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Abstract

Multi-country risk management of longevity risk provides new opportunities to hedge mortality and interest rate risks in guaranteed lifetime income streams. This requires consideration of both interest rate and mortality risks in multiple countries. For this purpose, we develop value-based longevity indexes for multiple cohorts in two different countries that take into account the major sources of risks impacting life insurance portfolios, mortality and interest rates. To construct the indexes we propose a cohort-based affine model for multi-country mortality and use an arbitrage-free multi-country Nelson-Siegel model for the dynamics of interest rates. Index based longevity hedging strategies have the advantages of efficiency, liquidity and lower cost but introduce basis risk. Graphical risk metrics are a way to effectively capture the relationship between an insurer's portfolio and hedging strategies. We illustrate the effectiveness of using a value–based index for longevity risk management between two countries using graphical basis risk metrics. To show the impact of both interest rate and mortality risk we use Australia and UK as domestic and foreign countries, and, to show the impact of mortality only, we use the male populations of the Netherlands and France with common interest rates and basis risk arising only from differences in mortality risks.

Keywords: Value-based longevity indexes; Multi-country mortality model; Arbitrage-free Nelson-Siegel model; Basis risk; Graphical risk metric.

JEL Classification Numbers: C13, G22, G23, J11.

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1 Introduction

Longevity risk is an important risk factor for life insurance companies and defined benefit (DB) pension plans that provide lifetime annuity-type payouts. It is a systematic risk and cannot be averaged out by applying the law of large numbers. Reinsurers have been reluctant to accommodate this risk due to liquidity constraints. Blake and Burrows [2001] suggest transferring this risk to financial markets as such markets have proved to be effective in inducing liquidity, an important ingredient for mitigating other forms of risk such as equity, commodity, credit, interest risks, among others. The first financial market solution for longevity risk management was executed by Lucida and J.P. Morgan in the form of a q-forward linked to J.P. Morgan's LifeMetrics longevity index in 2008. Since then, there has been a gradual increase in the number of transactions involving longevity risk transfer occurring among pension funds, insurance companies and investment banks as presented in Table 1. Such transactions lay the foundations for a potential market in trading longevity risk.

Though the first longevity hedge was an index-based hedge, most of the hedging transactions that followed have been customized indemnity-based hedges. Compared to an indemnity-based hedge, an index-based hedge is more desirable due to its greater liquidity potential and lower transaction costs [Lin and Cox, 2005, Coughlan et al., 2011, Cairns and El Boukfaoui, 2017]. An index plays a crucial role of facilitating liquidity and providing a benchmark for pricing and hedging of securities. To promote a liquid longevity market, the development of underlying longevity indexes is critical in inducing liquidity and transparency.

Date	Hedger	Provider	Type	Description	Size (£m)
Jan-08	Lucida	J.P. Morgan	Value hedge	10-year q-Forward (LifeMetrics Index)	N/A
Jul-08	Canada Life	J.P. Morgan	Cash flow hedge	40-year survivor swap	500
Feb-09	Aviva	Royal Bank of Scotland	Cash flow hedge + value hedge	10-year collared survivor swap + final commutation payment	475
Jan-11	Pall UK Pension Fund	J.P. Morgan	Value hedge	10-year q-Forward (LifeMetrics Index)	70
Jul-14	British Telecom Pension Scheme	Prudential Insurance Co of America	Cash flow hedge	Longevity swap	16,000
Jan-15	Merchant Navy Officers Pension Fund	Towers Watson	Cash flow hedge	Longevity swap	1,500

Table 1: Selected market transactions of longevity-linked instruments.

Source: http://www.longevity-risk.org/Pres_Coughlan.pdf

An initial effort to construct a longevity index was undertaken by Credit Suisse in 2005, who launched the first market-wide longevity index based on historical and projected life expectancy for the US population. This index is no longer being actively maintained by Credit Suisse. Two influential longevity indexes are still operational, namely the LLMA index by Life & Longevity Markets Association (LLMA)[1] and the Xpect index by Deutsche Börse.[2] J.P. Morgan developed an internal toolkit called LifeMetrics which is now the intellectual property of LLMA. It has been designed for pension plans, their sponsors, insurers, reinsurers and investors affiliated with J.P. Morgan to manage their longevity risk portfolios. The LLMA longevity index is a body of data relating to the mortality, survivorship and life expectancy of a specified group of individuals, calculated according to robust and well defined algorithms and processes for England & Wales, Germany, Netherlands and the US. The Deutsche Börse's main purpose in launching its longevity indexes has been the desire to evaluate liability-based longevity risks and create a basis for financial instruments, such as securitization of life insurance and pension insurance risks. Xpect provides mortality data and indexes for Germany, Netherlands and England & Wales on a monthly basis. Xpect also developed the Xpect-Club Vita indexes which are based on mortality data of three different groups of UK pensioners. For the forecasting of future cohort mortality rates, Xpect uses the Lee-Carter model as the underlying mortality model.

Developing a robust value-based longevity index, which can be used as a reference by market participants to design more effective financial instruments for managing longevity risk remains a significant impediment to the efficient and effective pricing and hedging of longevity risk. Despite the different indexes which have been proposed, none has proved to be universally effective as the basis for financial market transactions. Liabilities of insurance companies and pension funds involve interest rate risk as well as longevity risk. To manage this risk, there is a need to consider both interest rates and mortality. An initial step in constructing a value-based longevity index was proposed by Chang and Sherris [2015], who focus on the Australian market. They show that using a value-based longevity index is more effective for hedging purposes than mortality indexes alone. Multi-country risk management of longevity risk provides new opportunities to hedge mortality and interest rate risks in guaranteed lifetime income streams. This requires consideration of both interest rate and mortality risks in multiple countries.

We develop value-based longevity indexes for multiple cohorts in two different countries that take into account the major sources of risks impacting life insurance portfolios; mortality and interest rates. The index is defined as the discounted value of lifetime income of a unit of currency per annum for a specific cohort in a domestic country and a foreign country. Such indexes quantify

[1] http://www.llma.org

[2] http://www.xpect-index.com

changes in the costs and risks of longevity across countries, opening up an additional opportunity for managing these risks. Our focus on value-based longevity indexes is practically motivated. They not only capture the impact of the most significant risks, but the underlying models can be used as a basis for pricing and hedging longevity-linked securities.

We use a multi-country continuous time affine model for the mortality dynamics and an arbitrage-free Nelson-Siegel (AFNS) model for interest rate evolutions for both the domestic and foreign countries. For the mortality, we propose a multi-factor joint affine term structure mortality model on a cohort basis with mortality intensities in domestic and foreign countries affected by common risk factors as well as country specific factors. This modelling approach has the benefit of producing closed form survival curves that can readily be applied in practice and can be efficiently estimated using maximum likelihood and Kalman filters.

For interest rates, we use the arbitrage-free Nelson-Siegel (AFNS) model developed by Christensen et al. [2011]. Diebold and Li [2006] apply these dynamics to the yield curve model of Nelson and Siegel [1987] and show that the model provides a good empirical fit. Christensen et al. [2011] prove that with a time-invariant yield-adjustment term, the empirically successful dynamic Nelson-Siegel (DNS) model can be made arbitrage free. The AFNS model combines the DNS factor loading structure and the arbitrage free property of an affine term structure model. We use the independent-factor AFNS model since it outperforms the correlated-factor AFNS model in out-of-sample forecasts [Christensen et al., 2011].

A major factor when hedging longevity risk with instruments based on indexes is the quantification of basis risk. Basis risk refers to the mismatch between the experience of the hedged exposure and that of the underlying index. In recent years the pace of globalization has dramatically increased. Globalization of the economy creates new markets and opportunities as well as new risk exposures. For example, longevity indexes from foreign countries may be used to hedge longevity risk in a domestic country when there are no local longevity indexes. The basis risk from a cross–country index–based hedge arises from the differences in the future mortality of the different country populations, the difference in the risk exposure being hedged from the country population as well as differences in interest rate developments. We are motivated by studies that show that there is positive correlation between the mortality experience of two different populations [Li and Hardy, 2011].

Various ways of quantifying basis risk have been extensively discussed in literature. Coughlan et al. [2011] develop a framework to analyse longevity basis risk by proposing the use of metrics such as mortality rates, life expectancies and liability cash flows. Other authors try to assess its impact on the effectiveness of longevity hedges. For example, Ngai and Sherris [2011] adopt expected shortfall to examine the impact of different levels of the ratio between annuitant mortality rates and population mortality rates; Cairns et al. [2014] use variance as the risk measure, and quantify basis risk arising from using England & Wales mortality as the underlying index to hedge the Continuous Mortality Investigation (CMI) mortality. In an alternative approach, Chan et al. [2016] propose a graphical risk metric to assess population basis risk. The graphical risk metric is a visual approach to showing hedging outcomes, which can be readily interpreted and well suited to practical applications.

We adopt the graphical risk metric developed in Chan et al. [2016] in assessing the basis risk arising from cross–country hedging of longevity exposure of a domestic country with a foreign country. We use examples to illustrate the implementation and performance of the multi-country indexes. We use the UK as the foreign country with Australia as the domestic since the UK market has a larger potential market for longevity-linked instruments and existing longevity indexes and an obvious candidate to hedge longevity exposure in the Australian market with a much smaller market. In so doing, we simultaneously hedge both mortality and interest rate risk exposures. We also illustrate the hedge effectiveness of a portfolio exposed to mortality risk only by considering the Netherlands and France with common interest rates but different mortality experiences.

We contribute to the literature in three important ways. First, from a modelling perspective, the paper develops a multi-country affine continuous–time mortality model allowing the construction of longevity indexes for different cohorts in both domestic and foreign countries. Second, from a practical perspective, the value-based longevity indexes allows the simultaneous hedging of mortality and interest rate risks. Third, we assess basis risk in the index-based longevity hedges using a graphical risk metric allowing visual assessment of the interaction between the portfolio to be hedged and the hedging instruments that can be more informative than a single measure of hedge effectiveness.

The remainder of this paper is organized as follows. Section 2 develops the multi-country continuous-time mortality model. Section 3 introduces the arbitrage-free Nelson-Siegel (AFNS) model used to model the dynamics of interest rate and provides the calibration results. Based on the mortality and interest rate models, Section 4 covers the construction of a value-based longevity index and applies the graphical basis risk metric using Australia and UK data. Finally we illustrate the hedging result for populations from the Netherlands and France with common interest rates and only mortality risk. Concluding remarks are presented in Section 5.

2 Multi-Country Mortality Model

2.1 Model Setup

There has been a shift in mortality modelling from deterministic models to stochastic models in recent years. There are several reasons behind the need for an effective stochastic mortality model. Unanticipated mortality improvements have proved to be a significant challenge for insurance companies and pension funds [Cairns et al., 2006]; solvency and accounting requirements have required a risk and market-based approach; and, the development of longevity-linked transactions, indexes, securities and derivatives requires the integration of mortality risk analysis into stochastic valuation models [Biffis et al., 2010].

In respect of stochastic financial models, the affine term structure models [Duffie et al., 1996, Dai and Singleton, 2000] have proved invaluable for interest rate risk management. Their flexibility and analytic tractability have been shown to be fundamental for valuation and risk management. In particular, with the increasing pace of financial integration and cooperation, the empirical performance and application of multi-country affine term structure models have gained popularity in the international financial markets [Ahn, 2004, Egorov et al., 2011, Hodrick and Vassalou, 2002, Tang and Xia, 2007].

In the longevity market, a variety of cross-country mortality models have been considered in the literature, most of which are discrete-time models (see Enchev et al. [2016] for an overview). Continuous-time models have significant benefits for the valuation and hedging of life insurance liabilities when the mortality model is to be integrated with interest rate risk models. The continuous-time affine term structure models (ATSMs) have recently been adapted for mortality modelling and have been shown to be successful in capturing the dynamics of mortality improvements [Blackburn and Sherris, 2013, Jevtić et al., 2013, Biffis, 2005, Schrager, 2006, Xu et al., 2015]. Regardless of the need of a multi-country mortality model, there has not been any development of such a model in the affine framework. To fill this gap, we develop a multi-country ATSM to model mortality dynamics. Furthermore, we assess if the multi-country ATSM is consistent with empirical data while maintaining tractability. Our analysis makes both theoretical and empirical contributions to the literature.

This section presents a multi-factor joint ATSM for a domestic country, d, and a foreign country,

f. The multi-factor model considered is a three-factor model with one common factor and two local country-specific factors. We define a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \bar{Q})$, where Ω is the set of possible states of nature, $\mathbf{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions of right continuous and $\bar{Q}$ completeness where $\bar{Q}$ is interpreted as the best-estimate probability measure which is estimated from observed mortality data.

Assume that the domestic and foreign cohort instantaneous mortality intensities, $\mu^d(x,t)$ and $\mu^f(x,t)$ respectively for an individual age x at time t are affine functions of latent state variables

$$\begin{aligned} \mu^d(x,t) &= \delta_0 + \delta_1' \mathbf{Y}_x^d(t), \\ \mu^f(x,t) &= \delta_0 + \delta_1' \mathbf{Y}_x^f(t), \end{aligned} \tag{1}$$

where $\delta_0 \in \mathbb{R}$ and $\delta_1 \in \mathbb{R}^2$. Here, $\mathbf{Y}_x^d(t)$ and $\mathbf{Y}_x^f(t)$ are vectors of two factors defined as

$$\mathbf{Y}_x^d(t) = \begin{pmatrix} X(t) \\ Z_x^d(t) \end{pmatrix} \text{ and } \mathbf{Y}_x^f(t) = \begin{pmatrix} X(t) \\ Z_x^f(t) \end{pmatrix}, \tag{2}$$

where $X(t)$ is the common factor that affects both countries, $Z_x^d(t)$ and $Z_x^f(t)$ are local factors that only affect a specific country. For notational convenience, we suppress the dependence on x in the following definitions of all the factors. We use a cohort model for mortality so that we follow a cohort through time and as time passes an individual ages.

Rather than assuming no connection between the mortality rates of the two populations, this model allows dependence through the common factor driving mortality developments of both populations. The state variables $(X(t), Z^d(t), Z^f(t))'$ follow affine diffusions under the best-estimate measure $\bar{Q}$ such that

$$\begin{pmatrix} dX(t) \\ dZ^d(t) \\ dZ^f(t) \end{pmatrix} = - \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{pmatrix} \begin{pmatrix} X(t) \\ Z^d(t) \\ Z^f(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dW_\mu^{\bar{Q}}(t) \\ dW_\mu^{d,\bar{Q}}(t) \\ dW_\mu^{f,\bar{Q}}(t) \end{pmatrix}, \tag{3}$$

where ϕ_1, ϕ_2, ϕ_3, σ_1, σ_2 and σ_3 represent the best-estimate parameters, $W_\mu^{\bar{Q}}(t)$, $W_\mu^{d,\bar{Q}}(t)$ and $W_\mu^{f,\bar{Q}}(t)$ are independent standard Wiener processes.

A two-country joint mortality model as specified above is said to be decomposable as it can be decomposed into two single-country mortality models [Egorov et al., 2011]. Egorov et al. [2011] states that a two-country joint affine model, like the one presented above, is decomposable if the common factors do not depend on local factors. Since our two-country model is an independent-factor model (factors do not depend on each other), it can be decomposed into two single-country term structure models.

Given the dynamics of $(X(t), Z^d(t))'$ under $\bar{Q}$, the domestic cohort survival probabilities at initial age x and time t can be represented as[3]

$$
\begin{aligned}
S^d(x,t,T) &= E^{\bar{Q}}[e^{-\int_t^T \mu^d(x,s)ds}|\mathcal{F}_t] \\
&= e^{B_1^d(t,T)X(t)+B_2^d(t,T)Z^d(t)+A^d(t,T)},
\end{aligned} \tag{4}
$$

where $B_1^d(t,T)$, $B_2^d(t,T)$ and $A^d(t,T)$ are governed by ordinary differential equations

$$
\begin{pmatrix} \frac{dB_1^d(t,T)}{dt} \\ \frac{dB_2^d(t,T)}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \begin{pmatrix} B_1^d(t,T) \\ B_2^d(t,T) \end{pmatrix}, \tag{5}
$$

$$
\frac{dA^d(t,T)}{dt} = -\frac{1}{2}\sum_{j=1}^{2}((\Sigma^d)'B^d(t,T)B^d(t,T)'(\Sigma^d))_{j,j}, \tag{6}
$$

with boundary conditions $B_1^d(T,T) = B_2^d(T,T) = A^d(T,T) = 0$ and Σ^d being a diagonal matrix with elements, σ_j for $j = 1, 2$. The solution to the above system of ODEs can be represented as

$$
\begin{aligned}
B_1^d(t,T) &= -\frac{1-e^{-\phi_1(T-t)}}{\phi_1}, \\
B_2^d(t,T) &= -\frac{1-e^{-\phi_2(T-t)}}{\phi_2}, \\
A^d(t,T) &= \frac{1}{2}\sum_{i=1,2}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i(T-t)}) - 2(1-e^{-\phi_i(T-t)}) + \phi_i(T-t)].
\end{aligned} \tag{7}
$$

The corresponding domestic average force of mortality curve is an affine function of the state variables $X(t)$ and $Z^d(t)$ which can be represented as

$$
\begin{aligned}
\bar{\mu}^d(x,t,T) &= -\frac{1}{T-t}\log[S^d(x,t,T)] \\
&= \frac{1-e^{-\phi_1(T-t)}}{\phi_1(T-t)}X(t) + \frac{1-e^{-\phi_2(T-t)}}{\phi_2(T-t)}Z^d(t) - \frac{A^d(t,T)}{T-t}.
\end{aligned} \tag{8}
$$

[3]See Dai and Singleton [2000] for the analysis of a single-country model.

Similarly, the foreign average force of mortality curve is given by

$$
\begin{aligned}
\bar{\mu}^f(x,t,T) &= -\frac{1}{T-t}\log[S^f(x,t,T)] \\
&= \frac{1-e^{-\phi_1(T-t)}}{\phi_1(T-t)}X(t) + \frac{1-e^{-\phi_3(T-t)}}{\phi_3(T-t)}Z^f(t) - \frac{A^f(t,T)}{T-t},
\end{aligned} \tag{9}
$$

where

$$
A^f(t,T) = \frac{1}{2}\sum_{i=1,3}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i(T-t)}) - 2(1-e^{-\phi_i(T-t)}) + \phi_i(T-t)].
$$

The corresponding foreign cohort survival function is

$$
S^f(x,t,T) = E^{\bar{Q}}[e^{-\int_t^T \mu^f(x,s)ds}|\mathcal{F}_t]. \tag{10}
$$

Note that all the coefficients ϕ_1, ϕ_2, ϕ_3, ψ_1, ψ_2, ψ_3, σ_1, σ_2 and σ_3 depend on age x. As a result, the domestic and foreign survival functions depend also on age x through $B_1^d(t,T)$, $B_2^d(t,T)$, $A^d(t,T)$, $B_1^f(t,T)$, $B_2^f(t,T)$ and $A^f(t,T)$.

2.2 Empirical Analysis

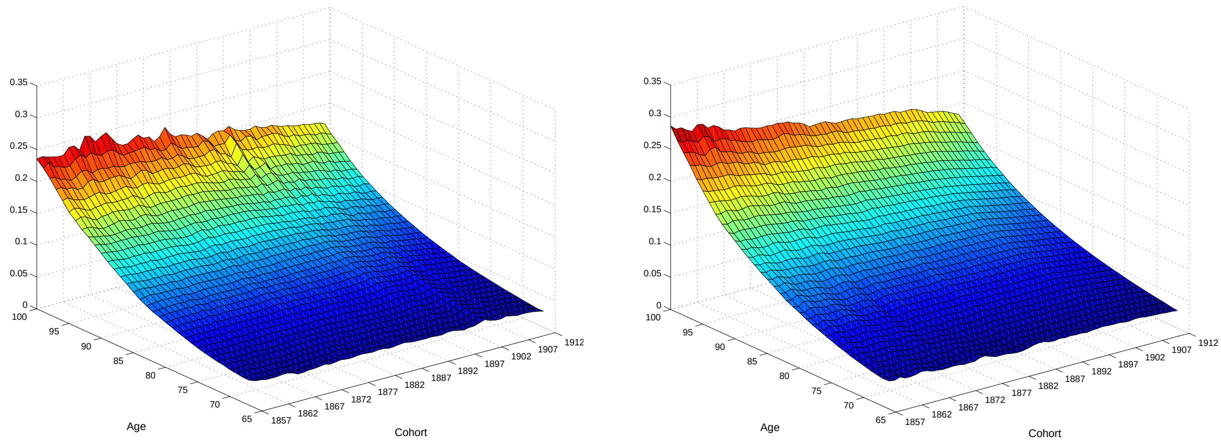
For the empirical analysis, we calibrate the joint mortality model by taking Australia as the domestic country and the UK as the foreign country. We choose the UK as the foreign country for two reasons. Firstly, the UK is one of the largest economies in the world with a well developed financial and life insurance market. Secondly, the study of UK mortality shows that there are clear signs of cohort effects in the UK [Gallop, 2008] which leads to the need for an internally consistent mortality model of cross-country cohort mortality. We show that the joint affine mortality model is rich enough to address this issue.

We use male mortality data for the age range 65 to 100 for cohorts born from 1857 to 1911 with 5-year intervals for both countries as full data for these cohorts are available in both countries over this period. Since the sample for ages above 100 is rather small, we take 100 as the maximum age. Thus we use 11 cohort groups with ages from 65 to 100.

The data source is the Human Mortality Database[4] (HMD). The average force of mortality as defined in Equation (8) and (9) is shown in Figure 1. From Figure 1 we note that the average force of mortality of Australian cohorts is generally lower and more volatile than that of UK cohorts. In

[4]http://www.mortality.org/

Figure 2 we compare the average force of mortality of four Australian and UK cohorts (from the oldest cohort to the youngest cohort), where we observe lower mortality rates in Australia across all cohorts. This figure also reveals that there has been significant mortality improvements through time, with higher rates associated with the 1857 cohort and much lower rates associated with the 1911 cohort.



(a) Australian cohort mortality data

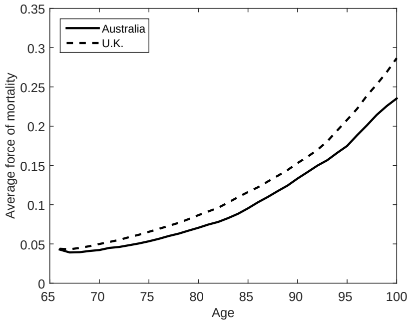
(b) UK cohort mortality data

Figure 1: Average force of mortality for males born from 1857 to 1911, aged 65 to 100.

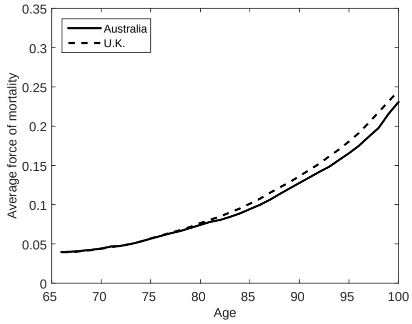
This joint affine mortality model is rewritten in a state space form and the parameters estimated using the Kalman Filter algorithm [Harvey, 1990]. The Kalman Filter method is widely used in the estimation of term structure models that can be rewritten in a state space form [Babbs and Nowman, 1999, Jong, 2000, Christensen et al., 2011]. The state space form involves the measurement equation and the state transition equation. The measurement equation represents the affine relationship between the average force of mortality (or yield rates) and the state variables, whilst the state transition equation describes the dynamics of the state variables. Details of the Kalman Filter algorithm are provided in Appendix A.

In our joint affine mortality model, the measurement equation is

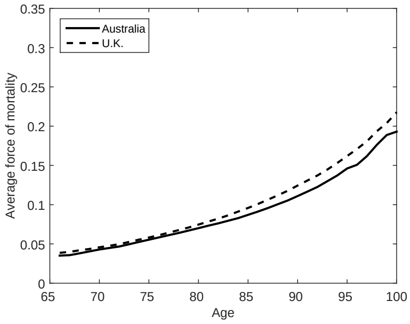
$$\bar{\mu}_t = -BY_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{11}$$



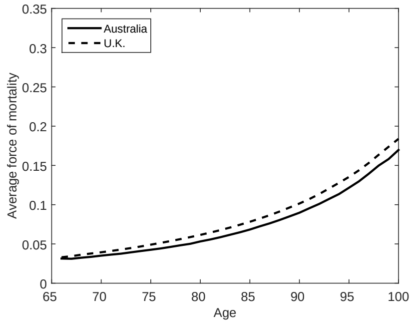
(a) 1857 cohort



(b) 1877 cohort



(c) 1897 cohort



(d) 1911 cohort

Figure 2: Average force of mortality of selected cohorts, aged 65 to 100 for Australia and UK respectively.

where

$$\bar{\mu}_t = \begin{pmatrix} \bar{\mu}_t^d(\tau_1) \\ \vdots \\ \bar{\mu}_t^d(\tau_k) \\ \bar{\mu}_t^f(\tau_1) \\ \vdots \\ \bar{\mu}_t^f(\tau_k) \end{pmatrix}, \quad B = -\begin{pmatrix} \frac{1-e^{-\phi_1\tau_1}}{\phi_1\tau_1} & \frac{1-e^{-\phi_2\tau_1}}{\phi_2\tau_1} & 0 \\ \vdots & \vdots & \vdots \\ \frac{1-e^{-\phi_1\tau_k}}{\phi_1\tau_k} & \frac{1-e^{-\phi_2\tau_k}}{\phi_2\tau_k} & 0 \\ \frac{1-e^{-\phi_1\tau_1}}{\phi_1\tau_1} & 0 & \frac{1-e^{-\phi_3\tau_1}}{\phi_3\tau_1} \\ \vdots & \vdots & \vdots \\ \frac{1-e^{-\phi_1\tau_k}}{\phi_1\tau_k} & 0 & \frac{1-e^{-\phi_3\tau_k}}{\phi_3\tau_k} \end{pmatrix}, \quad Y_t = \begin{pmatrix} X(t) \\ Z^d(t) \\ Z^f(t) \end{pmatrix},$$

$$A = \begin{pmatrix} \frac{1}{2\tau_1}\sum_{i=1,2}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i\tau_1}) - 2(1-e^{-\phi_i\tau_1}) + \phi_i\tau_1] \\ \vdots \\ \frac{1}{2\tau_k}\sum_{i=1,2}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i\tau_k}) - 2(1-e^{-\phi_i\tau_k}) + \phi_i\tau_k] \\ \frac{1}{2\tau_1}\sum_{i=1,3}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i\tau_1}) - 2(1-e^{-\phi_i\tau_1}) + \phi_i\tau_1] \\ \vdots \\ \frac{1}{2\tau_k}\sum_{i=1,3}\frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i\tau_k}) - 2(1-e^{-\phi_i\tau_k}) + \phi_i\tau_k] \end{pmatrix},$$

and H is the covariance matrix for the Gaussian observation noise. Since the volatility of the measurement error varies with age, we assume H to be an n-dimensional diagonal matrix with elements $\sigma_\varepsilon^2(\tau_i)$ $(i = 1, 2, ..., N)$ taking an exponential form

$$\sigma_\varepsilon^2(\tau_i) = \varepsilon_1 \exp(\varepsilon_2 \tau_i), \tag{12}$$

where ε_1 and ε_2 are two constants. With this specification the volatility of the measurement error is exponentially increasing with age reflecting the increasing volatility of observed mortality rates at older ages as the size of the population reduces. This is often referred to as the "Poisson" variation.

The state transition equation is

$$Y_t = \Psi Y_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{13}$$

where

$$\Psi = \begin{pmatrix} e^{-\psi_1} & 0 & 0 \\ 0 & e^{-\psi_2} & 0 \\ 0 & 0 & e^{-\psi_3} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{\sigma_1^2}{2\psi_1}(1 - e^{-2\psi_{11}}) & 0 & 0 \\ 0 & \frac{\sigma_2^2}{2\psi_2}(1 - e^{-2\psi_2}) & 0 \\ 0 & 0 & \frac{\sigma_3^2}{2\psi_3}(1 - e^{-2\psi_3}) \end{pmatrix}.$$

The estimated parameters are reported in Table 2. Figure 3 plots the mean absolute percentage errors (MAPE) of survival probabilities where it can be noted that while the two country affine mortality model does a better job in the UK than in Australia, the model provides a satisfactory fit in both countries. Both countries have similar profiles over the age range in terms of model fit. Australian population sizes are smaller and this reflects in the slightly higher errors at older ages. Another reason for the higher errors at the older ages is mortality heterogeneity. The model assumes a single mortality rate for each age and does not account for a possible distribution of mortality rates, as would occur with mortality heterogeneity, which would lead to a higher variability at older ages. Modifications to the model to account for these factors are the topic of future research.

Table 2: Estimated multi-country mortality model parameters. The maximized log likelihood for the parameter estimates is 9847, $\varepsilon_1 = 1.51 \times 10^{-6}$ and $\varepsilon_2 = 0.01$.

i	ϕ_i	ψ_i	σ_i
1	-0.0847	0.0090	0.0004
2	-0.0329	0.0223	0.0012
3	-0.0302	0.0071	0.0011

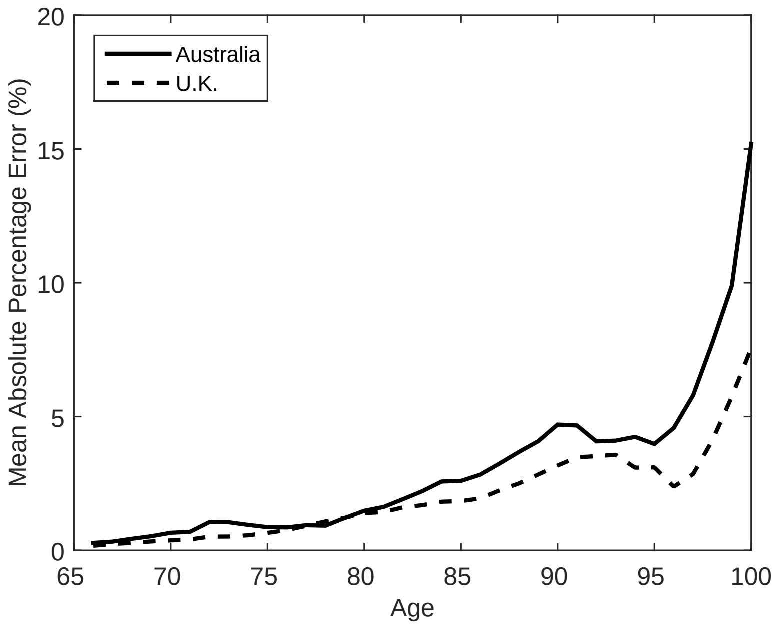


Figure 3: In-sample mean absolute percentage error of survival probabilities, for cohorts born from 1857 to 1911, aged 65 to 100

3 Interest Rate Model

In order to develop our value-based indexes, we require an interest rate term structure model for multiple countries. We require a term structure to value longevity-linked cash–flows, and we need to incorporate interest rate risk into the hedging. Several term structure models have been proposed in the literature, with one class being the affine term structure models (ATSMs). This class encompasses the models of Vasicek [1977], Cox et al. [1985], Duffie et al. [1996] among others. It is a tractable class of Markov arbitrage-free models. Another class is the Heath-Jarrow-Morton model [Heath et al., 1992] and its several extensions. Models belonging to this class are forward rate models and evolve the whole yield curve forward in time. These models are usually non-Markovian and in general computationally intractable. We adopt the ATSM class due to its theoretical tractability and ease of application.

We adopt the arbitrage-free Nelson-Siegel (AFNS) model developed in Christensen et al. [2011] which has proved to better capture the full effects of shifts in the term structure. We also adopt the assumption that the risk factors in the interest rate model are independent from those of the mortality model [Biffis, 2005].

The AFNS model belongs to the affine term structure model class while maintaining the yield curve representation introduced by Nelson and Siegel [1987]. The standard Nelson-Siegel model does not satisfy the no-arbitrage condition, but is well known for its good empirical fit of the observed term structure. We choose the well-specified AFNS model since it combines the no-arbitrage property of ATSMs and the good empirical fit of Nelson-Siegel models. On the one hand it is flexible enough to cover most yield curve dynamics; on the other hand it has closed-form expressions for zero-coupon prices which can greatly facilitate pricing applications [Dai and Singleton, 2000].

3.1 Model Setup

Denoting the domestic and foreign zero-coupon bond prices contracted at time t with $T-t$ to maturity by $P^d(t,T)$ and $P^f(t,T)$, by definition we have

$$P^d(t,T) = e^{-(T-t)y^d(t,T)}, \tag{14}$$

$$P^f(t,T) = e^{-(T-t)y^f(t,T)}, \tag{15}$$

where $y^d(t,T)$ and $y^f(t,T)$ are the corresponding domestic and foreign zero-coupon yield rates.

To specify the dynamics of the AFNS model, we take the domestic country as an example (full details of the AFNS model can be found in Christensen et al. [2011]). Christensen et al. [2011] show that the domestic yield function of the AFNS model is given by

$$y^d(t,T) = L^d(t) + \frac{1-e^{-\lambda^d(T-t)}}{\lambda^d(T-t)}S^d(t) + \left[\frac{1-e^{-\lambda^d(T-t)}}{\lambda^d(T-t)} - e^{-\lambda^d(T-t)}\right]C^d(t) - \frac{V^d(t,T)}{T-t}, \tag{16}$$

where λ^d is the Nelson-Siegel parameter, $L^d(t)$, $S^d(t)$ and $C^d(t)$ are the time-varying level, slope and curvature factors, $\frac{V^d(t,T)}{T-t}$ is the yield-adjustment term. In the independent-factor case the yield-adjustment term is

$$\begin{aligned}
\frac{V^d(t,T)}{T-t} =& \frac{(T-t)^2}{6}(s_1^d)^2 \\
&+ \left[\frac{1}{2(\lambda^d)^2} - \frac{1}{(\lambda^d)^3}\frac{1-e^{-\lambda^d(T-t)}}{(T-t)} + \frac{1}{4(\lambda^d)^3}\frac{1-e^{-2\lambda^d(T-t)}}{(T-t)}\right](s_2^d)^2 \\
&+ \left[\frac{1}{2(\lambda^d)^2} + \frac{1}{(\lambda^d)^2}e^{-\lambda^d(T-t)} - \frac{1}{4\lambda^d}(T-t)e^{-2\lambda^d(T-t)} - \frac{3}{4(\lambda^d)^2}e^{-\lambda^d(T-t)}\right. \\
&\quad \left. -\frac{2}{(\lambda^d)^3}\frac{1-e^{-\lambda^d(T-t)}}{(T-t)} + \frac{5}{8(\lambda^d)^3}\frac{1-e^{-2\lambda^d(T-t)}}{(T-t)}\right](s_3^d)^2,
\end{aligned}$$

where s_1^d, s_2^d and s_3^d are elements in the volatility matrix of the three factors.

Under the risk-neutral Q-measure, $L^d(t)$, $S^d(t)$ and $C^d(t)$ have the following dynamics

$$\begin{pmatrix} dL^d(t) \\ dS^d(t) \\ dC^d(t) \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^d & -\lambda^d \\ 0 & 0 & \lambda^d \end{pmatrix} \begin{pmatrix} L^d(t) \\ S^d(t) \\ C^d(t) \end{pmatrix} dt + \begin{pmatrix} s_1^d & 0 & 0 \\ 0 & s_2^d & 0 \\ 0 & 0 & s_3^d \end{pmatrix} \begin{pmatrix} dW_1^{d,Q}(t) \\ dW_2^{d,Q}(t) \\ dW_3^{d,Q}(t) \end{pmatrix}, \tag{17}$$

while under the real-world P-measure

$$\begin{pmatrix} dL^d(t) \\ dS^d(t) \\ dC^d(t) \end{pmatrix} = \begin{pmatrix} \kappa_1^d & 0 & 0 \\ 0 & \kappa_2^d & 0 \\ 0 & 0 & \kappa_3^d \end{pmatrix} \left[\begin{pmatrix} \theta_1^d \\ \theta_2^d \\ \theta_3^d \end{pmatrix} - \begin{pmatrix} L^d(t) \\ S^d(t) \\ C^d(t) \end{pmatrix} \right] dt + \begin{pmatrix} s_1^d & 0 & 0 \\ 0 & s_2^d & 0 \\ 0 & 0 & s_3^d \end{pmatrix} \begin{pmatrix} dW_1^{d,P}(t) \\ dW_2^{d,P}(t) \\ dW_3^{d,P}(t) \end{pmatrix}, \tag{18}$$

with κ_1^d, κ_2^d, κ_3^d, θ_1^d, θ_2^d and θ_3^d being the real-world parameters for the domestic country.

Similar assumptions are made for the financial market in the foreign country so that the foreign yield to maturity is

$$y^f(t,T) = L^f(t) + \frac{1 - e^{-\lambda^f(T-t)}}{\lambda^f(T-t)} S^f(t) + \left[\frac{1 - e^{-\lambda^f(T-t)}}{\lambda^f(T-t)} - e^{-\lambda^f(T-t)} \right] C^f(t) - \frac{V^f(t,T)}{T-t}. \tag{19}$$

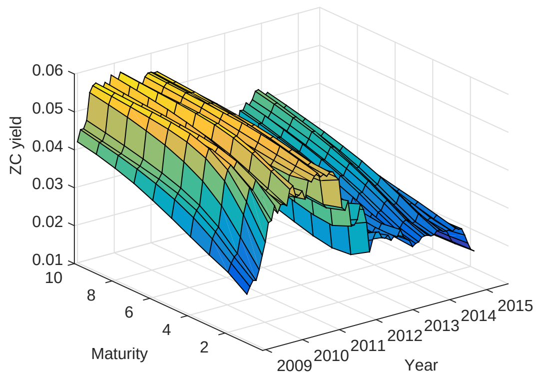
Equation (16) and (19) will be used in the estimation of AFNS model in Section 3.2.

3.2 Empirical Analysis

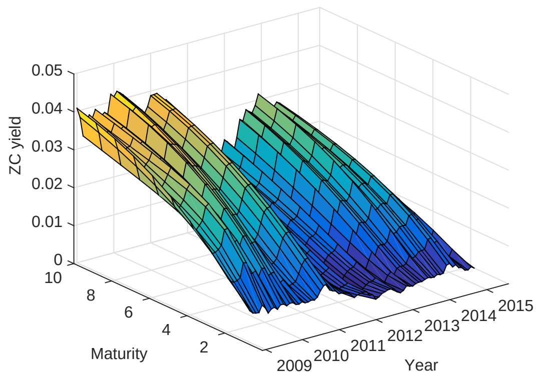
Following the mortality framework presented in Section 2, we model a financial market consisting of Australian and UK treasury bonds. We use zero-coupon yields provided by the Reserve Bank of Australia and the Bank of England. To calibrate the interest rate model, we consider the period from January 2009 to March 2015, using end-of-month observations of 1-, 2-, $\cdots$, 10-year yields from Australia and the UK. Since the yield rates are only provided for maturities up to 10 years in Australia, we employ maturities up to 10 years for both countries to maintain consistency.

Figure 4 provides time series plots of zero coupon yields for both Australia and UK. The yield curves are typically upward sloping but change unpredictably depending on the economic outlook between the respective countries. Descriptive statistics for the two data sets are presented in Table 3. We can also observe the upward sloping trend from the mean of yields. In both countries the mean increases with maturity. We can also observe that yields are highly persistent, there are sizeable autocorrelations at 1 month and even at 6 months.

The AFNS model is represented in a state-space form and estimated using a Kalman filter algorithm.



(a) Australian zero-coupon yield rate curve



(b) UK zero-coupon yield rate curve

Figure 4: Time series of zero coupon yields in Australia and UK

Table 3: Summary statistics for zero coupon yields. All yield data are monthly, from January 2009 through to March 2015. $\hat{\rho}(\tau)$ denotes the autocorrelation at displacement τ.

Australia							
Maturity	Mean	Standard Deviation	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(6)$	$\hat{\rho}(12)$
1Y	3.31	0.9691	1.75	4.92	0.9449	0.7414	0.4853
2Y	3.37	1.0046	1.71	5.05	0.9444	0.7402	0.5287
3Y	3.51	1.0493	1.69	5.21	0.944	0.7283	0.5202
4Y	3.65	1.0736	1.71	5.39	0.945	0.7192	0.4990
5Y	3.78	1.0726	1.78	5.49	0.9458	0.7129	0.4783
6Y	3.90	1.0575	1.88	5.55	0.9463	0.7087	0.4604
7Y	4.00	1.035	1.99	5.6	0.9457	0.7049	0.4447
8Y	4.09	1.0096	2.1	5.64	0.9442	0.6992	0.4311
9Y	4.18	0.9859	2.2	5.68	0.942	0.6933	0.4182
10Y	4.26	0.9659	2.29	5.72	0.9397	0.6864	0.4074
UK							
Maturity	Mean	Standard Deviation	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(6)$	$\hat{\rho}(12)$
1Y	0.50	0.1933	0.17	0.95	0.8851	0.6396	0.4912
2Y	0.79	0.4305	0.07	1.72	0.9369	0.6722	0.4069
3Y	1.14	0.5918	0.16	2.38	0.9443	0.6812	0.3967
4Y	1.48	0.6798	0.35	2.80	0.9443	0.6847	0.4021
5Y	1.78	0.7295	0.57	3.08	0.9419	0.6830	0.4074
6Y	2.06	0.7606	0.80	3.33	0.9386	0.6769	0.4094
7Y	2.30	0.7824	1.02	3.62	0.9353	0.6692	0.4085
8Y	2.52	0.7978	1.22	3.85	0.9323	0.6613	0.4063
9Y	2.71	0.8084	1.31	4.05	0.9301	0.6549	0.4040
10Y	2.88	0.8145	1.39	4.24	0.9285	0.6501	0.4022

To display the state-space representations of the domestic and foreign AFNS model, we drop the superscript d and f for simplicity. The measurement equation is

$$y_t = -BY_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{20}$$

where

$$y_t = \begin{pmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_k) \end{pmatrix}, \quad B = -\begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_k}}{\lambda\tau_k} & \frac{1-e^{-\lambda\tau_k}}{\lambda\tau_k} - e^{-\lambda\tau_k} \end{pmatrix}, \quad Y_t = \begin{pmatrix} L(t) \\ S(t) \\ C(t) \end{pmatrix}, \quad A = \begin{pmatrix} \frac{V(\tau_1)}{\tau_1} \\ \vdots \\ \frac{V(\tau_k}{\tau_k} \end{pmatrix}.$$

The state transition equation is

$$Y_t = (I - \mathrm{e}^{-K\Delta t})\Theta + \mathrm{e}^{-K\Delta t} Y_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{21}$$

where

$$K = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Since we use monthly data, we have $\Delta t = \frac{1}{12}$ and $Q = \int_0^{\Delta t} \mathrm{e}^{-Ks}\Sigma\Sigma^{\mathrm{T}}\mathrm{e}^{-(K^{\mathrm{T}}s)} ds$ where

$$\Sigma = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}.$$

Estimates for the independent-factor AFNS model are presented in Table 4. They are generally consistent with the estimates presented in Christensen et al. [2011].

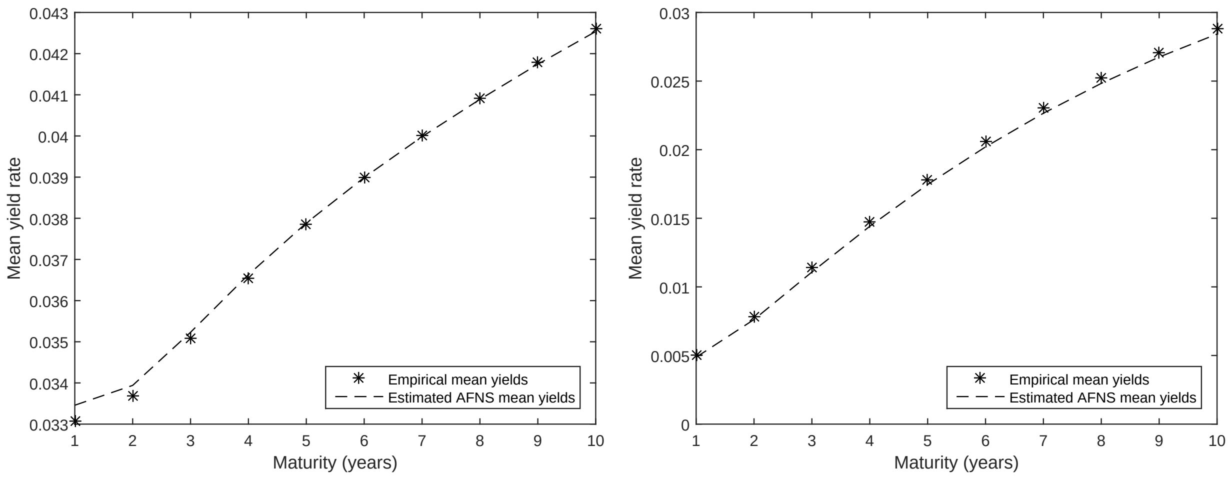
Table 4: Estimates for the independent-factor AFNS model. The maximized log likelihood is 4672.62 for Australia and 4144.24 for UK.

	Australia				UK			
i	κ_i^d	θ_i^d	s_i^d	λ^d	κ_i^f	θ_i^f	s_i^f	λ^f
1	0.0576	0.0702	0.0062	0.5232	0.0436	0.0702	0.0068	0.7449
2	0.2312	-0.0252	0.0046		0.2313	-0.0252	0.0046	
3	1.0623	-0.0136	0.0059		1.0623	-0.0115	0.0046	

For the in-sample fit, residual means and their root mean square errors (RMSE) are provided in Table 5. The lower log likelihood value together with the generally larger residual means and RMSEs obtained for UK market show that the AFNS model performs better for the Australian market, but overall the AFNS model shows good in-sample performance in both countries. Additional evidence on goodness-of-fit is provided in Figure 5 which plots the mean yield curves of the empirical data and estimated AFNS models for both countries. From Figure 5 we can directly see that the calibrated curves perform well.

Table 5: Residual means and their root mean square errors for maturities measured in years. Means and RMSEs are in basis points.

	Australia		UK	
Maturity	Mean	RMSE	Mean	RMSE
1Y	1.16	3.15	3.53	4.93
2Y	-1.62	3.04	2.21	2.92
3Y	-0.49	3.02	-1.87	4.35
4Y	0.62	2.76	-5.50	5.12
5Y	0.62	2.56	-6.86	4.45
6Y	0.12	2.05	-5.76	3.01
7Y	-0.33	1.42	-2.76	1.74
8Y	-0.48	1.23	1.29	2.30
9Y	-0.13	1.82	5.70	4.01
10Y	0.53	2.65	9.96	5.88



(a) Australia (b) UK

Figure 5: Empirical and estimated mean yield curves in Australia and UK, from January 2009 to March 2015.

4 Longevity Indexes and Cross Country Hedging

In this section we present the value-based indexes for both domestic and foreign countries. Li and Hardy [2011] show that the lack of a multi-country mortality model may lead to assessed basis risk being inaccurate. This arises from a positive correlation between the two populations that should not be neglected. Li and Hardy [2011] assess this problem using variants of the Lee-Carter model [Lee and Carter, 1992] that allow dependence between two populations (see Enchev et al. [2016] for other multi-country mortality models). We use our multi-country continuous time mortality model with a common factor capturing the correlation structure between the two populations to capture the positive correlation between two populations.

We also use a graphical risk metric to assess the basis risk arising from hedging both longevity risk and interest rate risk in a domestic country with foreign indexes. To show the impact of both interest rate and mortality risk we use Australia and UK as domestic and foreign countries, respectively.

4.1 Value-based Longevity Index

The longevity index we use is value-based, allowing for better quantification of risk, and the index includes both longevity risk and interest rate risk. Our index is the value of a unit stream of life-time nominal income commencing at age 65. It is similar to an annuity value quoted by a life insurer except that it includes no risk loading, no expense loading and no profit loading. It captures the impact of mortality and interest rates through the models we have proposed and calibrated.

The value-based index also reflects the liability of a life insurer issuing a life annuity, or a pension fund paying a non-indexed defined benefit pension. As mortality is currently a non-tradable asset, the value-based longevity index provides an index that reflects the cost of providing retirement income and the liability of an annuity provider. The development of a value-based longevity index, including multi-country indexes, along with an improved framework for modelling and managing longevity risk has the potential to reduce the capital costs of providing long-term retirement products and the solvency risk for life annuity and pension providers through a better understanding of the risk and more effective risk reduction techniques.

With the calibrated mortality and interest rate models we compute the value-based longevity indexes for both countries as

$$I_x^{id}(t) = \sum_{j=1}^{x^*-x} P^d(t, t+j) S^d(x, t, t+j), \tag{22}$$

$$I_x^{if}(t) = \sum_{j=1}^{x^*-x} P^f(t, t+j) S^f(x, t, t+j), \tag{23}$$

where x^* is the maximum age. Each index is defined as the discounted value at t of lifetime annual income of one unit of currency for each cohort in country d or f.

4.2 Basis Risk Metric

The value-based longevity index is designed to track the cost of lifetime annual income on retirement. We use UK longevity indexes associated with the following cohorts; 1950, 1945, 1940 and 1935 (whose members have already retired aged 65, 70, 75 and 80 respectively at the end of December 2015) and assume that mortality swaps can be obtained referenced to these UK indexes. Although we construct value-based longevity indexes for Australia, we assume that there are no liquid contracts that can be referenced to such indexes.

We consider an annuity provider exposed to life annuity payments for a pool consisting of an Australian cohort and hedging the longevity risk arising from this Australian cohort by trading mortality swaps indexed to the UK mortality experience, whose survivor index is represented in Equation (23). Because in practice there are multiple cohorts to hedge, we assess three alternatives on how the annuity provider can select the most appropriate UK cohort for hedging the Australian cohort.

Table 6 shows the three examples considered; for Australian cohorts born in 1940 and 1945, we use UK cohorts which are of the same age, 5 years older and 5 years younger. For the Australian cohort born in 1950, we use UK cohorts born in 1949, 1945 and 1950 since the 1950 cohort is the youngest UK cohort available in retirement.

Table 6: Selected domestic cohort and foreign cohorts for Example I, II and III, which are presented below.

Example	Australian Cohort	UK Cohort		
I	1950	1940	1945	1950
II	1945	1940	1945	1950
III	1940	1935	1940	1945

The Australian annuity provider is exposed to the risk of paying $1 annually to each annuitant until the last remaining annuitant dies. The survivor index for each domestic cohort is presented in Equation (22). Let

$$H^d = I_x^{id}(t) - E(I_x^{id}(t)), \tag{24}$$

and

$$H^f = I_x^{if}(t) - E(I_x^{if}(t)), \tag{25}$$

be the exceedances of $I_x^{id}(t)$ and $I_x^{if}(t)$ over their expected values respectively.

As suggested in Chan et al. [2016], we base the graphical risk metric on H^d and H^f rather than $I_x^{id}(t)$ and $I_x^{if}(t)$, as in practice we are primarily interested in deviations from the expected outcomes, and also because the use of H^d and H^f ensures all resulting risk metrics are centred at the origin, thereby enabling a direct comparison of the risk metrics for different foreign cohorts.

Denoting h^f as the hedge ratio, we use h^f units of the foreign index for each unit of domestic index. Our primary hedging objective is to maximise hedge effectiveness, that is, to minimize the uncertainty in $(H^d - h^f H^f)$. Thus the optimization problem becomes that of finding the optimal h^f units of foreign index that minimizes the variance of $(H^d - h^f H^f)$, that is

$$\begin{aligned}\text{Var}(H^d - h^f H^f) &= \text{Var}(H^d) - 2h^f \text{Cov}(H^d, H^f) + (h^f)^2 \text{Var}(H^f) \\ &= \text{Var}(H^f) \left[h^f - \frac{\text{Cov}(H^d, H^f)}{\text{Var}(H^f)} \right]^2 + \text{Var}(H^d) - \frac{\left[\text{Cov}(H^d, H^f)\right]^2}{\text{Var}(H^f)}. \end{aligned} \tag{26}$$

To minimize the above equation, we have

$$h^f = \frac{\text{Cov}(H^d, H^f)}{\text{Var}(H^f)}. \tag{27}$$

The risk metric is illustrated with joint prediction regions at different confidence levels (see Chan et al. [2016]). For a confidence level of $1 - \alpha$ (where $0 \leq \alpha \leq 1$)

$$Pr\left[\left(H^d, h^f H^f \right) \in \mathbf{J}_\alpha \right] = 1 - \alpha, \tag{28}$$

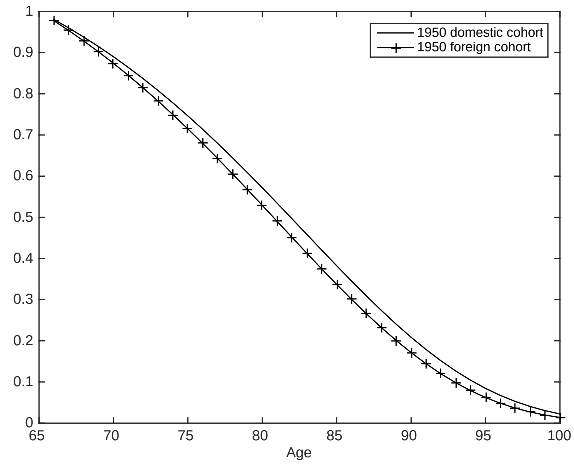
where $\mathbf{J}_\alpha$ is the joint prediction region at the $1 - \alpha$ confidence level.

We use simulation to generate future paths of mortality rates as well as interest rates. The basis risk metric is constructed using the following steps:

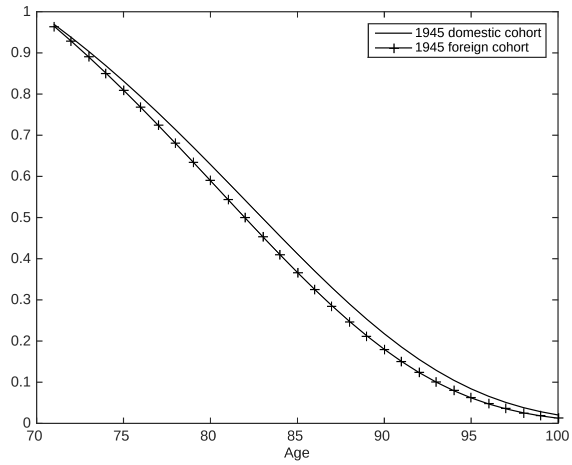
- Step 1: Simulate 20,000 best-estimate domestic and foreign cohort survival curves for the cohorts presented in Table 6, using parameter estimates for the multi-country mortality model (see Table 2).
- Step 2: Simulate 20,000 yield rates for Australia and the UK at the end of December 2015, using parameter estimates for the independent-factor AFNS model as shown in Table 4, and then calculate the corresponding discount bond prices.
- Step 3: Calculate values of domestic and foreign indexes (Equation (22) and (23) respectively) using the cohort survival curves and zero-coupon bond prices obtained in previous steps, and

then calculate realized H^d and H^f.

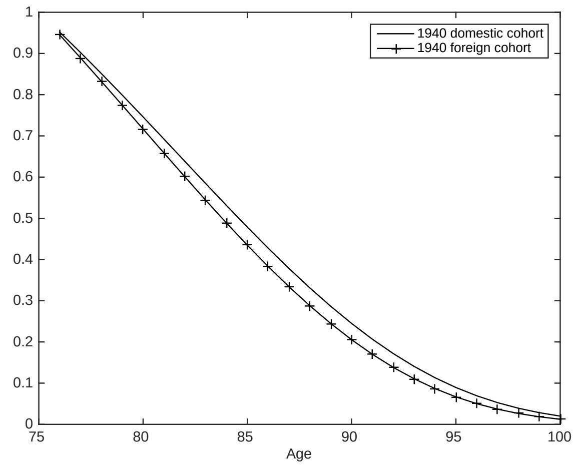
- Step 4: Calculate h^f such that $h^f = \frac{\text{Cov}(H^d, H^f)}{\text{Var}(H^f)}$, the results are presented in Table 7.
- Step 5: For each realized $x = (H^d, h^f H^f)'$ calculate its Mahalanobis distance to the best estimate $\mu = (0,0)'$. Mahalanobis distance measures the distance between an observation of x and the distribution characterized by mean μ and covariance matrix S, $D_M(x) = \sqrt{(x-\mu)' S^{-1} (x-\mu)}$ (details of Mahalanobis distance can be found in Mahalanobis [1936] and Gnanadesikan and Kettenring [1972]). With H^d and H^f defined in Equation (24) and (25), $\mu = (0,0)'$ in our case.
- Step 6: Draw a convex hull that encloses $20,000(1-\alpha)$ realizations with the shortest Mahalanobis distances.



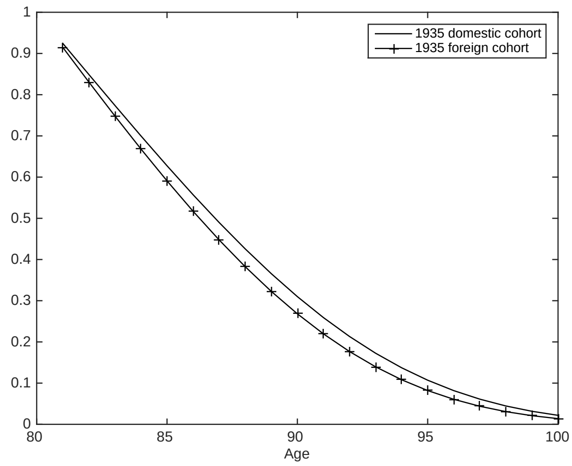
(a) 1950 domestic and foreign cohorts



(b) 1945 domestic and foreign cohorts



(c) 1940 domestic and foreign cohorts



(d) 1935 domestic and foreign cohorts

Figure 6: The means of simulated cohort survival curves for selected cohorts. 20,000 simulations are performed using the three-factor joint ATSM developed in Section 2.

The means of simulated survival curves in the first step are shown in Figure 6. We note that survival probabilities of an Australian cohort are consistently higher than that of the UK cohort born in the same year. This is consistent with the trend in historical data as plotted in Figure 1 and Figure 2, in which the average force of mortality of an Australian cohort is generally lower than that of the UK cohort born in the same year. The lower average force of mortality produces the higher survival probabilities in the Australian population.

For each domestic cohort, we consider three existing foreign indexes around similar ages as specified in Table 6. Table 7 reports the optimal hedge ratios for each cohort which are calculated from Equation (27).

Table 7: The calculated values of h^f, where h^f refers to the optimal hedge ratio.

Example	Domestic Cohort i	Foreign Index	$h^f = \frac{\text{Cov}(H^d, H^f)}{\text{Var}(H^f)}$
I	1950	$I_{75}^{1940f}(t)$	0.9997
		$I_{70}^{1945f}(t)$	0.8540
		$I_{65}^{1950f}(t)$	0.7507
II	1945	$I_{75}^{1940f}(t)$	0.8423
		$I_{70}^{1945f}(t)$	0.7188
		$I_{65}^{1950f}(t)$	0.4815
III	1940	$I_{75}^{1935f}(t)$	0.8423
		$I_{75}^{1940f}(t)$	0.6882
		$I_{70}^{1945f}(t)$	0.4359

Figure 7 shows an example of the basis risk metric with α being 10%, 20%, 30%, 40% and 50%. In this example, we consider the risk exposures to be both longevity risk and interest rate risk of the 1950 domestic cohort. These risk exposures are hedged by a foreign longevity index born in the same year which is $I_{65}^{1950f}(t)$. The red dots are the 20,000 realized $(H^d, h^f H^f)$. For each realization, if a perfect hedge occurs $(H^d = h^f H^f)$, the red dot will lie on the 45-degree line; if an under hedge occurs $(H^d > h^f H^f)$, the red dot will lie below the 45-degree line; if an over hedge occurs $(H^d < h^f H^f)$, the red dot will lie above the 45-degree line. The convex hull (the smallest convex set) drawn is a $100(1-\alpha)$ per cent joint prediction region for H^d and $h^f H^f$, by construction it contains a randomly selected pair of H^d and $h^f H^f$ in the simulated sample with a probability of $(1-\alpha)$. To be more specific, the innermost part (surrounded by the circle of the lightest colour) encompass 50% of possible pairs of H^d and $h^f H^f$ (where $\alpha = 50\%$ and the confidence interval is $1-\alpha = 50\%$, see Equation (28)), surrounded by 60%, 70%, 80% and 90% (surrounded by the circle of the darkest colour) respectively. With this risk metric, the probability of occurrence is related to the ranges of possible hedging outcomes.

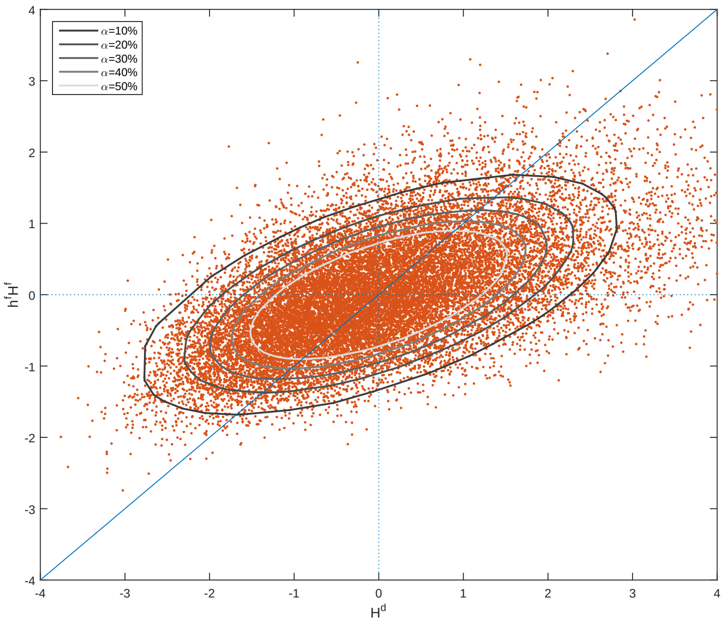
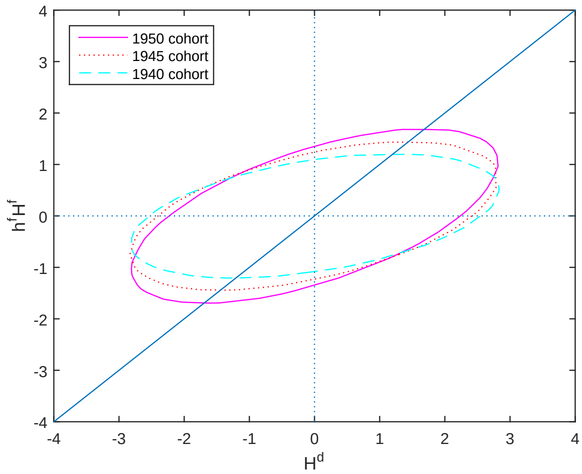


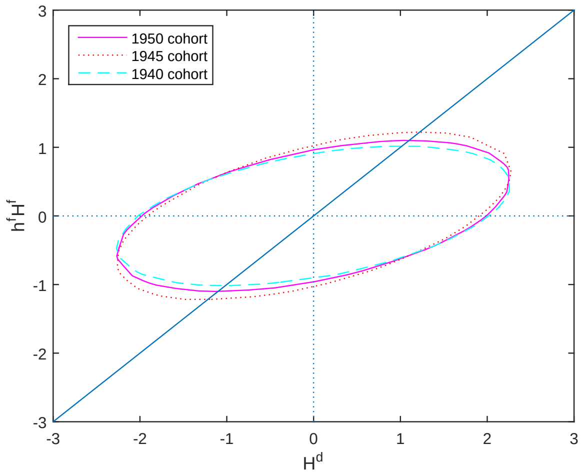
Figure 7: Basis risk metric using $I_{65}^{1950f}(t)$ to hedge risks associated with the domestic 1950 cohort. The optimal hedge ratio is 0.7507.

Given $\alpha = 10\%$, Figure 8 compares risk metrics of three foreign indexes for each domestic cohort. Note that the 45-degree line indicates the perfect hedge, and the deviation from the 45-degree line indicates the extent of over- or under-hedge. We can conclude from Figure 8 that, while with the optimally chosen h^f, all foreign indexes provide satisfactory hedges to the domestic cohorts, the best hedge is the foreign index of the same cohort since it best matches the 45-degree line.

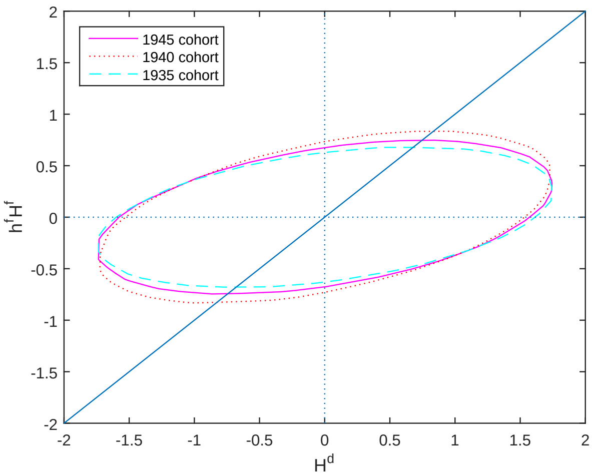
To further explore this, we take the 1950 domestic cohort as an example. Figure 9 gives plots of the simulated surplus distribution of the 1950 domestic cohort hedged with 1950, 1945 and 1940 foreign cohorts respectively. From Figure 9 we can see that the surplus is more centred around zero if the 1950 domestic cohort is hedged with the 1950 foreign cohort, and the distribution deteriorates if an older foreign cohort is used. Figure 9 shows that the 1950 foreign cohort dominates the other foreign cohorts in providing an effective hedge against both mortality and interest rate risks.



(a) Example I: 1950 domestic cohort

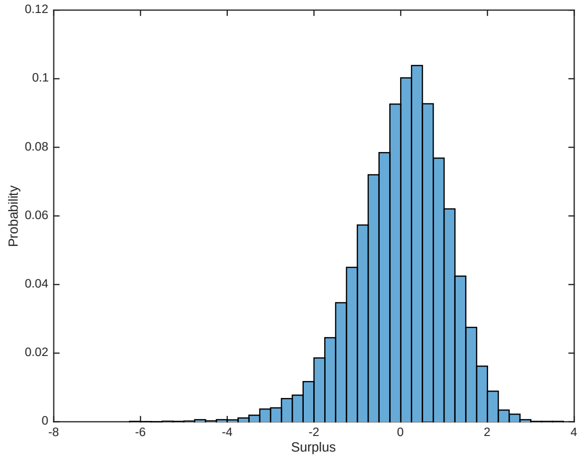


(b) Example II: 1945 domestic cohort

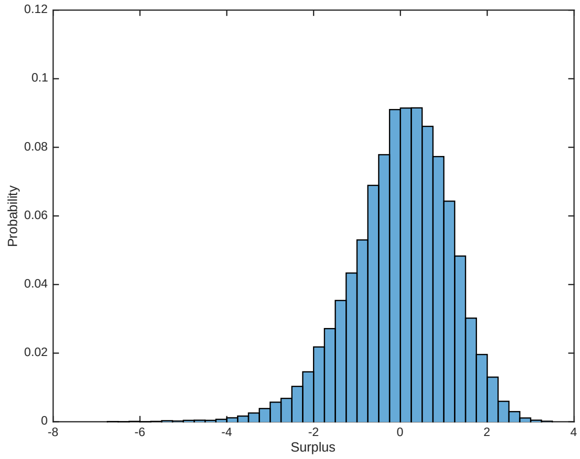


(c) Example III: 1940 domestic cohort

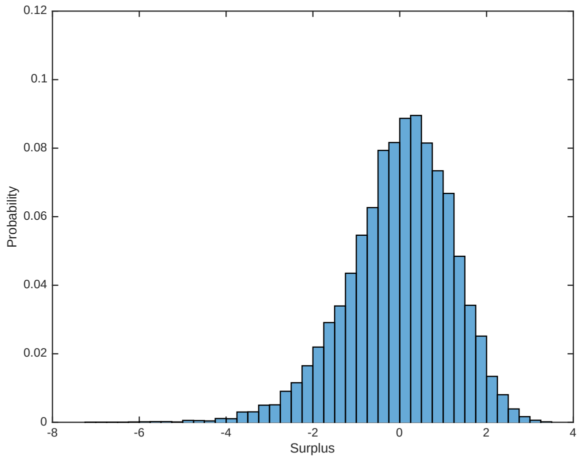
Figure 8: Comparison of basis risk metrics. Each sub-figure shows a comparison of basis risk metrics hedged with three different foreign indexes.



(a) Hedged with 1950 foreign cohort



(b) Hedged with 1945 foreign cohort



(c) Hedged with 1940 foreign cohort

Figure 9: Surplus distribution of risks associated with 1950 domestic cohort hedged using foreign indexes.

4.3 Stochastic Mortality Basis Risk with Deterministic Interest Rates

Most longevity indexes include only mortality risk. They effectively assume that interest rate is fully hedged in any application. To consider only basis risk associated with stochastic mortality evolutions, we redo the analysis presented in Subsection 4.2 where in place of the stochastic interest rates we fix the initial yield curves for both Australia and the UK across various maturities. We use the yield rates for the two countries as of the 31^{st} of December 2015. We maintain the stochastic mortality evolutions as presented in (1) with the resulting survival functions presented in equations (4) and (10) for the Australian and UK populations respectively.

Figure 10 shows the convex hulls for varying levels of significance, α. In comparison with Figure 7, we note that the red dots in Figure 10 are more concentrated around the the 45-degree line highlighting the reduction in basis risk associated with the assumption of deterministic interest rates. However, besides the concentration around the 45-degree line, the two plots are very consistent with each other across all levels of significance with $\alpha = 10\%$ encompassing most of the realizations. Even though interest rates are assumed to be deterministic, the difference in the term structure between the two countries has a significant influence on the basis risk arising from differences in the interest rates.

From Figure 11 we note consistence with findings in Figure 8a where the best hedging instrument for the 1950 domestic cohort is 1950 UK index.

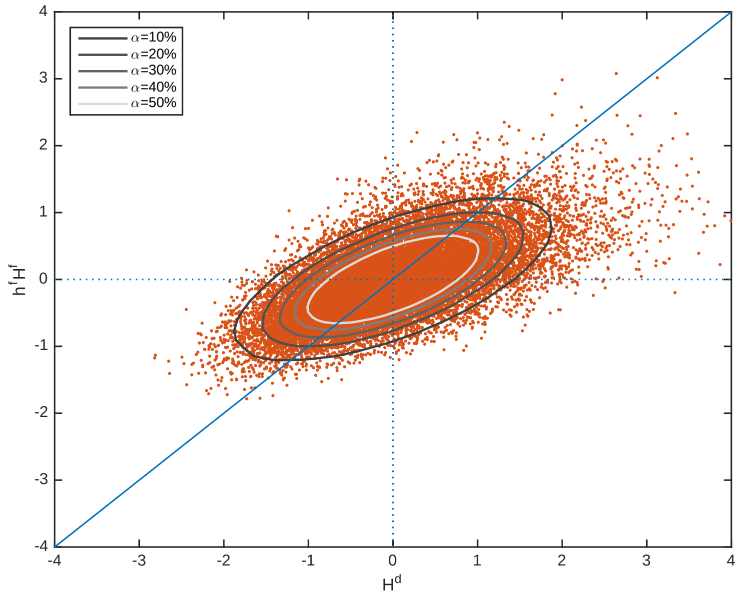


Figure 10: Basis risk metric using $I_{65}^{1950f}(t)$ to hedge risks associated with the domestic 1950 cohort when the interest rates are extracted from the initial forward curve as of the 31^{st} of December 2015.

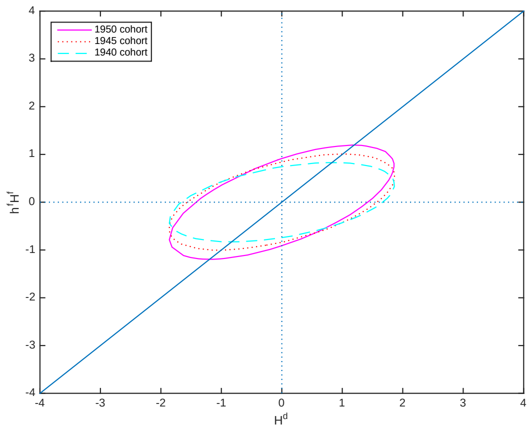


Figure 11: Comparison of basis risk metrics when the 1950 domestic cohort is hedged with three different foreign indexes.

4.4 Common Interest Rates and Different Mortality Experiences

Since many countries in Europe share a common currency and interest rate market we use France and Netherlands data to show the impact on basis risk of a common interest rate term structure. We use the Netherlands as the domestic country and France as the foreign country. Since both countries are in the eurozone where interest rates are set by the European Central Bank, the basis risk only arises from different mortality experiences. Instead of calibrating an interest rate model, we use a constant discount rate of 5% as suggested in Coughlan et al. [2011]. This allows us to focus on hedging effectiveness where basis risk is purely caused by differences in mortality. We recalibrate the mortality model described in Section 2 using mortality experiences in France and Netherlands. We use male mortality data aged 65 to 100 for cohorts born from 1857 to 1911 for both countries which are available in the Human Mortality Database.

Table 8: Estimated multi-country mortality model parameters. The maximized log likelihood for the parameter estimates is 9321, $\varepsilon_1 = 1.41 \times 10^{-6}$ and $\varepsilon_2 = 0.05$.

i	ϕ_i	ψ_i	σ_i
1	-0.0792	0.0088	0.0004
2	0.0237	0.0070	0.0011
3	0.0975	0.0295	0.0014

Table 8 reports the estimated parameter values and Figure 12 plots the in-sample mean absolute percentage errors of survival probabilities for France and Netherlands (equivalent to Table 2 and Figure 3). Comparing Table 8 with Table 2 it is interesting to note that, although local factors behave differently, the parameter values related to the common factor of France and Netherlands are similar to those of the common factor for Australia and the UK. The mean absolute percentage errors for cohorts in France and Netherlands shown in Figure 12 are generally consistent with those in Australia and UK.

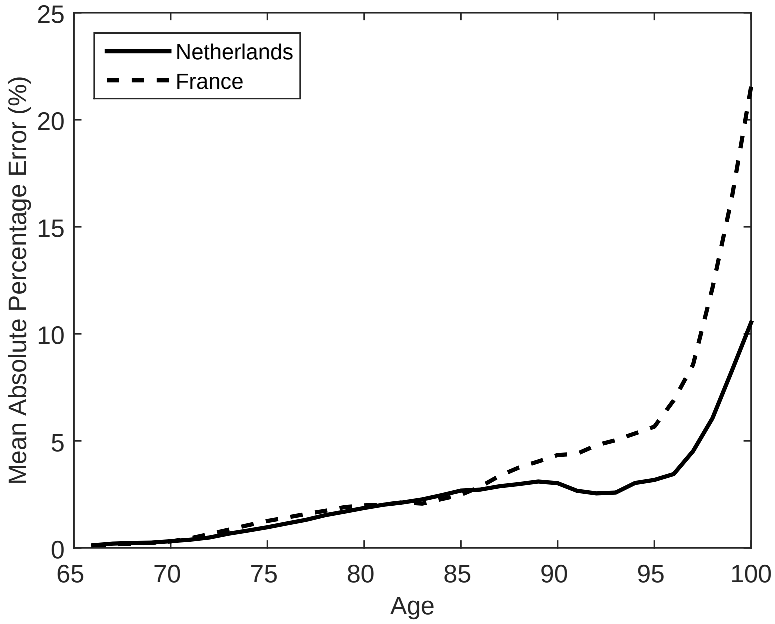


Figure 12: In-sample mean absolute percentage error of survival probabilities, for cohorts born from 1857 to 1911, aged 65 to 100

The French longevity indexes are used to hedge the risk exposures of an annuity provider in the Netherlands. We consider the case where French longevity indexes for cohorts 1940, 1945 and 1950 are available. The calculated optimal hedge ratios are reported in Table 9. The approach for calculating optimal hedge ratios is as described in Section 4.2. Because France and Netherlands have common interest rates, we note that the correlation between H^d and H^f is much higher as compared to that in the Australia and UK case.

Figure 13 shows the ranges of possible hedging outcomes using the three French longevity indexes with a confidence interval of 90% (where $\alpha = 10\%$). In Figure 13 we observe that the best hedge is obtained when the 1945 domestic cohort is hedged with the 1945 foreign index. Comparing Figure 13 with Figure 8(b), we note that the results obtained in the France and Netherlands case are consistent with those obtained in the Australia and UK case where the best hedging instrument is the foreign index of the same cohort. As expected, basis risk is relatively smaller in the France and Netherlands case as the hedging outcomes are much closer to the 45-degree line and this is also attributed by the absence of interest rate risk.

A specific example of basis risk metric, where the hedging instrument is the French longevity index linked to the cohort born in 1945, is given in Figure 14 (equivalent to Figure 7). The area inside the circle with the lightest colour consists of 50% of possible hedging outcomes, and is surrounded by 60%, 70%, 80% and 90% respectively.

Table 9: The calculated values of h^f, where h^f refers to the optimal hedge ratio.

Example	Domestic Cohort i	Foreign Index	$h^f = \frac{\mathrm{Cov}(H^d, H^f)}{\mathrm{Var}(H^f)}$
IV	1945	$I_{75}^{1940f}(t)$	1.0585
		$I_{70}^{1945f}(t)$	0.9428
		$I_{65}^{1950f}(t)$	0.6298

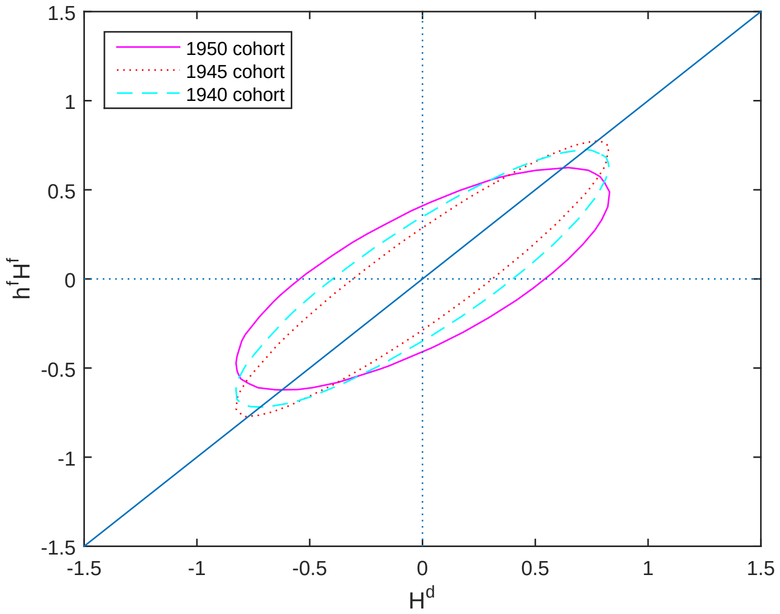


Figure 13: Comparison of basis risk metrics, $\alpha = 10\%$. The risks associated with 1945 domestic cohort are hedged with the 1940, 1945 and 1950 foreign indexes.

5 Conclusion

This paper has proposed and developed a multi-country continuous-time mortality model which allows for positive correlation of longevity improvements in different countries. Along with a multi-country affine Nelson-Seigel term structure model, we propose and assess value-based longevity indexes for multiple cohorts in different countries. The value-based longevity index can be used to measure and hedge longevity risk and interest rate risk. It tracks the cost of lifetime nominal annual income on retirement and provides a better measure of longevity risks for both individuals as well as pension funds and life insurers.

We apply the models and longevity indexes to assess risk exposures of an Australian annuity provider hedging using instruments based on UK longevity indexes. We use a graphical risk metric to assess the basis risk arising from the index-based hedge. We show how optimal hedge ratios

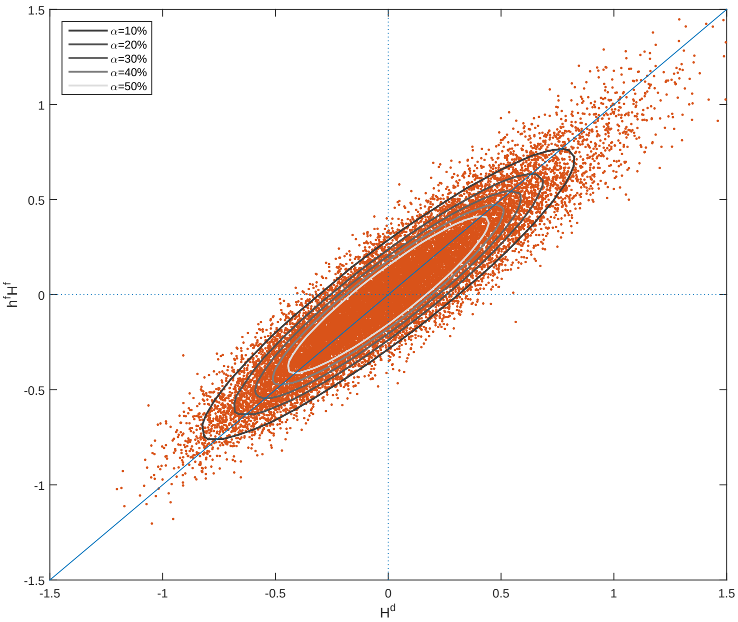


Figure 14: Basis risk metric using $I_{70}^{1945f}(t)$ to hedge risks associated with the domestic 1945 cohort. The optimal hedge ratio is 0.9428.

based on foreign longevity indexes can hedge domestic risk exposure where no hedge instruments on domestic indexes exist in the domestic country. We consider different domestic cohorts and each of them is hedged with foreign longevity indexes on different cohorts. The simulation results show that for all the three domestic cohorts considered, the best hedge is the foreign index of the same cohort. It provides a better match to the 45-degree line in the graphical basis risk metric and a more centred surplus distribution. The graphical approach provides a more complete summary of the hedging effectiveness than any single risk measure.

We also apply the graphical risk metric to consider two countries with the same interest rate in the eurozone. The populations used are the French population and the Netherlands population. French longevity indexes are used to hedge risk exposures of an annuity provider in the Netherlands. Because of the common interest rates the basis risk is reduced considerably compared to the UK and Australia case. The value-based index can be used to assess hedging instruments if the two populations share the same interest rates.

We assume a static hedging strategy using longevity swaps, which is a practical approach given the illiquid longevity market [Ngai and Sherris, 2011]. Future research will consider dynamic hedging strategies allowing for flexible hedge ratios and a wider range of hedging instruments.

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Appendix

A. Kalman Filter Algorithm

The measurement equation is

$$y_t = -BY_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{29}$$

where A and B are given by (11) and (20), H is a diagonal matrix with elements $\sigma_\varepsilon^2(\tau_i)$. The state transition equation can be represented as

$$Y_t = a + bY_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{30}$$

where a, b and Q are given by (13) and (21). Denote the filtered values of the state variables and their corresponding covariance matrix by $Y_{t|t}$ and $S_{t|t}$, and further denote the unknown parameters by θ. In the forecasting step, we forecast unknown values of state variables conditioning on the information at time $t-1$ such that

$$Y_{t|t-1} = a + bY_{t-1|t-1}, \quad S_{t|t-1} = b' S_{t-1|t-1} b + Q_t(\theta). \tag{31}$$

In the next step we use the information at time t to update our forecasts

$$Y_{t|t} = Y_{t|t-1} - S_{t|t-1} B(\theta) F_{t|t-1}^{-1} v_{t|t-1}, \tag{32}$$

$$S_{t|t} = S_{t|t-1} - S_{t|t-1} B(\theta) F_{t|t-1}^{-1} B(\theta)' S_{t|t-1}, \tag{33}$$

where

$$v_{t|t-1} = y_t + A(\theta) + B(\theta) X_{t|t-1}, \quad \text{and} \quad F_{t|t-1} = B(\theta)' S_{t|t-1} B(\theta) + H.$$

Every iteration will yield a value for the log-likelihood function shown below

$$\log l(y_1, ..., y_T; \theta) = \sum_{t=1}^{T} \left(-\frac{N}{2} \log(2\pi) - \frac{1}{2} \log(F_{t|t-1}) - \frac{1}{2} v'_{t|t-1} F_{t|t-1}^{-1} v_{t|t-1} \right), \tag{34}$$

where N is the number of observed time series. The estimated parameter set $\hat{\theta}$ is determined as the one which maximizes the log-likelihood function.