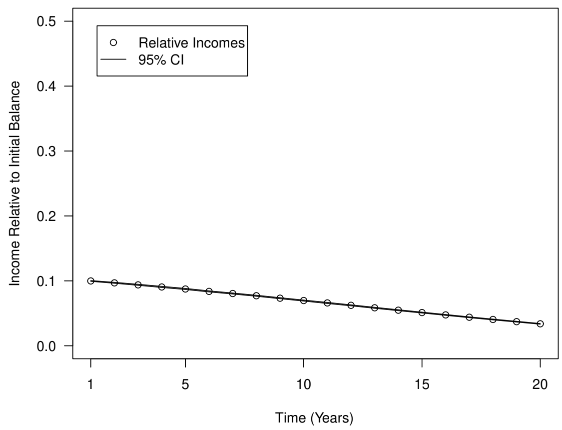


(c) D state

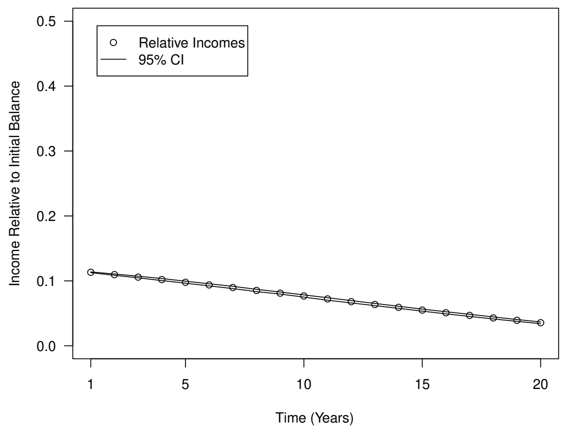


(d) MD state

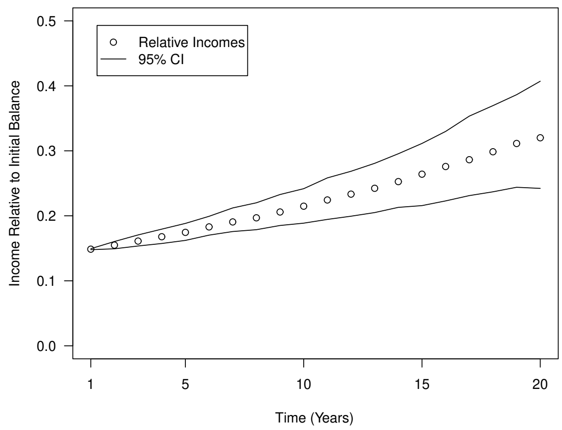
Figure 12: Relative income payments: 95% confidence interval for individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$).



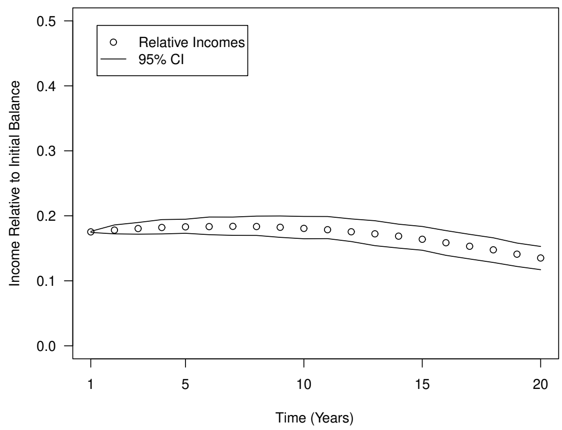
(a) H state



(b) M state



(c) D state



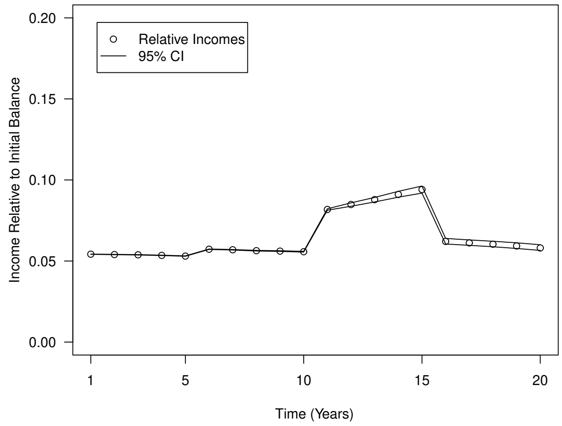
(d) MD state

Figure 13: Relative income payments: 95% confidence interval for individuals joined at age 80 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$).

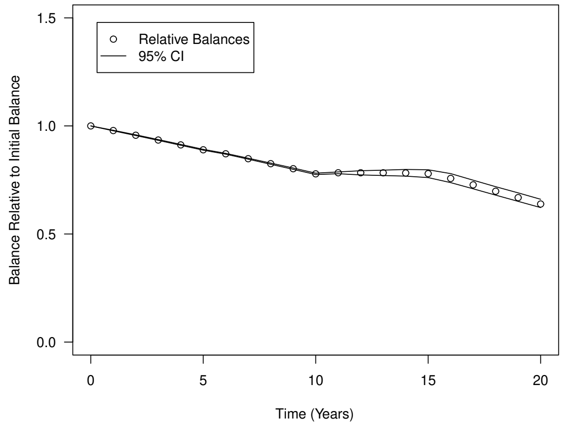
4.4.4 Multiple Transitions

Furthermore, we study the income payments when multiple transitions between health states happen. We observe an individual aged 60 at time 0 and initially healthy. Then, the individual transits into State M at time 6, stays until time 11 then transits to State MD, recovers from the MD state back to the M state at time 16, and stays in State M until time 20.

The mean income payments and balances relative to the initial balance of this individual are displayed in Figure 14. We can see from Figure 14(a) that the income payment ratio starts at around 5% in the healthy state, then slightly increases when the transition to the M state happens at time 6. At time 11, there is a sudden and significant increment in the income payment ratio to around 8% due to the transition to the MD state, and the income payment ratio keeps increasing after time 11 until time 15 to around 10%. At time 16 when the individual recovers from the MD state to the M state, the income payment rapidly reduces to slightly above 5% to reflect the lower incurred LTC costs. Therefore, the proposed LTC pooling product can quickly adjust to the different needs for LTC costs in different health states.



(a) Mean relative income payments



(b) Mean relative balances

Figure 14: Individuals joined at age 60 at time 0 initially healthy, transits to State M at time 6, transit to state MD at time 11, and transits back to state M at time 16 (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.02, 1.08\}$).

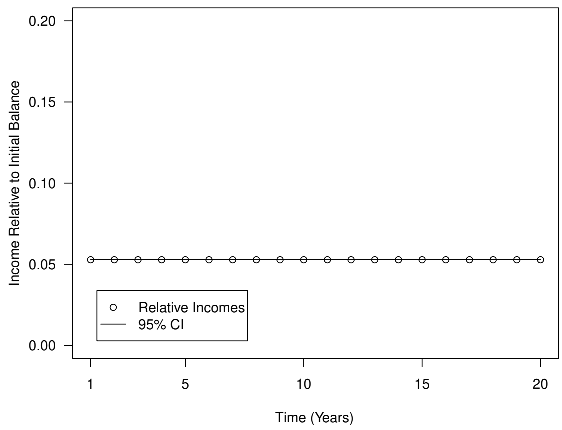
4.5 Level Payments: Risk-sharing, Open Pool, and State-dependent Accumulation Factors

We keep the assumptions as in Subsection 4.4, but we change the decumulation plan from Equation (13) to Equation (14). The predetermined drawdown payments in Equation (14) we use are:

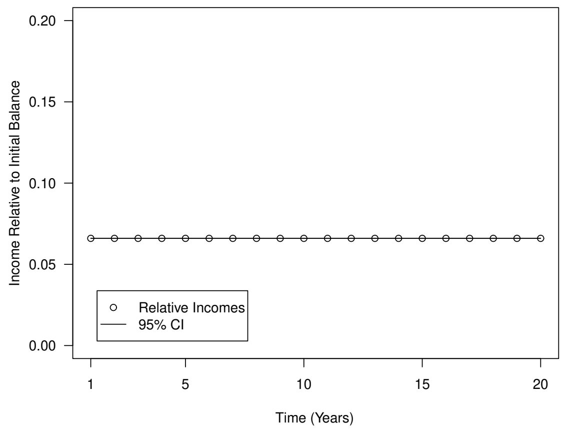
$$B_i^h(t+1)^{Predetermined} = \begin{cases} r_i^{base} F_i(\mathring{t}_i) & \text{if } h_i(t+1) = H, \\ 1.25 r_i^{base} F_i(\mathring{t}_i) & \text{if } h_i(t+1) = M, \\ 3.00 r_i^{base} F_i(\mathring{t}_i) & \text{if } h_i(t+1) = D, \\ 3.25 r_i^{base} F_i(\mathring{t}_i) & \text{if } h_i(t+1) = MD, \end{cases} \tag{15}$$

where $r_i^{base} = \frac{1}{\ddot{a}^H_{x_{i,t_{i,join}}}}$ and $\ddot{a}^H_{x_{i,t_{i,join}}}$ is the annuity due of individual i aged x at the time of joining if in the healthy state, $\mathring{t}_i$ is the initial time of joining of individual i, and $F_i(\mathring{t}_i)$ is thus the initial contribution of individual i. Note that $\ddot{a}^H_{x_{i,t_{i,join}}}$ and $F_i(\mathring{t}_i)$ do not change over time once the age, joining time, and initial contribution of the individual i are known. For example, in one path of the simulation, the annuities due values for individuals who join at time 0 aged 60 and 80 if being healthy are 19.3386 and 10.5082 respectively. This results in the base drawdown rates being equal to 5.2% and 9.5% for those who join at time 0 age 60 and 80 respectively. Then, if one individual who joins at time 0 aged 60 transits into D state, the drawdown rate becomes $3 \times 5.2\% = 15.6\%$. To match the significantly higher drawdown rates in D and MD states compared with annuity-like payments, we also change the proportion of the accumulation factors to $c_i^h = \{1, 1.01, 1.40, 1.45\}$ for $h = 1, 2, 3, 4$, every individual i, and every time period.

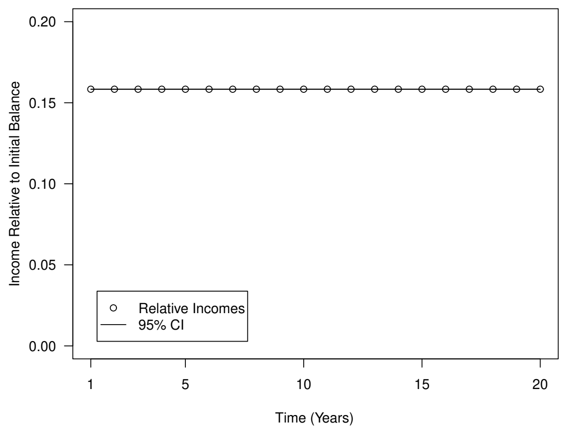
With the settings above, Figure 15 plots the 95% confidence interval of the relative income payments of individuals aged 60 joined at time 0 and then transfers and stays in the H, M, D, and MD states. We can see that with level payments, the income payments are straight lines over time as long as the balance does not completely run out because the risk is absorbed in the remaining balances. The income payments in the M, D, and MD states are $1.25, 3$, and 3.25 times the income payments in the H state as described in Equation (15).



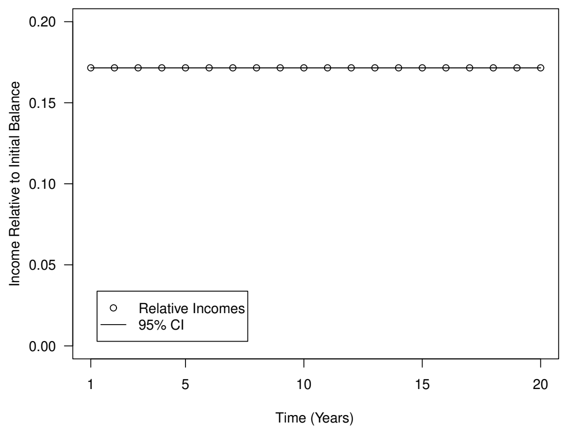
(a) H state



(b) M state



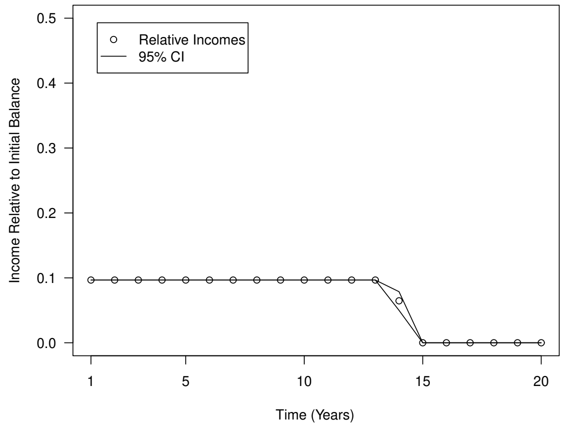
(c) D state



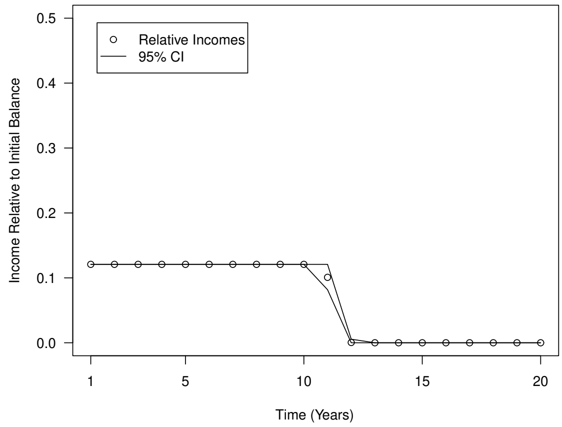
(d) MD state

Figure 15: Relative income payments: 95% confidence interval for individuals joined at age 60 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.40, 1.45\}$, predetermined level payments).

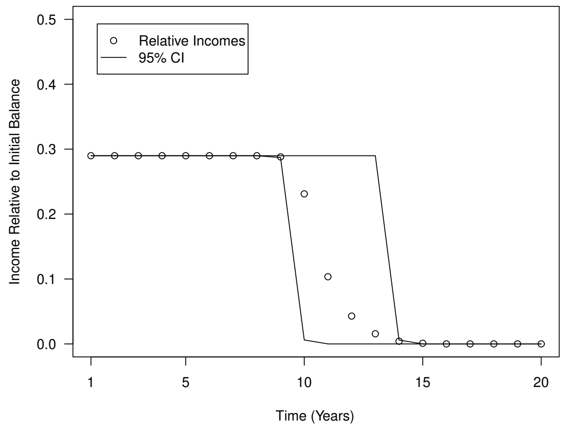
Figure 16 plots the 95% confidence interval of the relative income payments of individuals aged 80 joined at time 0 and then transfers and stays in the H, M, D, and MD states. A similar pattern is observed in that the relative income payments are stable until the balances run out, reflected as the payments drop to zero. We can also see from Figure 16 that for individuals aged 80 joined at time 0, the income payments in the D and MD states are 3 and 3.25 times the payments in the H state but last for a shorter period of time, which corresponds with the shorter life expectancy at age 80 in the D and MD states.



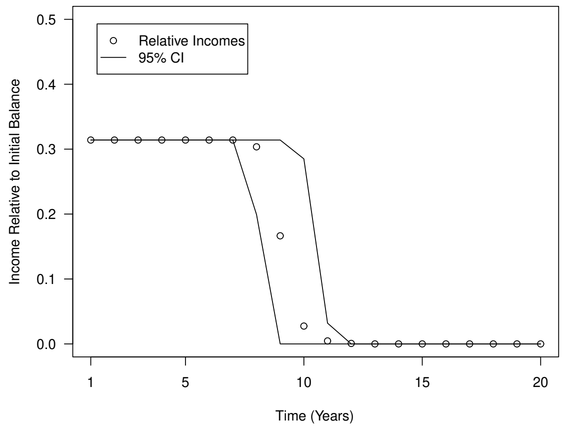
(a) H state



(b) M state



(c) D state



(d) MD state

Figure 16: Relative income payments: 95% confidence interval for individuals joined at age 80 at time 0 initially healthy and transit into different health states (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.40, 1.45\}$, predetermined level payments).

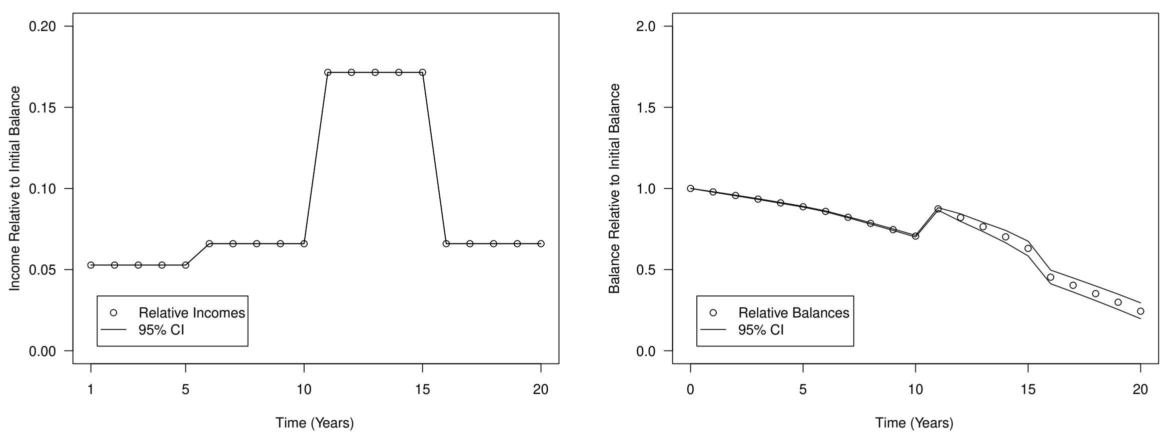
Table 3 summarises the sum of EPV of income payments for aged 60 and 80 members who join at time 0 and stay in H, M, D, and MD states respectively. We can see that the EPV in D and MD states are more than twice the initial contribution for both ages 60 and 80, indicating the effectiveness of hedging against LTC risk. Compared with Table 2, we can see that for aged 60 individuals, the EPV in D and MD states are higher using level income payments compared with annuity-like payments. Meanwhile, for aged 80 individuals, the EPV in D and MD states using level income payments are lower compared with annuity-like payments. This is because level income payments are good at immediately providing a higher income payment in the more unhealthy state since it is a pre-specified amount paid out from the account balance, while annuity-like payments need to wait for the higher proportion from the mortality credits and higher health-state-dependent return to accumulate. However, using level income payments can drain the balance faster, especially in the D and MD states at older age, which explains why the EPVs are lower using level payments. Nevertheless, old individuals in the D and MD states still benefit a lot from this product since they have more than twice their initial contribution.

Table 3: Sum of the expected present value of level income payments relative to initial contribution for cohorts in different health states with state-dependent accumulation factors proportional to $c_i^h = \{1, 1.01, 1.40, 1.45\}$.

Age	Health States			
	H	M	D	MD
60	0.7852	0.9815	2.3557	2.5520
80	1.0704	1.1035	2.5466	2.3482

4.5.1 Multiple Transitions with Level Payments

We also study the case when multiple transitions between health states happen with level payments. Figure 17 displays the mean relative income payments and mean relative balances along with the 95% confidence intervals when the individual is age 60 and joined at time 0, initially in State H, transits to State M at time 6, transits to State MD at time 11, and transits back to State M at time 16. From Figure 17, we observe the pattern that the relative income payments change immediately when a health-state transition happens, and there is no volatility in the income payments as long as the balance does not completely run out.



(a) Mean relative income payments

(b) Mean relative balances

Figure 17: Individuals joined at age 60 at time 0 initially healthy, transits to State M at time 6, transit to State MD at time 11, and transits back to State M at time 16 (Risk sharing with open pool, state-dependent factors proportional to $c_i^h = \{1, 1.01, 1.40, 1.45\}$, predetermined level payments).

5 Conclusions

In conclusion, this paper proposes a general framework for health-contingent mortality pooling products allowing for recovery from disability, heterogeneous members, dynamic pooling, actuarial fairness, and self-sustainability. When a transition to a less healthy state happens, the proposed health-contingent product can immediately provide a higher income payment at the end of the year to cover the higher long-term care (LTC) cost incurred. This rise in income payments persists if the individual stays in this less healthy state. Moreover, if the member recovers from disability, the income payments of the proposed product will decrease to reflect the less required LTC cost in a healthier state.

There are three mechanisms resulting in higher income payments in less healthy states: the proposed health-contingent accumulation, a higher proportion of mortality credits due to the higher probability of death, and higher payout ratios in less healthy states. We examine how the different settings on the risk-sharing pool can affect the income payments and balances. We find that the proposed state-dependent risk-sharing rule is useful in providing much higher income payments to members in a less healthy state to cover their higher LTC cost, at a relatively low cost to the healthy members. Moreover, a larger pool size helps reduce the volatility in the income payments and balances.

Furthermore, the proposed framework has the flexibility to accommodate different types of decumulation plans, while annuity-like payments and level payments are studied in the paper. We find that the payments of the proposed products are stable and can sustain the lifetime of most individuals. For both decumulation plans, the total expected present value of all income payments will be higher for members in the less healthy states. With level payments, people being functionally disabled will receive higher income payments for a shorter period of time, which corresponds with their shorter life expectancy.

6 Acknowledgements

Yuxin Zhou gratefully acknowledges financial support from the Society of Actuaries Center of Actuarial Excellence Research Grant 2016: Longevity Risk: Actuarial and Predictive Models, Retirement Product Innovation, and Risk Management Strategies, the Australian Research Council Centre of Excellence in Population Ageing Research (CEPAR) project number CE170100005, and the 2023 UNSW Business School PhD International Visiting Program. Jan Dhaene gratefully acknowledges funding from FWO and F.R.S.-FNRS under the Excellence of Science (EOS) programme, project ASTeRISK (40007517).

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Appendix 1: Multi-state Health Model Parameters

We use the Cox regression model with trend and frailty, along with the parameters calibrated in Sherris and Wei (2021) using the Health and Retirement Study (HRS) data from 1998 to 2014. The Cox regression model is represented as:

$$\ln\{\lambda_{i,s}(t)\} = \beta_s + \gamma_s^{\text{age}}\, x_i(t) + \gamma_s^{\text{female}}\, F_i + \phi_s w + \alpha_s \psi_w,$$

where

$$\psi_w = \psi_{w-1} + \epsilon_w, \psi_0 = 0.$$

We use $\lambda_{i,s}(t)$ to represent the transition intensity of individual i for transition type s at time t, β_s for the level of transition intensity of transition type s, $x_i(t)$ for the age of individual i at time t, F_i as an indicator of gender for individual i which equals 1 for female and 0 for male, γ_s^{age} and γ_s^{female} for the sensitivity of $\ln\{\lambda_{i,s}(t)\}$ with respect to $x_i(t)$ and F_i, w for the wave of investigation which follows $w = (t-1998)/2+1$, the latent factor ψ_w that is a random walk, and ϵ_w the error term that is standard normal distributed.

The parameters calibrated in Sherris and Wei (2021) are presented in Tables 4 and 5 along with the standard errors in parentheses. The parameters vary by transition types $s = 1, 2, 3, ..., 12$, where each transition type that s corresponds to is presented in the row above. We use *** for p-value< 0.01, ** for p-value< 0.05, and * for p-value< 0.10.

Table 4: Parameters for Cox model with trend and frailty (Part 1).

Transition Type	H→M		H→D		H→MD		H→Dead		M→MD		M→Dead	
$s =$	1		2		3		4		5		6	
β_s	-4.8819	***	-9.8858	***	-12.2858	***	-11.1111	***	-7.2376	***	-9.2753	***
	(0.0160)		(0.0261)		(0.0479)		(0.0252)		(0.0176)		(0.0170)	
γ_s^{age}	0.0254	***	0.0792	***	0.0979	***	0.1039	***	0.0540	***	0.0875	***
	(0.0002)		(0.0003)		(0.0006)		(0.0003)		(0.0002)		(0.0002)	
γ_s^{female}	-0.3234	***	0.2712	***	0.1458	**	-0.5462	***	0.3852	***	-0.2676	***
	(0.0213)		(0.0311)		(0.0573)		(0.0350)		(0.0225)		(0.0252)	
ϕ_s	0.0328	***	-0.0427	***	-0.0908	***	-0.0715	***	-0.0269	***	-0.0643	***
	(0.0035)		(0.0059)		(0.0110)		(0.0057)		(0.0036)		(0.0036)	
α_s	-0.0108		-0.0235		0.0454		-0.0014		-0.0058		-0.0358	**
	(0.0118)		(0.0217)		(0.0402)		(0.0027)		(0.0100)		(0.0153)	

Table 5: Parameters for Cox model with trend and frailty (Part 2).

Transition Type	D→H		D→M		D→MD		D→Dead		MD→M		MD→Dead	
$s =$	7		8		9		10		11		12	
β_s	0.4088	***	-1.9761	***	-4.3012	***	-7.9530	***	-0.0150		-6.2490	***
	(0.0351)		(0.0863)		(0.0603)		(0.0406)		(0.0211)		(0.0195)	
γ_s^{age}	-0.0312	***	-0.0195	***	0.0147	***	0.0741	***	-0.0300	***	0.0591	***
	(0.0005)		(0.0012)		(0.0008)		(0.0005)		(0.0003)		(0.0002)	
γ_s^{female}	-0.0300		-0.1695	*	0.1451	**	-0.4672	***	0.0011		-0.3161	***
	(0.0397)		(0.0982)		(0.0666)		(0.0496)		(0.0022)		(0.0250)	
ϕ_s	-0.0296	***	-0.0691	***	-0.0135		-0.0041		-0.0115	**	-0.0238	***
	(0.0084)		(0.0219)		(0.0119)		(0.0054)		(0.0052)		(0.0040)	
α_s	0.0855	***	-0.0667		0.1024	*	-0.0375		0.1029	***	0.0282	
	(0.0320)		(0.0716)		(0.0540)		(0.0329)		(0.0226)		(0.0173)	

Appendix 2: Outputs of Multi-state Health Model

With the multi-state health model and the parameters specified in Appendix 1, we use the Retirement Income Toolkit available at `https://github.com/RI-Toolkit/rit` to simulate one-year transition probabilities for different initial ages, genders, and calendar years. Tables 6 and 7 display two examples of simulated transition probabilities for USA males in 2022 aged 60 and 80 respectively. We can see that as age increases, the transition probabilities to the H state are lower, and the transition probabilities to the states with functional disability like the D and MD states are higher. The probabilities of death generally increase as age increases and as people move to a less healthy state.

Table 6: One-year transition probability matrix for USA males aged 60 in 2022.

Initial State\New State	H	M	D	MD	Dead
H	0.9409	0.0519	0.0030	0.0008	0.0034
M	0	0.9797	0	0.0120	0.0083
D	0.1343	0.0208	0.7950	0.0226	0.0273
MD	0	0.1245	0	0.8288	0.0467
Dead	0	0	0	0	1

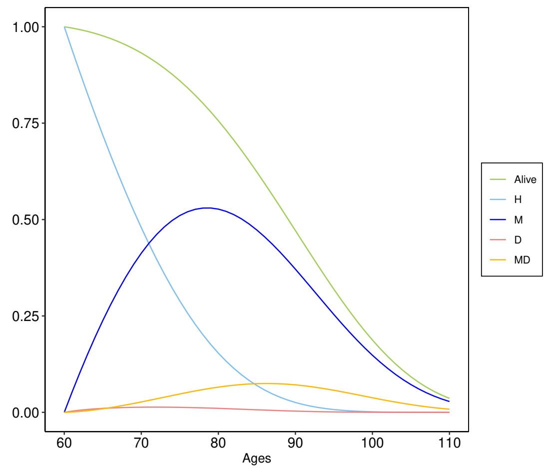
Table 7: One-year transition probability matrix for USA males aged 80 in 2022.

Initial State\New State	H	M	D	MD	Dead
H	0.8649	0.1098	0.0087	0.0032	0.0135
M	0	0.9508	0	0.0256	0.0236
D	0.0643	0.0099	0.7884	0.0340	0.1034
MD	0	0.0776	0	0.7959	0.1265
Dead	0	0	0	0	1

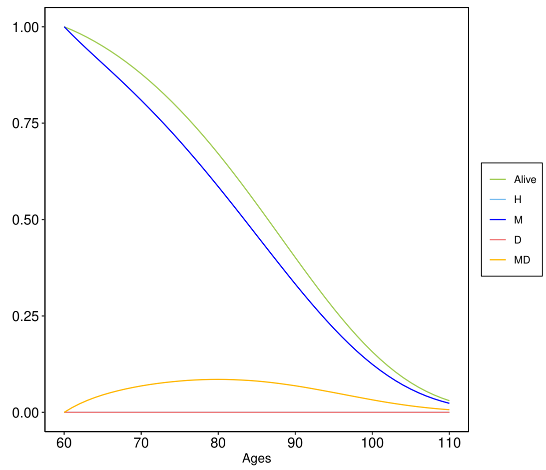
Figure 18 displays the transition probabilities over time of the aged 60 cohort initially in the H, M, D, or MD states in year 2022 to all the health states, along with the probability of being alive. From Figure 18(a), we can see that for those aged 60 initially in the healthy (H) state, the number of people in the healthy (H) state steadily decreases, the number of people with morbidity (M) increases until around age 80 and then decreases, and the number of people in the D state only slightly increases because most of the people have functional disability also have morbidity and thus go into the MD state resulting in a peak at around age 90. From Figure 18(b), we can see that for those aged 60 initially in the M state, since recovery from morbidity is not allowed, people cannot enter the H and D states with no morbidity. The number of people entering the MD state has a peak around age 80.

Figure 18(c) also displays the aged 60 cohort in 2020 but the initial health state is the D state with functional disability. Since recovery from functional disability is allowed, some people move back to the healthy (H) state, with a peak around age 68. Meanwhile, some people move to the M state, which can be a direct transition from the D state to the M state, recovery from the D state to the H state and then transit to the M state, going to the MD state first and then transiting to the M state, or any number of transitions as long as allowed in the model. The peak of people in the M state is around age 80. Moreover, we can also see some proportion of people in the MD state that keeps increasing until around age 65 then stays at a similar level and starts decreasing after around age 90.

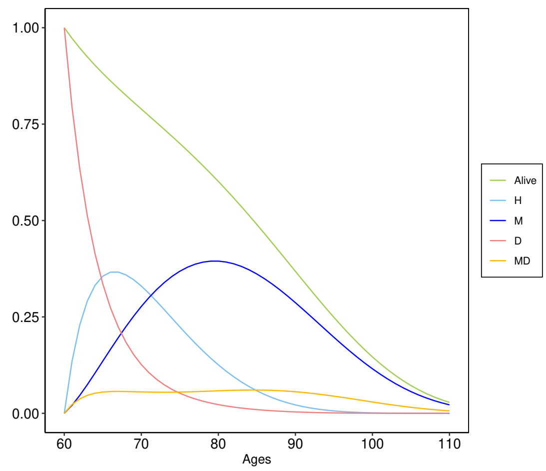
Finally, Figure 18(d) displays the aged 60 cohort in 2020 and initially in the MD state. We can see from Figure 18(d) that people can only transit between the MD state and the M state given initially in the MD state, except for the dead state. The survival probability decreases the fastest when the individual is initially in the MD state, compared with the H, M, and D states shown in Figure 18(a), Figure 18(b), and Figure 18(c) respectively. The number of people remaining in the MD state steadily decreases, while the number of people in the M state reaches a peak at around age 70.



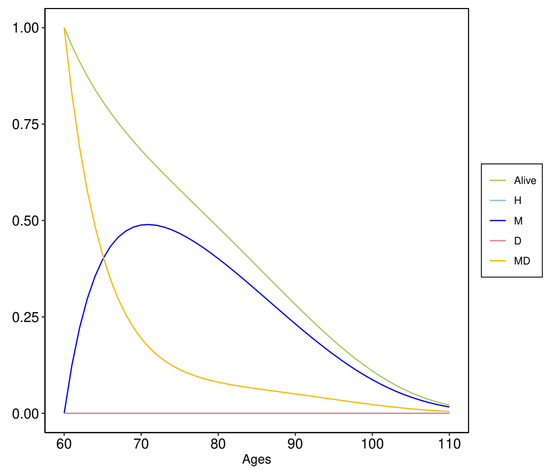
(a) H state



(b) M state



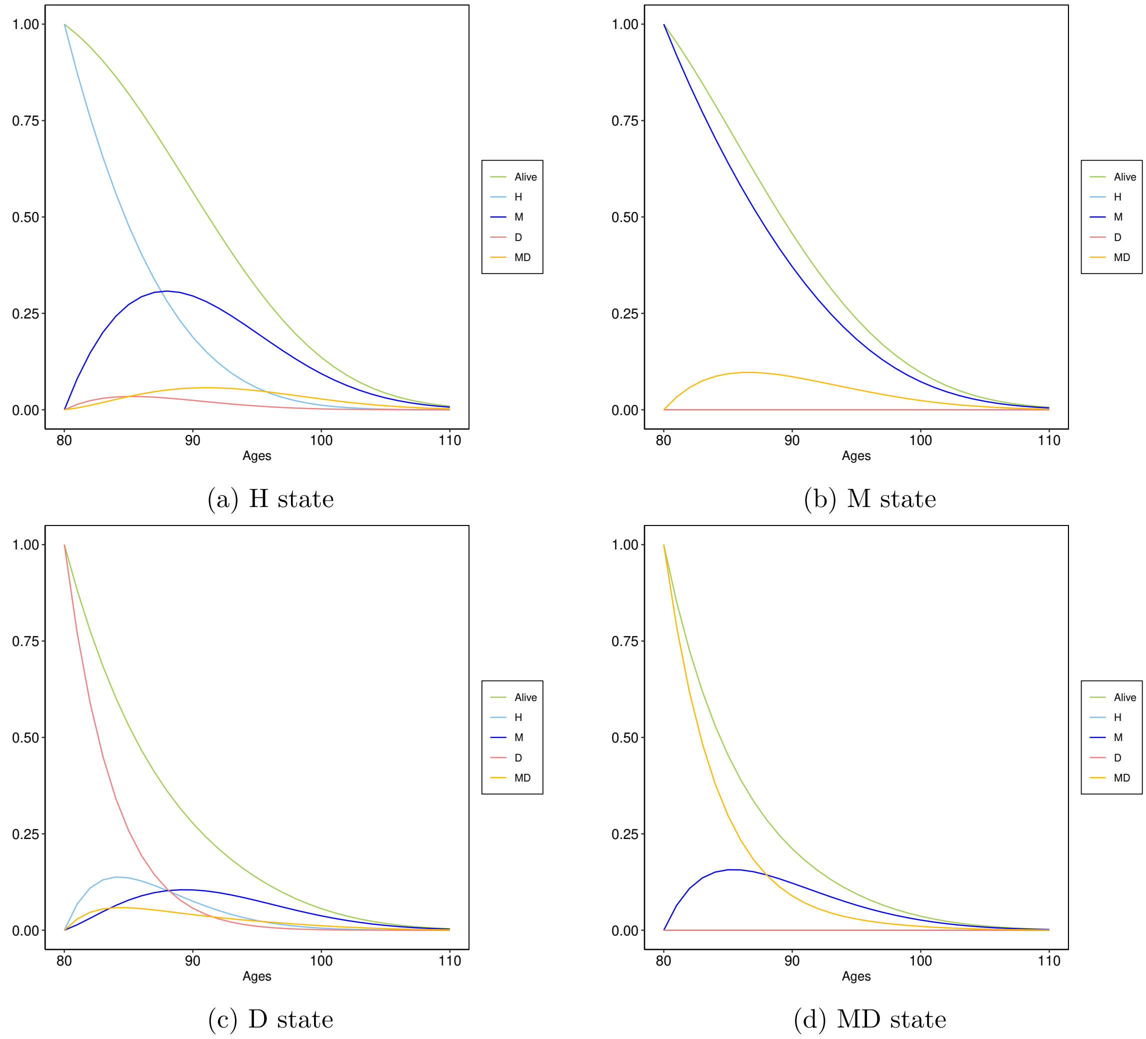
(c) D state



(d) MD state

Figure 18: Transition probabilities to different health states for aged 60 cohort in 2022 initially in the H state, M state, D state, or MD state.

Figure 19 shows the transition probabilities for the aged 80 cohort in 2022 initially in the H, M, D, and MD states. One common property we can observe by comparing the aged 80 cohort with the aged 60 cohort is that the survival probabilities reduce much faster compared with the aged 60 cohort. Then, comparing Figure 19(a) with Figure 18(a), we can see that the aged 80 cohort transits into the M, D, and MD states sooner than the aged 60 cohort. The proportion of people in the M, D, and MD states is not necessarily higher because more aged 80 people transit into the dead state. Comparing Figure 19(b) with Figure 18(b) for people initially in the M state, we can see that more aged 80 people transit into the MD state and the peak also comes faster. Then, from Figures 19(c) and 19(d), we can see that the mortality rates for the aged 80 cohort initially with functional disability and with or without morbidity are quite high, so the survival probabilities are decreasing rapidly.



(a) H state

(b) M state

(c) D state

(d) MD state

Figure 19: Transition probabilities to different health states for aged 80 cohort in 2022 initially in the H state, M state, D state, or MD state.

Table 8 shows the values of the annuity due for people aged from 60 to 100 in different health states in 2022. As time progresses, the same table of annuity due values is recalculated with the updated transition probabilities, which is used in the calculation of benefit payments in Equation (13). For example, the 20-year ahead estimated annuity due values in 2042 are displayed in Table 9. From Table 8 and Table 9, we can see that the annuity due values increase over time in all four health states, which reflects mortality improvements.

Table 8: Annuity due values for people aged from 60 to 100 in H, M, D, and MD health states in 2022.

Age	Health States			
	H	M	D	MD
60	19.34	18.09	16.77	14.73
61	18.95	17.65	16.22	14.23
62	18.51	17.18	15.65	13.74
63	18.10	16.81	15.13	13.26
64	17.68	16.32	14.64	12.76
65	17.26	15.87	14.13	12.26
66	16.80	15.41	13.62	11.83
67	16.40	14.97	13.16	11.38
68	15.91	14.57	12.65	10.94
69	15.47	14.11	12.14	10.48
70	15.05	13.68	11.67	10.07
71	14.60	13.26	11.17	9.67
72	14.20	12.80	10.68	9.28
73	13.73	12.37	10.24	8.89
74	13.31	11.96	9.80	8.53
75	12.85	11.56	9.41	8.18
76	12.41	11.13	8.99	7.85
77	11.99	10.75	8.55	7.53
78	11.56	10.32	8.19	7.21
79	11.16	9.94	7.83	6.92
80	10.76	9.55	7.45	6.63
81	10.35	9.21	7.13	6.36
82	9.94	8.81	6.76	6.10
83	9.55	8.45	6.47	5.86
84	9.15	8.15	6.19	5.63
85	8.77	7.81	5.89	5.41
86	8.40	7.48	5.63	5.20
87	8.04	7.17	5.36	5.00
88	7.70	6.88	5.13	4.81
89	7.35	6.59	4.90	4.63
90	7.03	6.31	4.69	4.46
91	6.71	6.04	4.48	4.30
92	6.41	5.77	4.28	4.15
93	6.11	5.54	4.12	4.01
94	5.83	5.30	3.95	3.88
95	5.56	5.08	3.79	3.75
96	5.30	4.86	3.64	3.63
97	5.05	4.65	3.50	3.51
98	4.82	4.46	3.37	3.41
99	4.59	4.27	3.25	3.30
100	4.38	4.10	3.14	3.21

Table 9: Annuity due values for people aged from 60 to 100 in H, M, D, and MD health states in 2042.

Age	Health States			
	H	M	D	MD
60	21.29	20.45	17.50	16.62
61	20.89	20.08	17.01	16.11
62	20.53	19.67	16.42	15.61
63	20.12	19.28	15.95	15.11
64	19.70	18.86	15.38	14.61
65	19.28	18.43	14.85	14.13
66	18.92	17.98	14.38	13.64
67	18.50	17.57	13.87	13.15
68	18.06	17.12	13.33	12.67
69	17.64	16.69	12.81	12.21
70	17.20	16.30	12.26	11.76
71	16.81	15.81	11.83	11.31
72	16.40	15.41	11.36	10.87
73	15.94	14.97	10.88	10.48
74	15.52	14.52	10.38	10.05
75	15.10	14.08	9.93	9.63
76	14.68	13.64	9.52	9.27
77	14.24	13.25	9.12	8.89
78	13.78	12.83	8.65	8.52
79	13.36	12.42	8.27	8.18
80	12.95	12.00	7.93	7.84
81	12.53	11.58	7.54	7.53
82	12.13	11.19	7.16	7.22
83	11.72	10.79	6.85	6.93
84	11.31	10.41	6.55	6.65
85	10.89	10.03	6.25	6.38
86	10.49	9.63	5.94	6.12
87	10.09	9.29	5.68	5.88
88	9.72	8.91	5.42	5.65
89	9.33	8.59	5.18	5.43
90	8.96	8.21	4.95	5.22
91	8.59	7.92	4.72	5.02
92	8.25	7.59	4.51	4.83
93	7.89	7.28	4.32	4.65
94	7.56	6.97	4.14	4.48
95	7.23	6.67	3.96	4.32
96	6.90	6.39	3.79	4.17
97	6.57	6.11	3.65	4.02
98	6.27	5.85	3.51	3.88
99	5.98	5.59	3.37	3.75
100	5.69	5.33	3.25	3.63