ARC Centre of Excellence in Population Ageing Research

Working Paper 2024/12

Assessing Public Pensions Using Ruin Probability: Pay-As-You-Go versus Mixed Schemes

Jennifer Alonso-García, M. Carmen Boado-Penas, and Julia Eisenberg

---

Assessing Public Pensions Using Ruin Probability: Pay-As-You-Go versus Mixed Schemes

Jennifer Alonso-García[1], M. Carmen Boado-Penas*[2], and Julia Eisenberg[3]

[1]Department of Mathematics, Université Libre de Bruxelles, Belgium , ARC Centre of Excellence in Population Ageing Research (CEPAR), UNSW Sydney, Netspar, The Netherlands
[2]Department of Actuarial Mathematics and Statistics, Heriot-Watt University and the Maxwell Institute for Mathematical Sciences, Edinburgh, UK
[3]Research Unit Risk Management in Financial and Actuarial Mathematics, TU Wien, Austria

Abstract

Pay-as-you-go (PAYG) pension schemes are heavily affected by demographic risks. To mitigate the financial burden, mixed pension schemes that combine elements of funding and PAYG have been proposed. In this paper, we introduce a mixed scheme framework designed for a shrinking working-age population given a specific level of pension expenditure. We evaluate its performance using both the one-year ruin probability and the Value at Risk of the accumulated deficits over time. We also examine the implications of guaranteeing a return of zero on the investments within the funding scheme. Furthermore, we explore the creation of a buffer fund that invests part of the capital in the financial markets, thereby alleviating the financial pressures of the PAYG part. Our findings indicate that, although the proposed mixed framework does not hedge against demographic risk, it enhances the financial health of the system, delaying the need for pension reforms as a result.

Keywords: public pensions, demographic risks, investment, sustainability, ruin probability, value-at-risk, investment

JEL: G22, G52, H55, J26

2020 Mathematics Subject Classification: 91G05, 91G80, 60J70

*Corresponding author: c.boado@hw.ac.uk

1 Introduction

Public pension systems are usually financed on a Pay-As-You-Go (PAYG) basis, where pensions for retirees are paid by the contributions of current workers. This scheme requires an equilibrium between the benefits paid to retirees and the contributions made by the working-age population. In practice, surplus funds generated during periods of economic prosperity are often set aside to create a buffer fund, which is used to cover potential future cash shortfalls.[1]

Population ageing is accelerating at an unprecedented rate. While the total populations in some countries are declining, the global population is experiencing a significant increase in the proportion of older individuals. In 2020, for the very first time in history, the number of people aged 60 and over surpassed the number of children younger than five. By 2050, the world's population of people aged 60 and over is expected to double (United Nations, 2023).

Ageing is the result of demographic trends in life expectancy and fertility. In OECD countries, life expectancy at age 65 is projected to increase by 3.9 years for women and 4.5 for men by 2065 while current fertility rates of 1.67 are well below the level that ensures population replacement.[2] These demographic trends significantly impact population structure, with an old-age to working-age ratio – defined as the number of people over 65 per 100 working-age individuals (ages 20 to 64) – rising from 20 in 1990 to 30 in 2020 across OECD countries. This ratio is expected to reach a value of 53 in the next 30 years (OECD Publishing, 2021).

The COVID-19 pandemic has caused by far the largest shock to European economies since World War II, significantly impacting employment across many countries onwards. Under a pre-pandemic scenario, approximately 5 million jobs were projected to be created in Europe over the next decade; however, the pandemic has reduced this value by up to one and a half million, see Ando et al. (2022).[3] This situation worsens the financial sustainability of the PAYG scheme with fewer contributors financing each retiree. While the financial consequences of an increase in life expectancy on pension sustainability are gradual and long-term, the decline in employment rates poses immediate risks to the equilibrium of the scheme.

The decreasing ratio of workers to pensioners puts pressure on existing pension systems and demands new approaches to guarantee long-term sustainability in a changing world. In Europe the common trend of the pension crisis is a wave of parametric adjustments such as increases in the retirement ages or a decrease in pension indexation, amongst others (see Whitehouse (2009a,b) and OCDE (2011); OECD (2013, 2012, 2017). Others, however, have implemented structural reforms and adopted non-financial defined

[1] See Stewart and Yermo (2009) for specific examples and pension fund governance.

[2] In developed countries, about 2.1 children per woman are needed to maintain a stable total population (OECD Publishing, 2021).

[3] For further financial consequences of the lockdown on the economy see Boado-Penas et al. (2022) and Caulkins et al. (2022).

contribution schemes that reproduce the logic of a funded defined contribution plan but under a PAYG framework. Examples of this approach include Italy, Latvia, Poland and Sweden.[4] Other countries such as Australia, Canada, Norway, Sweden, Latvia and Poland, tackle these challenges by combining funded and PAYG elements within the mandatory pension system. In particular, Sweden allocates 86.5% of the pension contributions to PAYG whereas Latvia and Poland allocate 70% and 62.6% respectively. The remaining part accrues funded pension rights and earns the market rate of return, which is in general greater than the PAYG rate of return, particularly in countries where the working population is not growing. Therefore, it could be argued that funded schemes should be preferred on a mean return basis only (see Boado-Penas et al. (2020)).

However, when variability of returns is considered, the choice between PAYG and funding less obvious and there might be advantages of mixing PAYG and funded schemes (Persson, 2002; De Menil et al., 2006). Indeed, Dutta et al. (2000); Devolder and Melis (2015), and Alonso-García and Devolder (2016) show that diversification benefits arise under a mean-variance setting. Boado-Penas et al. (2021) propose an alternative solution where the deficit of the scheme is covered by the state (sponsor of the plan) but in return the individuals have to invest an amount of money into a fund. This investment is designed to repay the deficit at a particular level of probability and provide, in expectation, some gains to contributors.

While pension reforms involve an inescapable trade-off between sustainability and adequacy, it is essential to address the challenges of demographic ageing by adopting measures that ensure a decent income in retirement.

The aim of this paper is to assess whether a mixed pension scheme could achieve financial equilibrium in the PAYG component - exposed to the demographic risk - while also ensuring that retirees receive a level of pension benefits comparable to those of a pure defined benefit PAYG scheme. Specifically, we analyse whether the expected returns generated by the fund of the non-PAYG component could, in expectation, fulfill the payment promises of the PAYG component. We present the ruin probability of the mixed scheme in presence and absence of a buffer fund that allows to lock away potential surpluses of the scheme. Then, we extend our analysis to incorporate both the timing and severity of the ruin by considering the value at risk of the accumulated deficit.

Following this introduction, the next section describes the mathematical modelling of a pure PAYG and our mixed pension framework. Section 3 presents the risk measures used to evaluate the effectiveness of the mixed scheme compared to the pure PAYG. Section 4 provides a numerical illustration for a stylized population, based on realistic long-term European ageing trends. Finally, Section 5 concludes.

[4] See Holzmann et al. (2012).

2 Mathematical modelling

Our model seeks to mitigate the impact of a declining working population by combining PAYG financing - where income from contributions finance pension expenditures - with funding, where assets are invested in financial markets. In this paper, all random variables and processes live on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For any process $\{Y_t\}$ we write $\mathbb{E}_y[.]$ for $\mathbb{E}[.|Y_0 = y]$ and $\mathrm{Var}_y[.]$ for $\mathrm{Var}[.|Y_0 = y]$.

2.1 Pure PAYG pension framework

In a balanced PAYG, the total income from contributions C_t must be sufficient to cover the pension expenditures P_t within the same year, that is,

$$C_t = P_t. \tag{2.1}$$

In particular, rewriting the total income from contribution with respect to the contribution rate and the aggregate wage,[5] the following relationship needs to be satisfied:

$$\pi_t \cdot S_t = P_t, \tag{2.2}$$

where π_t is the contribution rate at time t and S_t the aggregate wage of the working population at time t.
Mathematically, the total pension expenditures and aggregate wages are defined as follows:

$$P_t = \bar{p}_t \cdot r_t,$$
$$S_t = \bar{s}_t \cdot w_t,$$

where $\bar{p}_t$ and $\bar{s}_t$ represent the average pension and salary at time t, respectively, while r_t denotes the number of $r$etirees and w_t the number of $w$orkers at t.

Equation (2.2) shows that the contribution rate should satisfy

$$\pi_t = \frac{P_t}{S_t} = \frac{\bar{p}_t}{\bar{s}_t}\frac{r_t}{w_t} = BR_t \cdot DR_t, \tag{2.3}$$

where BR_t and DR_t represent the benefit ratio and dependency ratio at time t, respectively. The benefit ratio, BR_t, represents the financial factor and, as defined in the theory of aggregate accounting (Jimeno et al., 2008; Alonso-García and Rosado-Cebrian, 2021), illustrates the relationship between the average pension and salary. The dependency ratio, DR_t, is the demographic factor influenced by factors such as fertility, unemployment, and longevity.

As shown in Equation (2.3) when targeting a specific benefit ratio BR and maintaining a fixed number of retirees, r, a decrease in the number of workers, w_t, results in an

[5] In this paper, we use the terms wage, salary and contribution base interchangeably.

increase in the dependency ratio, DR_t. Consequently, the required contribution rate, π_t would need to rise to ensure the financial equilibrium of the scheme.

In this paper, we assume that the government aims to maintain a constant level of total pension expenditures, denoted as $P_t = P$, and a target level of benefit ratio, BR while S_t decreases over time as a result of a declining working population w_t. In our framework, the number of contributors is modelled through $w = \{w_t\}$, following an Ornstein-Uhlenbeck process. This means that the number of contributors at time t is represented by

$$w_t = w_0 \cdot e^{-at} + b(1 - e^{-at}) + \delta \int_0^t e^{-a(t-s)}\, \mathrm{d}W_s^w \ , \tag{2.4}$$

where $W^w = \{W_t^w\}$ is a standard Brownian motion, $a > 0$ is reversion speed, $\delta > 0$ is the volatility and b denotes the so-called long-term mean.

Note that for every t the random variable w_t is normally distributed with the mean

$$\mathbb{E}[w_t] = w_0 \cdot e^{-at} + b(1 - e^{-at}) = (w_0 - b)e^{-at} + b \ , \tag{2.5}$$

and the variance

$$\mathrm{Var}[w_t] = \frac{\delta^2}{2a}(1 - e^{-2at}) \ . \tag{2.6}$$

Therefore, increase in the dependency ratio is driven by a decrease in the working population.

The average salary $\bar{s}$, as well as the contribution rate π are assumed to remain constant throughout the analysed period.[6] From here onwards, $\bar{c} = \pi \cdot \bar{s}$ represents the total individual contribution to the pension scheme based on the individual wage $\bar{s}$.

2.2 A mixed pension framework

Under a mixed pension scheme the total contribution rate π is divided into two parts: a portion θ, which is invested into a risky fund F (funded component), and the remaining portion $1 - \theta$ which is allocated to the PAYG scheme.

To model the evolution of the fund, we use a geometric Brownian motion. The value of the fund at time t is then given by

$$F_t = F_0 e^{\mu t + \sigma W_t^f} \ , \quad F_0 = 1 \ , \tag{2.7}$$

[6] An alternative scenario where the working population remains stable while the retired population increases would produce similar results with respect to the dependency ratio. In the mathematical model of this paper, this adjustment would involve relocating the random component within the study of ruin probability, yet the interpretation would remain similar.

where $\mu, \sigma > 0$ and $W^f = \{W_t^f\}$ a standard Brownian motion. Note that W^f and W^w are independent Brownian motions. Further, we let $\mathbb{F} = \{\mathcal{F}_t\}$ be a filtration generated by (w, F).

At $t = 0$, the investment in the risky fund is represented by $\Upsilon_0 = \theta \cdot \bar{c} \cdot w_0$. The notation Υ has been introduced to distinguish between the new investments and the fund itself, F.

In this paper, we examine, as a baseline scenario, the impact of providing a nominal guarantee on Υ, ensuring a 0% return, so that the financial risk is not transferred to the individual, who would only benefit from positive returns.[7] We also consider the scenario without the guarantee to evaluate its impact on the financial equilibrium of the mixed scheme.

At the end of each period, we calculate the balance of the scheme, denoted as $R = \{R_t\}$, by comparing the income from contributions (both PAYG and funding) plus the non-negative returns on investments of the funding component with the pension expenditure. If the balance at time t is positive,[8] the surplus can either be placed into a buffer fund $B = \{B_t\}$ at a 0% return or reinvested into the risky fund F.

Mathematically, the discrete (yearly) balance process $R = \{R_t\}$ without a buffer fund is expressed as:

$$R_t := (1 - \theta) \cdot \bar{c} \cdot w_t + \Upsilon_{t-1} \cdot \max\{F_t/F_{t-1}, 1\} - P \ . \tag{2.8}$$

Including a buffer fund, the balance process $R^b = \{R_t^b\}$ is given by:

$$R_t^b := R_t + B_{t-1} = (1 - \theta) \cdot \bar{c} \cdot w_t + \Upsilon_{t-1} \cdot \max\{F_t/F_{t-1}, 1\} + B_{t-1} - P \ . \tag{2.9}$$

Note that the balance process with buffer fund at time t depends on the buffer fund B_{t-1} since the buffer fund evolves at 0% interest rate.[9] We assume that if either R_t or R_t^b is negative, the deficit of the scheme is absorbed by the government's general budget.

3 Risk measures for the pension system

This section introduces two risk measures to compare the two schemes: the 1-year ruin probability and the Value at Risk (VaR) of the deficit at predefined intervals.

[7] This guarantee is significant, especially considering that the inherent return of the PAYG scheme is tied to the growth of the total contribution base, which, in this instance, is negative due to the shrinking working-age population, see Settergren and Mikula (2005).

[8] In practice, the surplus could also be fully absorbed by the government's general budget; however, this alternative is not considered in our analysis.

[9] If the buffer fund would earn a different (deterministic) rate we would write Equation (2.9) as $R_t^b := R_t + B_{t-1}(1 + i^b)$ with i^b the yearly return on the buffer fund.

3.1 1-year ruin probability with and without buffer fund

Proposition 3.1
Let the amount invested in the risky fund F at time t be denoted by Υ_t, i.e. $\Upsilon_t = \theta \cdot \bar{c} \cdot w_t$. The 1-year ruin probability under a mixed pension scheme in absence of a buffer fund is given by:

$$\mathbb{P}[R_1 \leq y | w_0] = \mathbb{E}\Big[\Phi\Big(\frac{P + y - \Upsilon_0 e^{\max\{\mu+\sigma W_1^f, 0\}} - \mathbb{E}_{w_0}[w_1]}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}}\Big)\Big] \;, \tag{3.1}$$

whereas the 1-year ruin probability in presence of a buffer fund is given by

$$\mathbb{P}[R_1^b \leq y | w_0] = \mathbb{P}[R_1 + B_0 \leq y | w_0, \Upsilon_0] \tag{3.2}$$

$$= \mathbb{E}\Big[\Phi\Big(\frac{P + y - \Upsilon_0 e^{\max\{\mu+\sigma W_1^f, 0\}} - B_0 - \mathbb{E}_{w_0}[w_1]}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}}\Big)\Big] \;, \tag{3.3}$$

where Φ denotes the distribution function of the standard normal distribution, R_1 is balance at $t = 1$ as given in (2.8) and $W^f = \{W_t^f\}$ is a standard Brownian motion.

Proof: Starting at time $t = 0$, the random balance at time 1, without a buffer, R_1, is expressed as:

$$R_1 := (1-\theta) \cdot \bar{c} \cdot w_1 + \Upsilon_0 \cdot \max\{F_1/F_0, 1\} - P \;.$$

Since w_1 and F_1 are independent, the claim follows directly by the law of total probability:

$$\begin{aligned}
&\mathbb{P}[R_1 \leq y | w_0, \Upsilon_0] \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \mathbb{P}\big[(1-\theta) \cdot \bar{c} \cdot w_1 + \Upsilon_0 \cdot e^{\max\{\mu+\sigma z, 0\}} \leq P + y\big] \, \mathrm{d}z \\
&\quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \Phi\Big(\frac{P + y - \Upsilon_0 \cdot e^{\max\{\mu+\sigma z, 0\}} - \mathbb{E}_{w_0}[w_1]}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}}\Big) \, \mathrm{d}z \\
&\quad = \mathbb{E}\Big[\Phi\Big(\frac{P + y - F_0 \cdot e^{\max\{\mu+\sigma W_1^f, 0\}} - \mathbb{E}_{w_0}[w_1]}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}}\Big)\Big] \;.
\end{aligned} \tag{3.4}$$

Including a buffer at $t = 0$, denoted as B_0, which earns a 0% return, an applying the same arguments as in (3.4), would result in the following:

$$\mathbb{P}[R_1 + B_0 \leq y | w_0, F_0] = \mathbb{E}\Big[\Phi\Big(\frac{P + y - F_0 \cdot e^{\max\{\mu+\sigma W_1^f, 0\}} - B_0 - \mathbb{E}_{w_0}[w_1]}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}}\Big)\Big] \;.$$

□

Thus, from Proposition 3.1 it is clear that $\mathbb{P}[R_1^b \leq y|w_0] = \mathbb{P}[R_1 + B_0 \leq y|w_0]$ is strictly decreasing in B_0. In other words, including the buffer reduces the probability of a deficit occurring in the scheme.

Remark 3.2
A natural question arises: "Is the buffer fund sufficiently large to cover what the government might anticipate as expected future deficits within a pre-specific horizon, or, should the scheme reinvest into F to achieve a risky higher average return?" For instance, one *might require that the minimal buffer should be the amount that ensures the risk of having a deficit after 1 year at $\alpha \cdot 100\%$ probability:*

$$B_0^{\min} = \inf\left\{x > 0 : \, \mathbb{P}_{w_0}[R_1 < -x] \leq \alpha\right\} . \tag{3.5}$$

If $B_0 > B_0^{\min}$, the difference could either remain in the buffer fund earning 0% return or be invested in the risky asset.

Remark 3.3
In our setting, the state guarantees a 0% interest rate to contributors. Such a guarantee imposes a cost on the state when the financial returns are negative. By offering this guarantee, it is reasonable for the pension scheme (state) to participate in the gains of the funded part while maintaining a basic amount invested in the fund to ensure sufficient money is available to cover expected losses. Let $S_0^{\min}$ denote the annual investment by the state in addition to the contribution to the funded element Υ_0, ensuring that the guaranteed 0% return can be provided at a α probability level:

$$\begin{aligned} S_0^{\min} : &= \inf\left\{x > 0 : \, \mathbb{P}\big[(x + \Upsilon_0) \cdot e^{\mu+\sigma W_1} < \Upsilon_0 \,\big|\, \mu + \sigma W_1 < 0\big] \leq \alpha\right\} \\ &= \inf\left\{x > 0 : \, \mathbb{P}\big[x \cdot e^{\mu+\sigma W_1} < \Upsilon_0 \cdot (1 - e^{\mu+\sigma W_1}) \,\big|\, \mu + \sigma W_1 < 0\big] \leq \alpha\right\} . \end{aligned}$$

As for the participation in the gains, the state may require a percentage q, such that

$$\mathbb{E}\big[q \cdot (e^{\mu+\sigma W_1} - 1)\mathbb{1}_{[\mu+\sigma W_1^f > 0]} + (e^{\mu+\sigma W_1} - 1)\mathbb{1}_{[\mu+\sigma W_1^f \leq 0]}\big] = 0 \, .$$

In other words, the state would participate in the gains at the rate of q when the return is strictly positive, to compensate the event where the returns are negative, aiming for a mean global return of zero.

As we require that $\mathbb{E}[e^{\mu+\sigma W_1^f}] = e^{\mu+\frac{\sigma^2}{2}} > 1$, such a q exists and is strictly smaller than 1.

■

The choice of the parameter θ, representing the portion of contributions paid by scheme participants and invested in the funded part, should not solely depend on the expected financial sustainability of the mixed scheme, but also on the "no loss" condition for the state. For example, as noted in Remark 3.3, the amount invested by the state at time 0, $S_0^{\min}$, to guarantee a 0%-return can be given by:

$$S_0^{\min} := \inf\left\{x > 0 : \, \mathbb{P}\big[x \cdot e^{\mu+\sigma W_1} < \theta \cdot \bar{c} \cdot w_0 \cdot (1 - e^{\mu+\sigma W_1}) \,|\, \mu + \sigma W_1 < 0\big] \leq 0.05\right\} .$$

It is clear that increasing θ also increases $S_0^{\min}$.

Note that $B_0^{\min}$ and $S_0^{\min}$ can be seen as financial indicators reflecting the probability of an unfavourable event occurring. Translating ruin probabilities into monetary values enables us to assess the scheme's viability by comparing these values with the actual buffer fund level B or total income from contributions C_t.

3.2 Minimising the 1-year ruin probability through buffer reinvestment

In this section, we select the probability that the balance at time t R_t^p, including a buffer, will be negative after one year as our risk measure.

Over the course of the first year, the ruin probability depends on the initial value of the buffer B_0, the initial number of the working population w_0, and the percentage of the buffer $p \in [0, 1]$ that is invested in a risky fund with no guaranteed return. The buffer process $\{B_t\}$ at the end of the first year, with a constant investment proportion p, is given by:

$$B_1^p = (1-p) \cdot B_0 + p \cdot B_0 \cdot \frac{F_1}{F_0}\ . \tag{3.6}$$

The balance at time 1, R_1^p, using Equation (2.8) for $t = 1$, (3.6) and $\frac{F_1}{F_0} = e^{\mu+\sigma W_1^f}$, is expressed as follows:

$$\begin{aligned} R_1^p :&= R_1 + (1-p) \cdot B_0 + p \cdot B_0 \cdot e^{\mu+\sigma W_1^f} \\ &= (1-\theta) \cdot \bar{c} \cdot w_1 + \theta \cdot \bar{c} \cdot w_0 \cdot \max\left(e^{\mu+\sigma W_1^f}, 1\right) - P \\ &\quad + (1-p) \cdot B_0 + p \cdot B_0 \cdot e^{\mu+\sigma W_1^f}\ . \end{aligned} \tag{3.7}$$

Note that the portion of the buffer fund invested does not have a nominal guarantee and could therefore incur losses. The following Proposition 3.4 aims to determine when it is more beneficial to invest a part of the buffer into the fund rather than keeping the money frozen at a 0% interest rate.

Proposition 3.4
The 1-year ruin probability linked to the balance level (3.7) *is given by:*

$$\begin{aligned} \psi(B_0, w_0, p) &:= \mathbb{P}_{w_0,B_0}[R_1^p < 0] \\ &= \mathbb{E}\left[\Phi\left(\frac{P - \Upsilon_0 \max\{e^{\mu+\sigma W_1^f}, 1\} - B_0 - B_0 p (e^{\mu+\sigma W_1^f} - 1)}{(1-\theta) \cdot \bar{c} \cdot \sqrt{\mathrm{Var}_{w_0}[w_1]}} - \frac{\mathbb{E}_{w_0}[w_1]}{\sqrt{\mathrm{Var}_{w_0}[w_1]}}\right)\right], \end{aligned} \tag{3.8}$$

where Φ denotes the standard normal distribution function.

Proof: The 1-year ruin probability is denoted by ψ. For the distribution of R_1^p we have with $y \in \mathbb{R}$:

$$
\begin{aligned}
&\psi(B_0, w_0, p) := \mathbb{P}_{w_0,B_0,\Upsilon_0}[R_1^p \leq y] \\
&= \mathbb{E}\Big[\mathbb{P}_{w_0}\Big[(1-\theta)\cdot\bar{c}\cdot w_1 + \Upsilon_0 \max\{e^{\mu+\sigma W_1^f}, 1\} + (1-p)\cdot B_0 + p\cdot B_0\cdot e^{\mu+\sigma W_1^f} - P \leq y | W_1^f\Big]\Big] \\
&= \mathbb{E}\left[\Phi\left(\frac{P + y - \Upsilon_0 \max\{e^{\mu+\sigma W_1^f}, 1\} - B_0 - B_0 p(e^{\mu+\sigma W_1^f} - 1)}{(1-\theta)\cdot\bar{c}\cdot\sqrt{\mathrm{Var}_{w_0}[w_1]}} - \frac{\mathbb{E}_{w_0}[w_1]}{\sqrt{\mathrm{Var}_{w_0}[w_1]}}\right)\right].
\end{aligned}
$$

■

Remark 3.5
Given (3.8), *how does the ruin probability depend on the investment proportion* p*? Note that the expression multiplying* p *is given by* $B_0\big(e^{\mu+\sigma W_1^f} - 1\big)$*. Therefore:*

$$
\begin{aligned}
\psi_p(B_0, w_0, p) :&= \frac{\mathrm{d}}{\mathrm{d}p}\psi(B_0, w_0, p) \\
&= \mathbb{E}\Bigg[-\frac{B_0\big(e^{\mu+\sigma W_1^f} - 1\big)}{(1-\theta)\cdot\bar{c}\cdot\sqrt{\mathrm{Var}_{w_0}[w_1]}} \\
&\quad\times\varphi\left(\frac{P - F_0 \max\{e^{\mu+\sigma W_1^f}, 1\} - B_0 - B_0 p(e^{\mu+\sigma W_1^f} - 1)}{(1-\theta)\cdot\bar{c}\cdot\sqrt{\mathrm{Var}_{w_0}[w_1]}} - \frac{\mathbb{E}_{w_0}[w_1]}{\sqrt{\mathrm{Var}_{w_0}[w_1]}}\right)\Bigg],
\end{aligned}
$$

where φ *is the density of the standard normal distribution.*

Unfortunately, representation (3.8) does not enable the derivation of an explicit expression for the optimal investment strategy. Nonetheless, it is clear how to obtain the optimal proportion p numerically. This involves investigating the properties of the derivative ψ_p and finding its minimum with respect to $p \in [0, 1]$.

Indeed, the sign of $\psi_p(B_0, w_0, p) = \frac{\mathrm{d}}{\mathrm{d}p}\psi(B_0, w_0, p)$ will heavily depend on the combination of the parameters. A possible scenario is discussed in the example below.

Example 3.6
For the parameters $a = 0.055$, $b = 5.56 \cdot 10^6$, $\delta = 35000$, $w_0 = 10^7$, $\sigma = 0.2$, $\mu = 0.02$ *a possible behaviour of the 1-year ruin probability of the balance, is given in Figure 1. Here, it is assumed* $B_0 = 10^8$.
We observe that the investment of the buffer is lucrative in terms of the ruin probability. The lowest ruin probability is achieved when the entire buffer is invested in risky assets. Indeed, our ruin probability is reduced by a third from around 1% to 0.3%.

3.3 VaR of the deficit

A one-year time horizon can be seen as insufficient since it does not take into account changes in the number of contributors and/or potential market shocks. Furthermore,

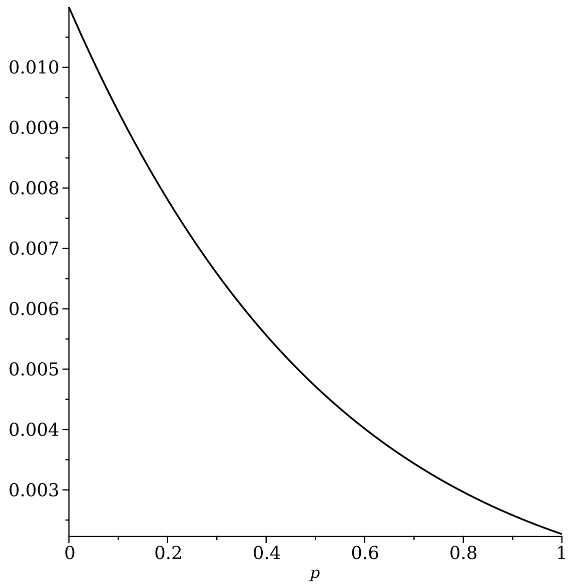


Figure 1: The behaviour of the ruin probability depending on the buffer investment proportion p for $B_0 = 10^8$.

using ruin probability fails to account for both the timing and severity of the ruin. To address these limitations, we consider the value at risk (VaR) of the accumulated deficit over a five-year period – that is $\sum_{i=1}^{5} D_i$ – compared to a one-year period. If the probability that the sum of the deficits of the next 5 years does not exceed a pre-specified level with a certain probability, the pension scheme is considered to be financially sustainable, and no changes of the contribution rate, retirement age or similar need to be introduced.

The Value at Risk (VaR) of the accumulated deficit over a 5-year period at a confidence level $\alpha \in (0, 1)$ can be expressed as follows:

$$\mathrm{VaR}_\alpha = \inf\left\{x :\ \mathbb{P}\Big[\sum_{i=1}^{5} D_i > x\Big] \leq 1 - \alpha\right\}, \tag{3.9}$$

where D_n is the deficit at the end of the n-th year, i.e.,

$$D_n := \max\{-R_n^p, 0\} \text{ for n} \geq 1; \tag{3.10}$$

with R_n^p representing the balance of the scheme at time n, assuming that the percentage p of the buffer to be invested into the fund is fixed at the beginning of the period and cannot be changed thereafter. Mathematically, for $n \geq 2$

$$\begin{aligned} R_n^p =& (1-\theta) \cdot \bar{c} \cdot w_n + \Upsilon_{n-1} \cdot \max\left(e^{\mu+\sigma(W_n^f - W_{n-1}^f)}, 1\right) - P \\ & + (1-p) \cdot B_{n-1} + p \cdot B_{n-1} \cdot e^{\mu+\sigma(W_n^f - W_{n-1}^f)} . \end{aligned}$$

We denote the new buffer at time $n-1$ by:

$$B_{n-1} = \max\{R^p_{n-1}, 0\} \; , \tag{3.11}$$

since we assume that the state covers any deficit whenever R^p_n becomes negative.

Similar to the case where ruin probability is the target functional, determining the optimal proportion p explicitly is not possible. However, since the distributions of a geometric Brownian motion and an Ornstein-Uhlenbeck process are well-known, the numerical calculations are straightforward. For our analysis, we need to determine the distribution of $\sum_{i=1}^{5} D_i$. This is done in Proposition 3.7.

First, we introduce the following notation where we relate the multi-year deficit distribution to the 1-year ruin probability G_R presented in Proposition 3.4. Let

$$G_R(y; w_0, \Upsilon_0, B_0, p) = \psi(B_0, w_0, p)$$

to explicitely bring forward the dependence on the investment Υ and the level of deficit y. With G_R given by ψ in (3.8) we let recursively for $y \in \mathbb{R}_+$:

$$\begin{aligned}
H_1(y; w_0, \Upsilon_0, B_0, p) &:= 1 - G_R(-y; w_0, \Upsilon_0, B_0, p) \; , \\
H_2(y; w_0, \Upsilon_0, B_0, p) &:= \mathbb{E}_{w_0}\Big[1 - G_R(-y - R^p_1 \wedge 0; w_1, \Upsilon_1, R^p_1 \vee 0, p)\Big] \\
&= \mathbb{E}_{w_0}\Big[H_1(y + R^p_1 \wedge 0; w_1, \Upsilon_1, R^p_1 \vee 0, p)\Big] \\
&\;\;\vdots \\
H_n(y; w_0, \Upsilon_0, B_0, p) &:= \mathbb{E}_{w_0}\Big[H_{n-1}(y + R^p_1 \wedge 0; w_1, \Upsilon_1, R^p_1 \vee 0, p)\Big] \; .
\end{aligned} \tag{3.12}$$

Proposition 3.7
For $n \geq 1$, $H_n(y; w_0, \Upsilon_0, B_0, p)$, given in (3.12), is the distribution of the sum $\sum_{i=1}^{n} D_i$.

Proof: For $y \in \mathbb{R}_+$ it holds that

$$\begin{aligned}
\mathbb{P}[D_1 \leq y] &= \mathbb{P}[\max\{-R^p_1, 0\} \leq y] = \mathbb{P}\big[[R^p_1 \geq 0] \cup [0 < -R^p_1 \leq y]\big] \\
&= \mathbb{P}[R^p_1 \geq 0] + \mathbb{P}[0 > R^p_1 \geq -y] \\
&= 1 - G_R(0; w_0, \Upsilon_0, B_0, p) + G_R(0; w_0, \Upsilon_0, B_0, p) - G_R(-y; w_0, \Upsilon_0, B_0, p) \\
&= 1 - G_R(-y; w_0, \Upsilon_0, B_0, p) = H_1(y; w_0, \Upsilon_0, B_0, p) \; .
\end{aligned}$$

The process $\{w_t\}$ is a Markov process. Furthermore, the increment $W^f_2 - W^f_1$ is independent of W^f_1. Therefore, the distribution of D_2 given w_1 and W^f_1 for $y \geq 0$ is:

$$\mathbb{P}\big[D_2 \leq y | w_1, W^f_1\big] = 1 - G_R(-y; w_1, \Upsilon_1, B_1, p) \; .$$

Note that in our case $\Upsilon_1 = \theta \cdot \bar{c} \cdot w_1$, i.e., the additional variable Υ is not necessary. However, to emphasise the dependence on the amount of the investment Υ_1, we keep Υ_1 as a separate variable.

By the law of total probability we get:

$$\begin{aligned}
\mathbb{P}\big[D_1 + D_2 \leq y\big] &= \mathbb{E}_{w_0}\Big[\mathbb{P}\big[D_1 + D_2 \leq y \big| w_1, W_1^f\big]\Big] \\
&= \mathbb{E}_{w_0}\Big[\mathbb{P}\big[D_2 \leq y - D_1 \big| w_1, W_1^f\big]\Big] \\
&= \mathbb{E}_{w_0}\Big[1 - G_R(-y + D_1; w_1, \Upsilon_1, B_1, p)\Big] \\
&= \mathbb{E}_{w_0}\Big[1 - G_R(-y - R_1^p \wedge 0; w_1, \Upsilon_1, R_1^p \vee 0, p)\Big] = H_2(y; w_0, \Upsilon_0, B_0, p) \ .
\end{aligned}$$

Analogously, one can proceed for D_3, D_4 and D_5 etc. and derive the distribution of $\sum_{i=1}^{n} D_i$ for any $n \in \mathbb{N}$. Recursively, we obtain for $n > 2$:

$$\mathbb{P}\Big[\sum_{i=1}^{n} D_i \leq y\Big] = \mathbb{E}_{w_0}\Big[H_{n-1}(y + R_1^p \wedge 0; w_1, \Upsilon_1, R_1^p \vee 0, p)\Big] = H_n(y; w_0, \Upsilon_0, B_0, p) \ .$$

□

Using (3.8), the distribution can be easily tackled numerically, for example using Monte Carlo method.

4 Numerical illustration

In this section, we generate several scenarios for the development of the risky fund and of the working population and calculate the 1-year ruin probability and the VaR of the deficit at predefined intervals. Additionally, we examine the impact of not guaranteeing a 0% return on the funded component. Considering the worst-case scenario from the generated evolution paths, we analyse potential consequences and actions that should be taken.

4.1 Assumptions

1. The initial number of contributors at time 0, w_0, is set arbitrarily at 10^7.
2. The number of pensioners remains constant over time at $3.48 \cdot 10^6$. This value is derived by multiplying w_0 by the dependency ratio in 2020 from Eurostat (2021a).
3. The average annual salary, $\bar{s}$, is EUR $36 \cdot 10^3$, based on the gross mean annual earnings for the European Union – 27 countries (Eurostat, 2021b). This value is assumed to remain constant over time.

4. The average annual pension that needs to be covered by contributions is EUR $21 \cdot 10^3$.

5. The contribution rate is set at a constant 20.88%.[10]

6. The total pension expenditure at time $t = 0$, which remains constant throughout the entire period of analysis, is

$$P := 21 \cdot 10^3 \cdot 3.48 \cdot 10^6 = 73.08 \cdot 10^9 \ .$$

7. The total income from contributions at time $t = 0$ is

$$C_0 := \pi \cdot \bar{s} \cdot w_0 = 0.2088 \cdot 36 \cdot 10^3 \cdot 10^7 = 75.168 \cdot 10^9 \ ,$$

   with $\bar{c} = \pi \cdot \bar{s} = 75.168 \cdot 10^2$ EUR being the average contribution.

8. We assume that 95% of the contributions are allocated the PAYG part, while 5% are invested in the risky fund described below.

9. The scheme has access to one risky fund that follows a geometric Brownian motion according to Equation (2.7) with parameters $\sigma = 0.2$, $\mu = 0.02$, $F_0 = 1$. The initial investment corresponds to 5% of the total contribution C_0:

$$\Upsilon_0 = 0.05 \cdot C_0 = 0.05 \cdot 75.168 \cdot 10^9 = 37.584 \cdot 10^8 \text{ EUR.}$$

10. The evolution of the working population over time is modeled by an OU process as specified in Equation (2.4). By calibrating the number of contributors to the dependency ratio of Eurostat (2021a), we obtain $a = 0.055$, $b = 5.56 \cdot 10^6$, $\delta = 35000$, $w_0 = 10^7$. As shown in Figure 2 the number of workers decreases rapidly over a period of 80 years. Given the constant number of retirees, this substantial decrease results in a dependency ratio that almost doubles over the same time horizon, closely mimicking the projections for the European Union (27 countries) from Eurostat (2021a).

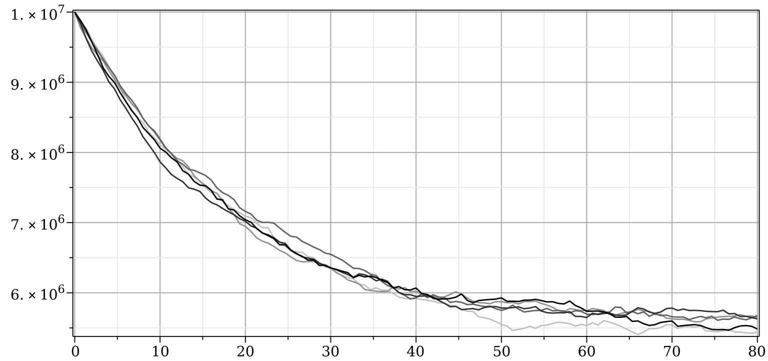
4.2 Ruin probability

4.2.1 1-year time horizon

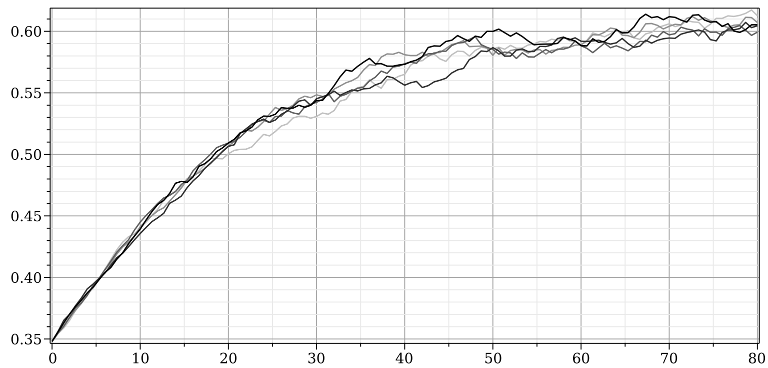
We analyse the impact of offering a nominal guarantee in the investment Υ under the worst-case scenario of our simulation on the scheme's balance and the 1-year ruin probability. The consequences of including a buffer fund, B, that provides a 0% return are also

[10] This contribution rate is slightly higher than the balanced contribution rate at time $t = 0$ to ensure the pension scheme has a small initial surplus at the start of the analysis. This rate corresponds to the balanced contribution rate that would provide a replacement rate of 60% of the salaries.

Figure 2: Simulated paths of the evolution of the working population and the dependency ratio over a period of 80 years



(a) Working population.



(b) Dependency ratio.

analysed. According to (3.4) in Proposition 3.1, the ruin probability is strictly decreasing in B_0 because the buffer earns a 0% return and can lock away potential surpluses to finance future deficit.

Under various rules of the scheme (with or without guarantees on the funded part and/or with or without a buffer fund), determining whether the mixed scheme is superior to the pure PAYG requires comparing the balance in each analysed case to the balance of the pure PAYG denoted by R_t^{PAYG}:

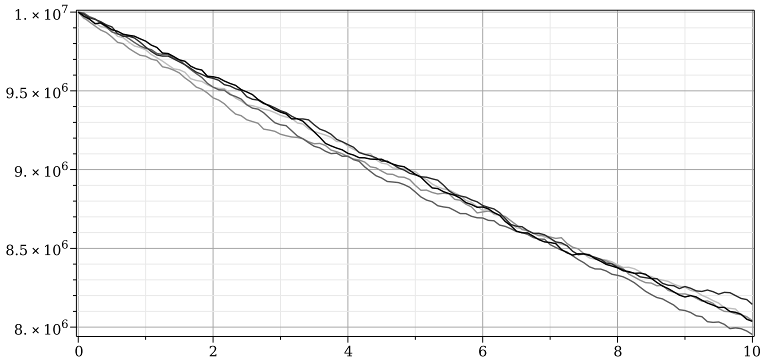
$$R_t^{\text{PAYG}} := \bar{c} \cdot w_t - P. \tag{4.1}$$

No guarantee on Υ *and no buffer fund*

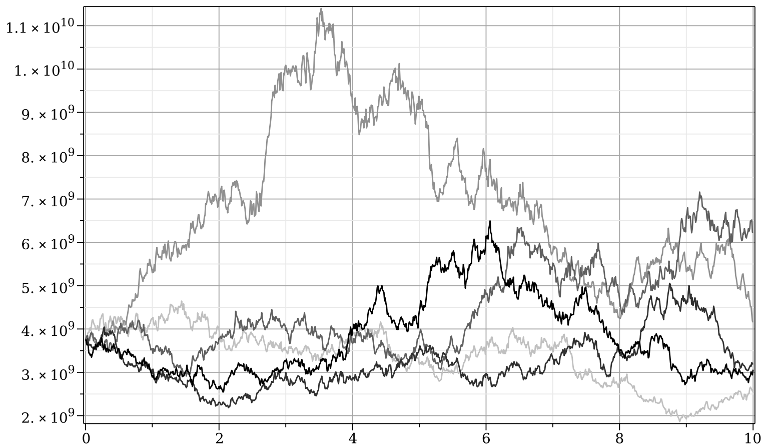
The balance at time t in the absence of a buffer fund and without any guarantee on the investment Υ can be expressed as:

$$R_t^{ng} := (1-\theta) \cdot \bar{c} \cdot w_t + \Upsilon_{t-1} \cdot F_t / F_{t-1} - P. \tag{4.2}$$

Figure 3: Simulated paths of the evolution of the working population and the fund over a period of 10 years



(a) Working population.



(b) GBM, with the initial value $\Upsilon_0 = 3.7584 \cdot 10^9$.

Nominal guarantee on Υ without a buffer fund

In this case, the balance at time t, R_t as per (2.8) is expressed as:

$$R_t := (1 - \theta) \cdot \bar{c} \cdot w_t + \Upsilon_{t-1} \cdot \max\{F_t/F_{t-1}, 1\} - P \ .$$

With respect of the filtration $\mathcal{F}_{t-1}$, the expressions above remain random variables as F_t and w_t are realizations of the stochastic working population and financial asset evolution.

Worst case scenario in our simulations (Figures 3a and 3b)

As shown in Figures 3a and 3b, the worst-case scenario yields $w_1 = 9.7 \cdot 10^6$ and $\Upsilon_0 \cdot F_1 = 3.1 \cdot 10^9$, where $\Upsilon_0 = 3.7584 \cdot 10^9$ as given in Assumption 9, Section 4.1. We initially assess whether the mixed system outperforms the pure PAYG in absence of a guarantee, that is whether R_1^{ng} (4.2) outperforms R_1^{PAYG} (4.1):

$$\begin{aligned}0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_1 + \Upsilon_0 F_1 &= 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot 9.7 \cdot 10^6 + 3.1 \cdot 10^9 \\ &= 72.3673 \cdot 10^9 < 73.08 \cdot 10^9 = P \ , \\ 0.2088 \cdot 36 \cdot 10^3 \cdot w_1 &= 72.9129 \cdot 10^9 < 73.08 \cdot 10^9 = P \ ,\end{aligned}$$

where P is calculated in Assumption 6, Section 4.1. This indicates a deficit in both the mixed scheme and the pure PAYG, as the scheme's income is lower than its pension expenditure. However, the deficit amounts to only 0.99% and 0.24% of pension spending P for the mixed and PAYG schemes, respectively.

If the return on investment, Υ, is guaranteed to be at least 0%, then the balance, R_t would exceed that of the pure PAYG scheme:

$$\begin{aligned}0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_1 + \Upsilon_0 \max\{F_1, 1\} &= 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot 9.7 \cdot 10^6 + 37.584 \cdot 10^8 \\ &= 73.0257 \cdot 10^9 < P \ .\end{aligned}$$

Nonetheless, it still remains below P for the worst-case scenario, with a deficit amounting to 0.09% of pension expenditure.

Given this shortfall, the government might need to invest additional funds on top of the regular investment Υ in other to avoid the deficit. The minimum amount at the confidence level $\alpha = 0.05$ invested by the state, as per Remark 3.3, equals

$$S_0^{\min} = 1.7322 \cdot 10^9 \ .$$

This quantity corresponds to approx 2.37% of the total pension expenditure.

Ruin probability

Whilst informative, assessing the effectiveness of the proposed scheme under a worst-case scenario ignores what occurs in all other less adverse situations. The ruin probability, in this case, is a more suitable measure. Indeed, by doing so, the following results are obtained:

$$\begin{aligned}&\mathbb{P}[0.2088 \cdot 36 \cdot 10^3 \cdot w_1 \le P] = 0.1191 \ , && \text{pure PAYG,} \\ &\mathbb{P}[0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_1 + \Upsilon_0 F_1 \le P] = 0.2669 \ , && \text{mixed, no guarantee,} \\ &\mathbb{P}[0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_1 + \Upsilon_0 \max\{F_1, 1\} \le P] = 0.0277 \ , && \text{mixed, with guarantee.}\end{aligned}$$

As observed in the *worst-case* scenario where the three cases result in a deficit, the overall probability of having a deficit after one year is strictly positive. Interestingly, the mixed scheme without guarantees appears to underperform, with a higher ruin probability compared to the pure PAYG scheme. Introducing a guarantee on Υ significantly reduces the probability of a deficit, now amounting to approximately one-fifth of the ruin probability in the PAYG scheme. The probability under the mixed scheme with guarantees is low; however, the state may aim to further decrease it by implementing a buffer fund. For $\alpha \geq 2.77\%$ the buffer fund (3.5) naturally equals 0, as the mixed scheme with guarantees and no buffer fund has a ruin probability of 0.0277. However, aiming for a confidence level of $\alpha = 1\%$, we obtain

$$B_0^{\min} = 1.098 \cdot 10^8.$$

In other words, by investing 0.15% of the pension expenditures P, we can reduce the ruin probability to just 1% in presence of guarantees.

However, given the long-term nature of pension schemes, it could be argued that assessing our proposed pension framework over a one-year period might be too short, as investments cannot unfold its potential.

4.2.2 t-year time horizon

In this subsection, we consider that the initial investment Υ_0 is left for t years to earn a financial return.[11] Under this assumption, the ruin probability of the mixed scheme at time t, financed through 95% of the PAYG contributions at time t and the initial investment Υ_0 that earns financial return over the period of t years is:

$$\mathbb{P}[0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_t + \Upsilon_0 \cdot \max\{F_t, 1\} \leq P]. \tag{4.3}$$

Firstly, let us compare the mixed scheme with the pure PAYG over a time horizon of $t = 10$ years. Retrieving the *worse-case scenario* presented in Figures 3b and 3a, we obtain $w_{10} = 7.9 \cdot 10^6$ and $\Upsilon_0 F_{10} = 2.6 \cdot 10^9$, with $\Upsilon_0 = 3.7584 \cdot 10^9$ as given in Assumption 9, Section 4.1. Then, this yields

$$\begin{aligned} 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} + \Upsilon_0 F_{10} &= 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot 7.9 \cdot 10^6 + 2.6 \cdot 10^9 \\ &= 59.0136 \cdot 10^9 < 73.08 \cdot 10^9 = P \ , \\ 0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} &= 59.3827 \cdot 10^9 < 73.08 \cdot 10^9 = P \ . \end{aligned}$$

Compared to the one-year time horizon, the deficit has increased over a ten-year period due to a declining population. It now represents 19.26% and 18.75% of pension spending, P, for the mixed scheme and PAYG, respectively. However, adding the guarantee, we obtain

$$\begin{aligned} 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} + \max\{\Upsilon_0 F_{10}, \Upsilon_0\} &= 0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot 7.9 \cdot 10^6 \\ &\quad + 37.584 \cdot 10^8 \\ &= 60.1720 \cdot 10^9 < 73.08 \cdot 10^9 = P \ . \end{aligned}$$

[11] In practice, we work on a rolling window basis, that is, investments done at year s are left every time to unfold over the fixed period of t years. In this case, the ruin probability of the mixed scheme at time $s+t$, is financed through 95% of the PAYG contributions in $s+t$ and the initial investment Υ_s, that earns financial return over the t-year period.

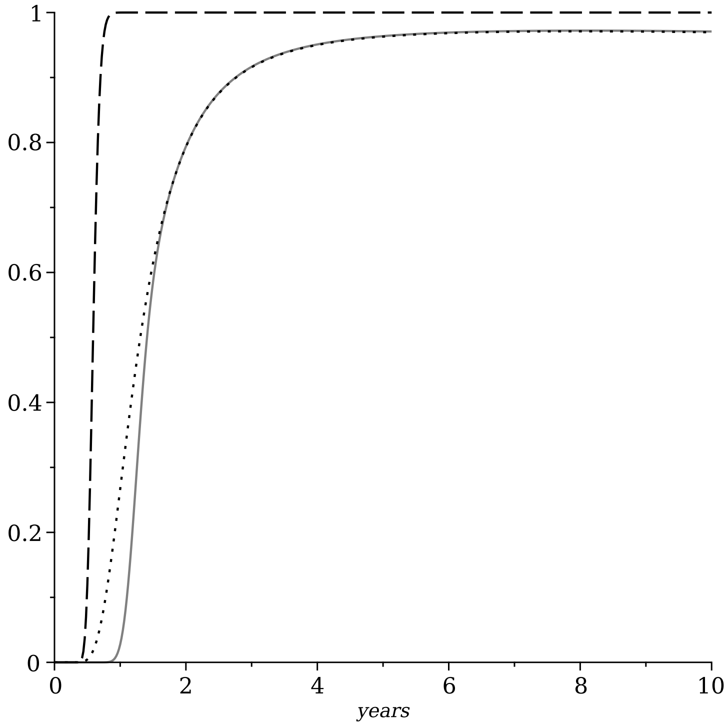


Figure 4: The ruin probability for PAYG (dashed line), with investments (no guarantees) (dotted line) and with investments (with guarantees) in dependence on the time t.

The scheme is still in deficit but at a much lower level than in the pure PAYG case. From a ruin probability perspective, we obtain the following results

$$
\begin{aligned}
\mathbb{P}[0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} \leq P] = 1 \;, \qquad & \text{pure PAYG,} \\
\mathbb{P}[0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} + \Upsilon_0 F_{10} \leq P] & \\
= 0.9696512673 \;, \quad & \text{mixed, no guarantee,} \\
\mathbb{P}[0.95 \cdot 0.2088 \cdot 36 \cdot 10^3 \cdot w_{10} + \max\{\Upsilon_0 F_{10}, \Upsilon_0\} \leq P] & \\
= 0.9696512669 \;, \quad & \text{mixed with guarantee.}
\end{aligned}
$$

The probability of encountering a deficit at time $t = 10$ in the mixed scheme, when the investment is held over a period of ten years, is approximately 97%. In contrast, the probability of having a deficit under the PAYG scheme is 100%. Naturally, the severity of the anticipated deficit must be taken into consideration. In Figure 4, we observe that the ruin probability for the case of the PAYG scheme lies above the ruin probability including investments (with and without guarantee) already for $t > 1$. This is due to the pronounced downward trend of the number of contributors. As the investment duration increases, the fund's drift has the potential to partially counteract the negative impact of the shrinking pool of contributors. We see that for longer time horizons, the guarantee almost does not have any impact on the ruin probability. The situation can be further improved by adding a buffer. For the above parameters we get for the 10-year's horizon and confidence level $\alpha = 5\%$:

$$B_0^{\min} = 1.2055 \cdot 10^{10} \ ,$$
$$S_0^{\min} = 7.7037 \cdot 10^{9} \ .$$

Given the high ruin probabilities presented, a higher amount of capital is needed compared to the one-year time horizon. However, an initial buffer fund $B_0^{\min}$ equivalent to 16.49% of pension expenditures reduces the ruin probability from 97% to 5%. Alternatively, if we choose to invest in the financial markets through $S_0^{\min}$ instead, we only need to invest 10.54% of pension expenditures to decrease the ruin probability to 5%.

4.2.3 Probability of PAYG outperforming the mixed scheme

In previous sections, we observed that including a nominal guarantee substantially decreases the probability of incurring a deficit. Our simulation study revealed that, the *worst-case* scenario of the mixed scheme in absence of guarantees underperforms a pure PAYG scheme. Furthermore, it is worth analysing the financial return in relation to the evolution of the income of contributions C_t over time. In this section, we analyse the 5% contribution allocated to the PAYG compared to investing the same amount in the financial markets over 1-year and 10-year time horizons. In absence of buffer fund, we obtain the following results for the 1-year and 10-year time horizon, respectively:

$$\mathbb{P}_{w_0}[0.05 \cdot w_1 \cdot 36 \cdot 10^3 \cdot 0.2088 > \Upsilon_0 F_1] = 41.28\%$$
$$\mathbb{P}_{w_0}[0.05 \cdot w_{10} \cdot 36 \cdot 10^3 \cdot 0.2088 > \Upsilon_0 F_{10}] = 25.94\% \ ,$$

which clearly indicates that the mixed scheme with a longer time horizon of 10-years outperforms the PAYG, with a probability of approximately 74%, even when no buffer fund is considered.

If the investment fund performs well and the return is greater than zero, the excess can be frozen in a buffer account. Figure 5, based on a set of simulated paths, shows the probability that the 5% allocated to PAYG outperforms the 5% invested in funding (with no guarantees) over 1-year and 10-year time horizons if a buffer was built. In particular, when the initial buffer fund x is lower than $4.7423 \cdot 10^8$ we find that an investment in the funding element outperforms the allocation of 5% into PAYG over a 10-year horizon. On the other hand, as shown in Figure 5, for a buffer $x > 4.7423 \cdot 10^8$, the one-year mixed scheme outperforms the PAYG with a higher probability than the 10-year scheme. Indeed, if $x = 5 \cdot 10^8$, the probabilities that the 5% allocated to PAYG outperforms the 5% invested in the financial market over 1-year and 10-year time horizons are as follows:

$$\mathbb{P}_{w_0}[0.05 \cdot w_1 \cdot 36 \cdot 10^3 \cdot 0.2088 > 5 \cdot 10^8 + \Upsilon_0 F_1] = 17.04\% \ ,$$
$$\mathbb{P}_{w_0}[0.05 \cdot w_{10} \cdot 36 \cdot 10^3 \cdot 0.2088 > 5 \cdot 10^8 + \Upsilon_0 F_{10}] = 17.67\% \ .$$

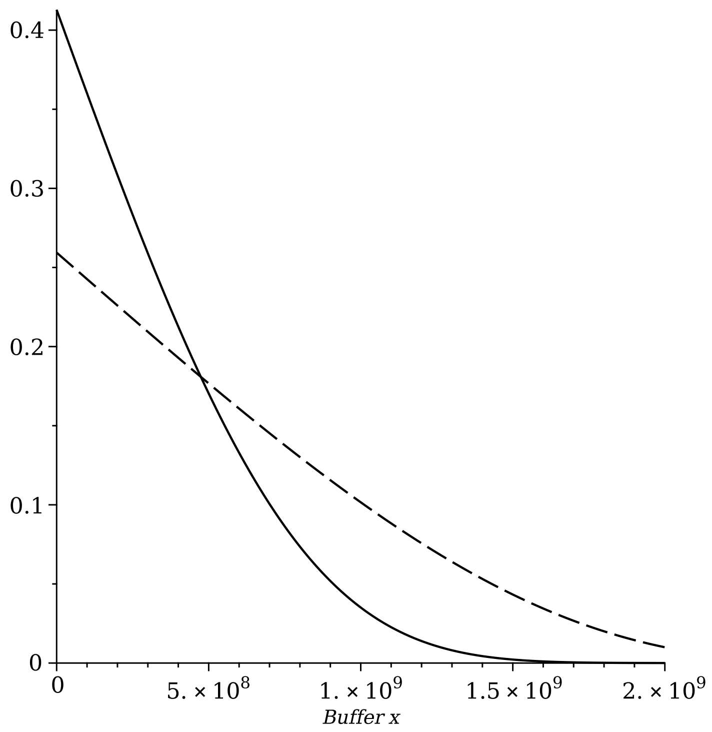


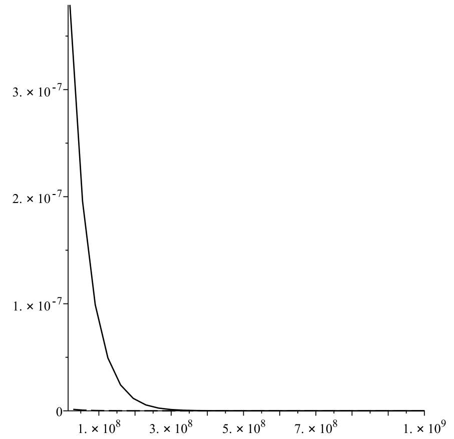
Figure 5: The probabilities that the PAYG outperforms the mixed scheme with no guarantee in dependence on the available buffer x for 10- (dashed line) and 1-year horizons.

Since the Brownian motions driving the number of contributors and the risky asset are assumed to be independent, there is no possibility to hedge the risk of a shrinking working population. For instance, in Cairns et al. (2014), the correlation between the hedge and the value of the pension liability was crucial. The difference in probabilities for the 1-year and 10-year horizons is rooted in the fact that, over the years, the working population shrinks more significantly than the fund's growth can compensate for.

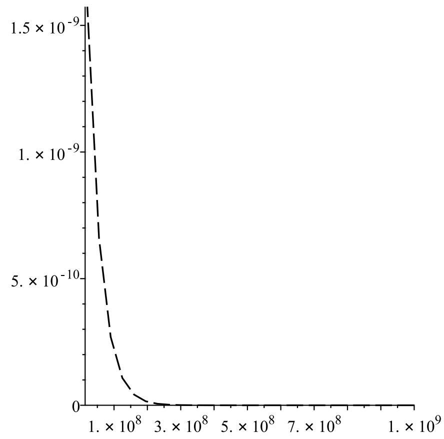
4.3 VaR of the deficit: 5-year time horizon

Assuming the initial buffer fund is $B_0 = 10^9$, or 1.37% of C_0, Figure 6a shows the probability of encountering a deficit in the scheme after a year, $\mathbb{P}[D_1 > x]$, for $p = 0$ (an almost invisible dashed line) and for $p = 1$ (solid line). The function $\mathbb{P}[D_1 > x]$ for $p = 0$ is shown separately in Figure 6b. It is clear that the risk increases when opting to invest the entire buffer ($p = 1$). For both cases $p = 1$ and $p = 0$, the probability of D_1 exceeding 0 (deficit after one period) is extremely small, approximately $4 \cdot 10^{-7}$ for $p = 1$ and $1.6 \cdot 10^{-9}$ for $p = 0$. Therefore, a one-period VaR would not be a very informative risk measure.

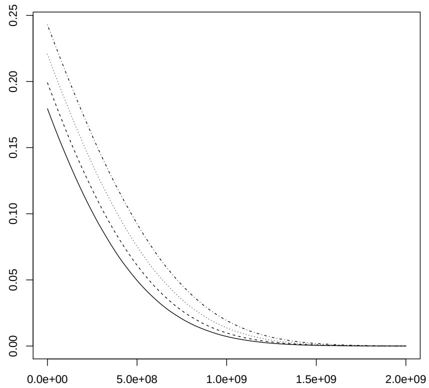
Figure 6: Deficit probability for different time horizons (1 year versus 5-year period) and portfolio allocations, p for $B_0 = 10^9$



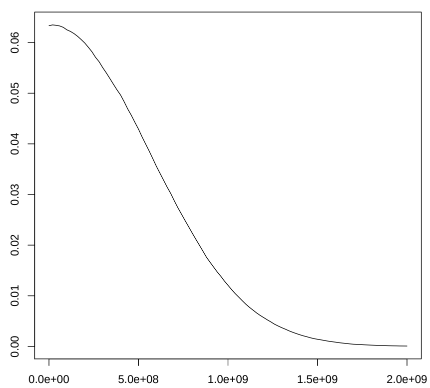
(a) $\mathbb{P}[D_1 > x]$ for $p = 0$ (dashed line) and $p = 1$ (solid line).



(b) $\mathbb{P}[D_1 > x]$ for $p = 0$ (dashed line).



(c) The function $\mathbb{P}\Big[\sum_{i=1}^{5} D_i > x\Big]$ for $p = 0$, $p = 1/3$, $p = 2/3$ and $p = 1$ (from the lowest to the highest curve).

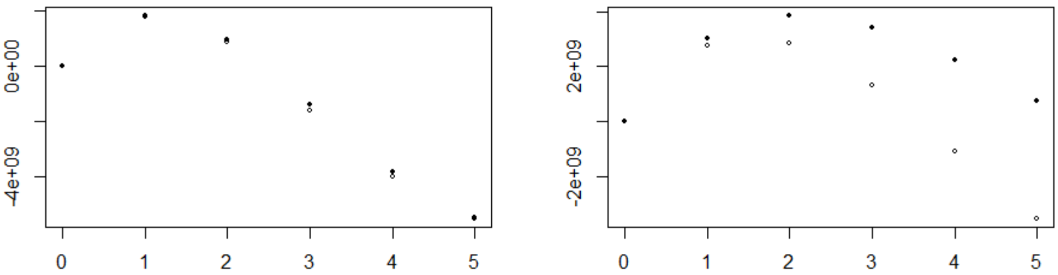


(d) The difference between $\mathbb{P}\Big[\sum_{i=1}^{5} D_i > x\Big]$ for $p = 1$ and for $p = 0$.

Figure 6c illustrates $\mathbb{P}\Big[\sum_{i=1}^{5} D_i > x\Big]$ for $p = 0$ (the lowest curve), $p = 1/3$, $p = 2/3$ and $p = 1$ (the highest curve). We observe that the probability of having a positive deficit (> 0) with $p = 1$ is greater than $p = 0$. In other words, the probability of having a deficit in our case increases with p. This implies that the smallest VaR_α is achieved for $p = 0$, indicating that the safest option is not to invest any part of the buffer.[12]

[12] The results have been obtained via Monte Carlo simulations involving 1,000,000 samples.

Figure 7: The expectations $\mathbb{E}[R_n^p]$ for $p = 1$ (black dots), $p = 0$ (circles) and $n = 1, .., 5$.



(a) $\mu = 0.02$ and $\sigma = 0.2$.

(b) $\mu = 0.1$ and $\sigma = 0.5$.

In Figure 6d, the difference between $\mathbb{P}\Big[\sum_{i=1}^{5} D_i > x\Big]$ for $p = 1$ and for $p = 0$ is depicted. This difference being positive indicates that investing the whole buffer fund into the risky asset increases the 5-year cumulative deficit probability. In particular, for $x = 0$ the difference is approximately 0.065 indicating a 6,5% excess ruin probability of having a deficit. Choosing, for instance, a confidence level of $\alpha = 0.95$, the difference in the VaR between $p = 1$ and $p = 0$ amounts to approximately $5 \cdot 10^8$, which corresponds to half of the initial buffer fund B_0 or 0.68% of the total pension expenditures. In other words, an additional $5 \cdot 10^8$ would have to be invested in the mixed system to achieve the same level of ruin probability in the event that everything is fully invested in the funding element.

However, it is important to consider the potential return on investment of the buffer and its sensitivity. The expectations $\mathbb{E}[R_1^p]$, ..., $\mathbb{E}[R_5^p]$ are illustrated in Figure 7 for $p = 1$ (black dots) and $p = 0$ (circles). Note that for $n = 3, 4, 5$ the expectations for both $p = 1$ and $p = 0$ are negative, indicating a deficit in the scheme for the period analysed. Given our assumptions, the downward trend in the number of contributors cannot be hedged by an investment when $\mu = 0.02$ and $\sigma = 0.2$. However, all other factors being equal, assuming that the fund is driven by $\mu = 0.1$ and $\sigma = 0.5$, the expectation $\mathbb{E}[R_n^1]$ remains positive for all $n = 1, ..., 5$, as shown Figure 7b, implying a surplus for the analysed period. The financial performance of the buffer fund is therefore crucial to the success of the mixed scheme.

Remark 4.1
One might choose to take on more risk in exchange for a greater expected income. The difference between the expectations $\mathbb{E}[R_1^0]$ *and* $\mathbb{E}[R_1^p]$*, see Tables 1a and 1b, for* $p > 0$*, can be easily calculated:*

$$\mathbb{E}[R_1^p] - \mathbb{E}[R_1^0] = pB_0\big(e^{\mu + \frac{\sigma^2}{2}} - 1\big)\ .$$

Table 1: The difference between the expectations $\mathbb{E}[R_n^1]$ and $\mathbb{E}[R_n^0]$ for $n = 1, ..., 5$.

year n	1	2	3	4	5
$\mathbb{E}[R_n^1]$	1,820,747,252	969,244,793	-1,415,308,672	-3,815,749,607	-5,480,620,643
$-$					
$\mathbb{E}[R_n^0]$	1,779,783,439	854,350,678	-1,618,404,588	-3,982,852,554	-5,509,376,935
$=$	40,963,813	114,894,115	203,095,916	167,102,947	28,756,292

(a) $\mu = 0.02$ and $\sigma = 0.2$

year n	1	2	3	4	5
$\mathbb{E}[R_n^1]$	3,033,243,495	3,853,940,515	3,394,034,334	2,218,280,832	754,651,794
$-$					
$\mathbb{E}[R_n^0]$	2,781,291,308	2,836,662,617	1,294,966,703	-1,087,694,686	-3,533,530,590
$=$	251,952,187	1,017,277,898	2,099,067,631	3,305,975,518	4,288,182,384

(b) $\mu = 0.1$ and $\sigma = 0.5$

As long as $\mu + \frac{\sigma^2}{2} > 0$ and $B_0 > 0$, the expected balance with investment of the buffer fund in the financial markets would exceed the expected balance with no investment – even if the latter yields a 0% return.

In Figure 8a, $\mu = 0.02$, $\sigma = 0.2$, we observe the distribution function of R_1^p for $p = 0$ (dashed line) and $p = 1$ (solid line). Figure 8b shows that the distribution functions intersect at approximately $1.7 \cdot 10^9$. Specifically, for values lower than $1.7 \cdot 10^9$, the distribution of R_1^1 lies above that of R_1^0, indicating that $p = 1$ represents a riskier strategy. However,

$$\mathbb{P}[R_1^0 \leq 0] = 2.56 \cdot 10^{-9} \quad \textit{and} \quad \mathbb{P}[R_1^1 \leq 0] = 5.59 \cdot 10^{-7}$$

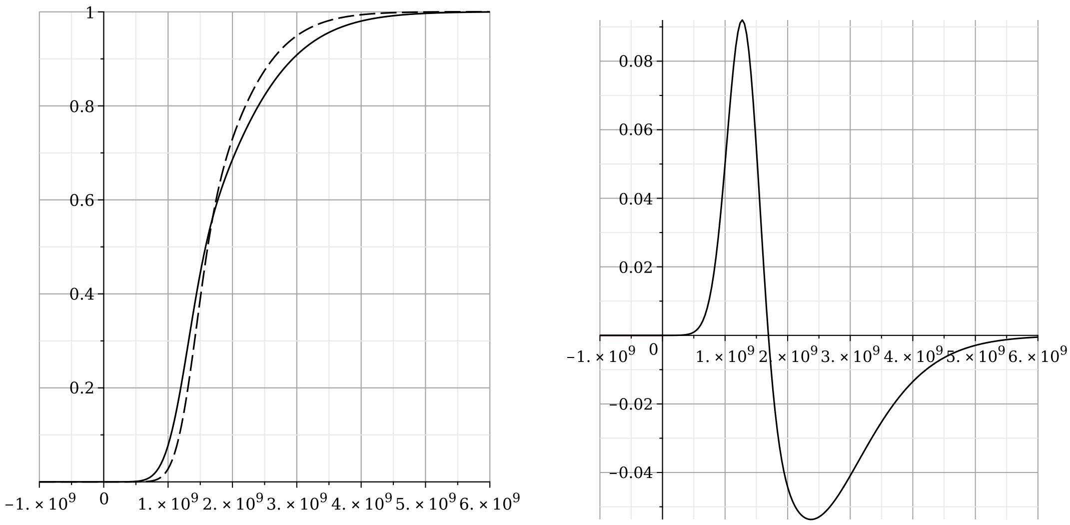
given that these probabilities are very small, it suggests that investing the entire buffer would likely be more beneficial in increasing the expected balance. For the chosen parameters, $B_0 = 10^9$, $\mu = 0.02$ and $\sigma = 0.2$, the difference is

$$\mathbb{E}[R_1^1] - \mathbb{E}[R_1^0] = B_0\left(e^{\mu+\frac{\sigma^2}{2}} - 1\right) = 10^9(e^{0.04} - 1) = 4.081 \cdot 10^7 \ .$$

For values higher than $1.7 \cdot 10^9$ the contrary is true, as the distribution of R_1^1 lies below that of R_1^0 indicating that not investing anything ($p = 0$) is a riskier strategy. Recall that Equation (3.7) is given by

$$R_1^p = (1-\theta) \cdot w_1 \cdot \bar{c} + \Upsilon_0 \max\left(e^{\mu+\sigma W_1^f}, 1\right) - P + B_1^p.$$

Figure 8: The distribution function of R_1^p.



(a) The distributions $\mathbb{P}[R_1^0 \leq x]$ (dashed line) and $\mathbb{P}[R_1^1 \leq x]$ (solid line).

(b) The difference $\mathbb{P}[R_1^1 \leq x] - \mathbb{P}[R_1^0 \leq x]$.

In this case, not investing implies earning a 0% return on the buffer fund, which is a less risky strategy but increases the probability of encountering a shortfall.

5 Conclusion

State pensions are normally financed on a PAYG basis, where current contributions are used to fund current pension expenditure. This financing method requires an annual balance between contributions and payouts. However, the forecasted demographic trends of an aging population and declining fertility rates jeopardize the sustainability of PAYG systems. Mixed schemes can be seen as an alternative to alleviate the demographic risks inherent in pure PAYG schemes by combining PAYG and funding schemes and investing a portion of the contributions in financial markets. The PAYG rate of return can be lower than the rate of return of funding schemes, especially in countries where the working population is not growing. Advocated by the World Bank, mixed systems are seen as a practical way to reconcile the higher returns from financial markets – compared to GDP growth – thus incorporating a greater funded element to enhance sustainability.

In this paper, we assess how a mixed scheme can temporarily alleviate the financial burden caused by an increase in the dependency ratio due to a decline in the number of contributors. We use ruin probability, i.e. the probability of a deficit, and Value-at-Risk (VaR) as risk measures to determine the financial imbalance and compare both schemes: pure PAYG versus a mixed scheme. Our findings indicate that, under the analysed scenarios, the probability to get ruined in one-year's time – to have a negative balance in the scheme – is higher for the mixed scheme. However, if a guarantee of a 0% return is provided on the invested part, the probability of a deficit is substantially reduced, becoming approximately one-fifth of the ruin probability in the PAYG scheme. Additionally, our analysis shows that over a longer time horizon, the ruin probability is lower for a mixed scheme, without a guarantee, compared to the PAYG scheme.

We consider ruin probability and VaR of the accumulated deficit over a period of several years. If the VaR at a particular confidence level is too big, it indicates that the system is too risky from the standpoint of financial sustainability in the near future. In the base case scenario, we show that the smallest VaR is achieved if the buffer fund is not invested in the financial markets. However, from an expected value perspective, investing a part of the fund is beneficial, given a reasonable choice of the return and volatility. Partial investment in the financial markets becomes more lucrative for higher return-to-volatility ratios. Special attention is given to the use of a buffer fund, whose returns could supplement contribution income and improve the financial health of the scheme. This approach would not completely hedge against demographic risk but would enhance the scheme's financial stability, thereby delaying the need for pension reforms. This measure could be a viable strategy for some governments when considering pension reforms.

Future research can be focused on optimising the percentage of contributions invested in financial markets to maximise expected returns while minimising risk. Additionally, the scheme should be analysed over a longer time horizon to fully understand its long-term financial sustainability.

References

Alonso-García, J. and Devolder, P. (2016), "Optimal mix between pay-as-you-go and funding for DC pension schemes in an overlapping generations model," *Insurance: Mathematics and Economics*, 70, 224–236.

Alonso-García, J. and Rosado-Cebrian, B. (2021), "Financial crisis and pension reform in Spain: The effect of labour market dynamics," *Journal of Economic Policy Reform*, 24, 201–218.

Ando, M. S., Balakrishnan, M. R., Gruss, B., Hallaert, M. J.-J., Jirasavetakul, L.-B. F., Kirabaeva, K., Klein, N., Lariau, A., Liu, L. Q., Malacrino, D., et al. (2022), *European labor markets and the COVID-19 pandemic: Fallout and the path ahead*, International Monetary Fund.

Boado-Penas, M. C., Eisenberg, J., and Korn, R. (2021), "Transforming public pensions: A mixed scheme with a credit granted by the state," *Insurance: Mathematics and Economics*, 96, 140–152.

Boado-Penas, M. d. C., Demarco, G., Eisenberg, J., Lundberg, K., and Şahin, Ş. (2022), *In Pandemics: Insurance and Social Protection by Boado-Penas, M.C., Eisenberg, J.*

and Şahin, Ş. (eds.), Springer International Publishing, chap. All-Hands-On-Deck!—How International Organisations Respond to the COVID-19 Pandemic., pp. 127–142.

Boado-Penas, M. d. C., Godínez-Olivares, H., Haberman, S., and Serrano, P. (2020), "Automatic balancing mechanisms for mixed pension systems under different investment strategies," *The European Journal of Finance*, 26, 277–294.

Cairns, A. J., Dowd, K., Blake, D., and Coughlan, G. D. (2014), "Longevity hedge effectiveness: A decomposition," *Quantitative Finance*, 14, 217–235.

Caulkins, J. P., Grass, D., Feichtinger, G., Hartl, R. F., Kort, P. M., Prskawetz, A., Seidl, A., and Wrzaczek, S. (2022), *In Pandemics: Insurance and Social Protection by Boado-Penas, M.C., Eisenberg, J. and Şahin, Ş. (eds.)*, Springer International Publishing, chap. COVID-19 and optimal lockdown strategies: The effect of new and more virulent strains., pp. 163–190.

De Menil, G., Murtin, F., and Sheshinski, E. (2006), "Planning for the optimal mix of paygo tax and funded savings," *Journal of Pension Economics and Finance*, 5, 1–25.

Devolder, P. and Melis, R. (2015), "Optimal mix between pay as you go and funding for pension liabilities in a stochastic framework," *ASTIN Bulletin*, 45, 551–575.

Dutta, J., Kapur, S., and Orszag, J. (2000), "A portfolio approach to the optimal funding of pensions," *Economics Letters*, 69, 201–206.

Eurostat (2021a), "Demographic balances and indicators by type of projection (PROJ 19NDBI)," `http://ec.europa.eu/eurostat/web/products-datasets/-/proj_18ndbi`.

— (2021b), "Mean annual earnings by sex, economic activity and collective pay agreement (EARN SES18 26)," `https://ec.europa.eu/eurostat/databrowser/product/page/earn_ses18_26`.

Holzmann, R., Palmer, E., and Robalino, D. (2012), *Nonfinancial Defined Contribution Pension Schemes in a Changing Pension World: Volume 1, Progress, Lessons, and Implementation*, vol. 1, World Bank Publications.

Jimeno, J. F., Rojas, J. A., and Puente, S. (2008), "Modelling the impact of aging on social security expenditures," *Economic Modelling*, 25, 201–224.

OCDE (2011), *Pensions at a Glance 2011*, OECD Publishing, Paris.

OECD (2012), *OECD Pensions Outlook 2012*, OECD Publishing, Paris.

— (2013), *Pensions at a Glance 2013*, OECD Publishing, Paris.

— (2017), *Pensions at a Glance 2017*, OECD Publishing, Paris.

OECD Publishing (2021), *Pensions at a Glance 2021: OECD and G20 Indicators*, Organisation for Economic Co-operation and Development OECD.

Persson, M. (2002), "Five fallacies in the social security debate," in *Social Security Reform in Advanced Countries: Evaluating Pension Finance*, eds. Ihori, T. and Tachibanaki, T., London: Routledge Press, pp. 39–51.

Settergren, O. and Mikula, B. D. (2005), "The rate of return of pay-as-you-go pension systems: a more exact consumption-loan model of interest," *Journal of Pension Economics and Finance*, 4, 115–138.

Stewart, F. and Yermo, J. (2009), "Pension fund governance: Challenges and potential solutions," *OECD Journal: Financial Market Trends*, 2008, 1–42.

United Nations (2023), "Leaving No One Behind In An Ageing World," *World Social Report*, 2023.

Whitehouse, E. (2009a), "Pensions during the crisis: Impact on retirement income systems and policy responses," *The Geneva Papers on Risk and Insurance-Issues and Practice*, 34, 536–547.

— (2009b), "Pensions, purchasing-power risk, inflation and indexation," *OECD Social, Employment and Migration Working Papers*, No. 77.