ARC Centre of Excellence in Population Ageing Research

Working Paper 2017/06

Life Annuities: Products, Guarantees, Basic Actuarial Models.

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Life Annuities
Products, Guarantees, Basic Actuarial Models

LECTURE NOTES

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October 2017

Contents

Preface

These Lecture Notes aim at introducing technical and financial aspects of the life annuity products, with a special emphasis on the actuarial valuation of life annuity benefits. The text has been planned assuming as target readers:

- advanced undergraduate and graduate students in Economics, Business and Finance;
- advanced undergraduate students in Mathematics and Statistics, possibly aiming at attending, after graduation, actuarial courses at a master level;
- professionals and technicians operating in insurance and pension areas, whose job may regard investments, risk analysis, financial reporting, and so on, hence implying communication with actuarial professionals and managers.

Given the assumed target, the use of complex mathematical tools has been avoided.

We assume that the reader has attended courses providing basic notions of Financial Mathematics (interest rates, compound interest, present values, accumulations, annuities, etc.) and Probability (probability distributions, conditional probabilities, expected value, variance, etc). As mentioned, Mathematics has been kept at a rather low level. Indeed, all topics are presented in a "discrete" framework, thus not requiring analytical tools like differentials, integrals, etc.

The Lecture Notes are organized as follows. Guarantees and options in life contingency products, and in life annuities in particular, are sketched in Chap. 1. In Chap. 2 the basic actuarial aspects are introduced with reference to conventional life annuity products, starting from expected present value definitions, then moving to premium and reserve calculations.

A more general framework is proposed in Chap. 3, in order to introduce a wide range of life annuity products. The guarantee structure implied by various products is analyzed in Chap. 4, while Chap. 5 focusses on the time profile of the annuity benefits.

Options and riders which can be added to life annuity products are dealt with in Chap. 6, while Chap. 7 focusses on annuity rates adopted to determine the annuity premiums. Cross-subsidy mechanisms working in life annuity portfolios are addressed in Chap. 8, where special attention is placed on "tontine" schemes.

Strategies, which can be adopted to get the post-retirement income and, to some extent, can constitute alternatives to the immediate full annuitization of resources available at retirement, are described in Chap. 9.

Long-term care insurance (LTCI) is briefly addressed in Chap. 10, with a special focus on life annuity benefits combined with LTCI benefits. Finally, Chap. 11 concludes with suggestions for further reading.

Chapter 1

Some preliminary ideas

Life annuities can be placed in the context of life contingency products, i.e. insurance products which pay out benefits depending, in particular, on the lifetime of one or more individuals. The structure of any insurance product is determined by the presence of guarantees and options.

1.1 Life contingency products

The term *life contingencies* is used, in the insurance context, to denote models that describe the survival of individuals and the related cash flows of benefits. Hence, the expression *life contingency products* denotes insurance products which provide benefits occurring, or starting, or stopping contingent upon survival of one or more individuals.

Some life contingency products are displayed in Fig. 1.1. Each product aims at satisfying a specific objective: protection, saving, periodic income.

A classical example of *protection* product is given by the *term insurance* (or *temporary insurance*), which pays the sum assured at the time of insured's death, provided that the death occurs before the policy term. Hence, this product faces the risk of a financial distress caused to a family by the early death of a member who provides the family with an income.

The *pure endowment insurance* pays to the beneficiary (who often coincides with the insured) a lump sum benefit at maturity, if the insured is alive at that time. The benefit can be financed by either a single premium or a sequence of periodic premiums. The nature of *saving* (or *accumulation*) product is apparent, in particular in the case of periodic premiums.

A life annuity provides a person, the annuitant, with a sequence of periodic amounts, i.e. with a *periodic income*, while he/she is alive. In particular, an *immediate life annuity* starts paying the benefits immediately after the policy issue. The term *annuitization* is commonly used to mean the purchase of a life annuity, that is, the exchange of an amount (a lump sum) available e.g. at retirement against a sequence of periodic amounts (an immediate life annuity).

Some life contingency products aim at satisfying more objectives. A typical example is given by the *endowment insurance*, which is defined as the combination of a term insurance and a pure endowment insurance, hence fulfilling both the protection and the saving needs.

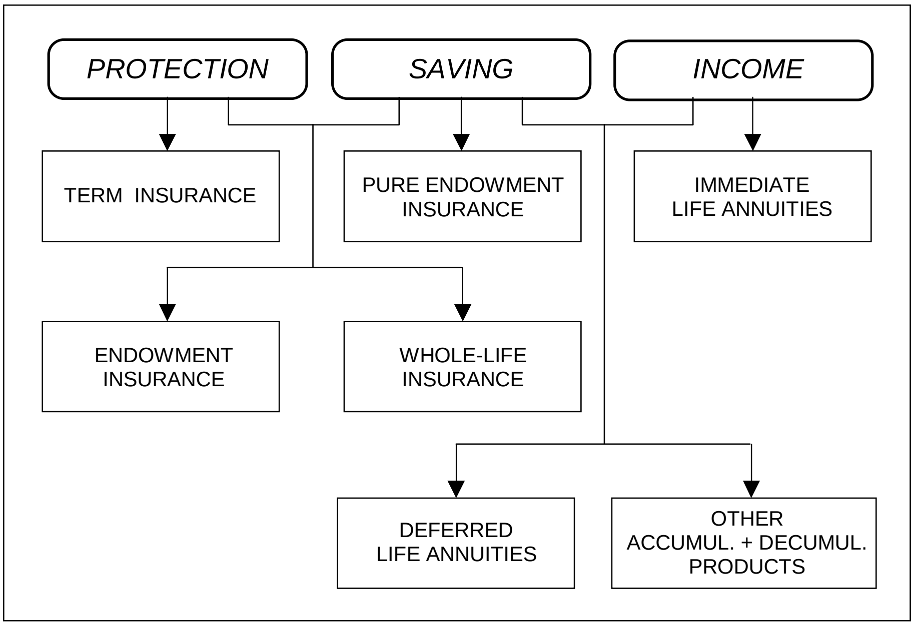

Figure 1.1: Life contingency products according to their main purpose

The *whole-life insurance* pays the sum insured at the time of insured's death, whenever it occurs, hence fulfilling the protection need. However, thanks to the possibility of surrendering the policy and hence cashing the amount corresponding to the surrender value, also the saving target can be achieved.

In order to get a post-retirement income, an *accumulation* phase is needed, which can coincide with the working part of the life. Then, the *decumulation* (or *payout*) phase starts. The classical *deferred life annuity* can be used, first as a saving instrument, then providing a post-retirement income. Because of the specific guarantee structure of the deferred life annuity, which implies a dramatic risk exposure of the annuity provider, various alternative products involving both the accumulation and the decumulation phase are available on the insurance markets.

1.2 Guarantees and options

Each life contingency product can be interpreted as a package of guarantees and options, implying risk transfers between the insured or annuitant on the one hand, and the insurer or annuity provider on the other.

In these Lecture Notes we address both *voluntary* (or *purchased*) *life annuities* and life annuities provided by occupational pension schemes, i.e. *pension life annuities*, focusing in particular on the guarantees provided by different types of life annuities. Indeed, each annuity product design determines a specific guarantee structure, which should carefully be considered when pricing the product itself.

Special attention will be placed on *biometric risks*, i.e. risks originated by the random lifetimes of the individuals, and related biometric guarantees. In particular, we will focus on the longevity risk which, at least to some extent, can be shared between the annuity provider and the annuitants according to the product design and the individual choices.

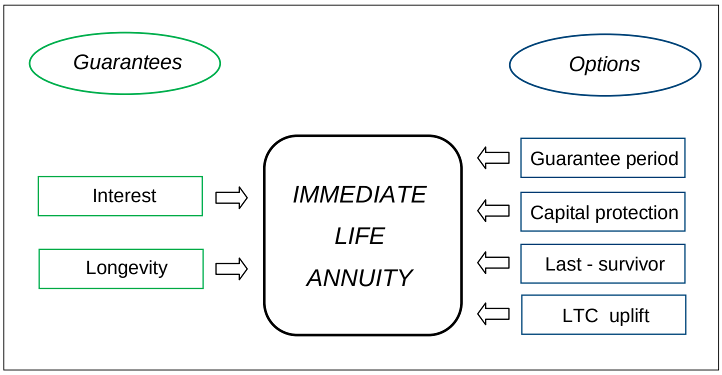

Figure 1.2: Guarantees and options in an immediate life annuity

Typical guarantees provided by a (conventional) immediate life annuity and some options which can be included in the annuity product are shown in Fig. 1.2.

The *interest guarantee* is a feature of most of the traditional life insurance and life annuity products. Of course, in a life annuity the importance of this guarantee is a straight consequence of the average long duration of the annuity itself.

Thanks to the *longevity guarantee*, the annuitant has the right to receive the stated annuity benefit as long as he/she is alive, and hence:

1. whatever his/her lifetime;
2. whatever the lifetimes of the annuitants in the annuity portfolio (or pension fund).

Because of feature 1, the annuity provider takes the *individual longevity risk*, originated by *random fluctuations* of the individual lifetimes around the relevant expected values. Feature 2 also implies the *aggregate longevity risk*: if the average lifetime in the portfolio is higher than expected, the annuity provider suffers a loss, because of *systematic deviations* of the lifetimes from the relevant expected values.

Various options can be added to the life annuity product. By exercising these options, other benefits and related guarantees are added to the basic life annuity product. The following options are of prominent interest.

- In a life annuity with a *guarantee period* the benefit is paid during the guarantee period regardless of whether the annuitant is alive or not (see Sect. 6.2).
- *Capital protection* is a rider benefit which, in case of early death of the annuitant, will pay to the annuitant's estate the difference (if positive) between the single premium and the cumulated benefits paid to the annuitant (see Sect. 6.3).
- A *last-survivor annuity* is an annuity payable as long as at least one of two (or more) individuals (the annuitants) is alive (see Sect. 6.4).
- *LTC uplift* is an increase in the benefit amount in the case the annuitant enters into a long-term care state (see Sect. 10.2).

The exercise of an option may be the effect of a *self-selection* process: people may be particularly interested in choosing one or more options because of their health conditions. For example, capital protection may be more attractive for people in non-optimal health conditions, whereas LTC uplift may be of interest to people who fear a future severe worsening of their health status. Thus, self-selection can result in an *adverse selection* (or *anti-selection)* from the point of view of the annuity provider, because of a higher probability of paying the benefits added to the basic life annuity product. To reduce the possible impact of self-selection, the exercise of the above options is only allowed before the start of the annuity payout period. In particular, in the case of deferred life annuities, the exercise is only allowed before the end of the deferment period, usually with some constraints, e.g. six months before the end of this period.

The range of guarantees (and options) provided by life annuities and the relevant features are strictly related to the type of the annuity product. For example, in a deferred life annuity both the accumulation and the payout phases are involved, so that some guarantees (e.g. the interest rate guarantee) can extend over a period of several decades. Moreover, the amount of longevity risk borne by the annuity provider depends on the time at which the annuity rate (which expresses the relation between premiums and benefits) is stated. These aspects will be addressed in Chap. 4.

1.3 The "annuity puzzle"

Decisions concerning the purchase of a life annuity can be analyzed in the framework of the life-cycle model of saving and consumption, in particular referring to the construction of an optimal retirement portfolio. This topic is beyond the scope of the present Lecture Notes, and hence we only focus on a particular issue, that is, the so-called *annuity puzzle*.

A life annuity provides protection against the risk of outliving the assets available at the time of retirement, because of a long lifetime or a poor investment performance. Hence, purchasing a life annuity, i.e. annuitizing (a part of) the available assets, should constitute a logical individual choice, especially if no other pension resources are available.

Market evidence however shows a low propensity to annuitize the assets available at retirement. Of course, good reasons work against the annuitization. In particular, the technical mechanism underpinning life annuities (described in Sect. 2.5) implies that, at the annuitant's death, the available fund related to the annuity must be shared among the surviving annuitants, so that nothing is credited to the annuitant's estate. This feature is clearly in contrast with a bequest motivation. Other features of life annuities, which might be perceived as weak characteristics of these products, will be singled out in Chap. 6.

The low propensity to annuitize is in contrast with the well known Yaari's theorem (see Yaari (1965)), which states that, under various assumptions, the optimal choice for an individual with a strong aversion to the risk of outliving his/her assets is the full annuitization of the wealth available at retirement.

The expression "annuity puzzle" is frequently used to denote that the common retirees' behavior stands in clear contradiction to the optimal choice (according to Yaari's theorem). The annuity puzzle has widely been analyzed in the economic literature; some bibliographic references are suggested in Chap. 11.

Chapter 2

Basic products and relevant actuarial aspects

While life annuity benefits can be defined and arranged in a number of ways, in this chapter we only focus on basic products. Expected present values, i.e. *actuarial values*, of benefits are presented, premiums and mathematical reserves are then introduced. The usual actuarial notation is adopted.

2.1 Actuarial values of life annuities

A life annuity provides the annuitant with a sequence of periodic amounts, while he/she is alive. The payment frequency may be monthly, quarterly, semi-annual, or annual. In the following, for the sake of brevity we only focus on annual payments, even though annuities payable more frequently than once a year can be of practical interest.

Consider a sequence of unitary amounts, payable at the beginning of each year as long as the annuitant is alive. This benefit is provided by the *whole-life annuity (paid in advance)*. Its actuarial value is given by:

$$\ddot{a}_x = \sum_{h=0}^{\omega-x} (1+i)^{-h} \, {}_h p_x \tag{2.1}$$

where:

- x denotes the annuitant's age at annuity commencement;
- i is the interest rate used to calculate present values, and hence $(1+i)^{-1}$ is the annual discount factor;
- ${}_h p_x$ denotes the probability for an individual age x of being alive at age $x+h$;
- ω is the limit age (for example, $\omega = 110$ can be assumed in numerical calculations).

If the annual amounts are payable for at most m years, we have the *temporary life annuity (paid in advance)*, whose actuarial value is given by:

$$\ddot{a}_{x:\overline{m}|} = \sum_{h=0}^{m-1} (1+i)^{-h} \, {}_hp_x \tag{2.2}$$

Conversely, if the annual amounts are payable as long as the individual is alive, but starting from time r, we have the *deferred life annuity (paid in advance)*. The actuarial value is then given by:

$$_{r|}\ddot{a}_x = \sum_{h=r}^{\omega-x} (1+i)^{-h} \, {}_hp_x = \ddot{a}_x - \ddot{a}_{x:\overline{r}|} \tag{2.3}$$

Combining the restrictions defined above, we obtain the *deferred temporary life annuity (paid in advance)*, whose actuarial value is given by:

$$_{r|}\ddot{a}_{x:\overline{m}|} = \ddot{a}_{x:\overline{r+m}|} - \ddot{a}_{x:\overline{r}|} \tag{2.4}$$

Formulae similar to the previous ones express the actuarial values of sequences of unitary amounts payable at the end of each year, namely the values of *life annuities paid in arrears*. We have:

$$a_x = \sum_{h=1}^{\omega-x} (1+i)^{-h} \, {}_hp_x = \ddot{a}_x - 1 \tag{2.5}$$

$$a_{x:\overline{m}|} = \sum_{h=1}^{m} (1+i)^{-h} \, {}_hp_x \tag{2.6}$$

$$_{r|}a_x = \sum_{h=r+1}^{\omega-x} (1+i)^{-h} \, {}_hp_x = a_x - a_{x:\overline{r}|} \tag{2.7}$$

$$_{r|}a_{x:\overline{m}|} = a_{x:\overline{r+m}|} - a_{x:\overline{r}|} \tag{2.8}$$

According to the well known *equivalence principle*, the *single premium* of any given life insurance or annuity product is given by the actuarial values of the relevant benefits. For example, the single premium of an immediate life annuity, whose annual unitary benefit is paid in arrears, is given by a_x (see Eq. (2.5)). For more details and numerical examples on premium calculation the reader is referred to Sects. 2.3 and 2.4.

2.2 Technical bases

The calculation of actuarial values requires the choice of:

1. the *interest rate*, i, to calculate present values;
2. the *life table*, $\{\ell_x\}$, from which the probabilities ${}_hp_x$ are derived as follows:

$$_hp_x = \frac{\ell_{x+h}}{\ell_x} \tag{2.9}$$

In particular, the life table we will adopt in the numerical examples is constructed by assuming that the age pattern of mortality follows the first Heligman-Pollard law. Let q_x denote the probability of a person age x dying before age $x+1$, and $\phi_x = \dfrac{q_x}{1-q_x}$ the mortality odds. The law is defined as follows:

$$\phi_x = A^{(x+B)^C} + D\,\mathrm{e}^{-E(\ln x - \ln F)^2} + GH^x \qquad (A,B,C,D,E,F,G,H > 0) \tag{2.10}$$

The first term on the right-hand side of Eq. (2.10) represents the infant mortality, and, via appropriate parameter values, it decreases as the age increases. The second term, which has a "Gaussian" shape, expresses the mortality hump (mainly due to accidents) at young-adult ages. Finally, the third term represents the senescent mortality at adult and old ages, and hence increases as the age increases.

From the mortality odds ϕ_x, all the biometric functions involved in actuarial calculations (e.g. ℓ_x, q_x, ${}_hp_x$, etc.) can be easily derived. In particular, we find:

$$q_x = \frac{\phi_x}{1+\phi_x} \tag{2.11}$$

and then:

$${}_hp_x = (1-q_x)\,(1-q_{x+1})\,\ldots\,(1-q_{x+h-1}) \tag{2.12}$$

Note that at adult and old ages we can accept the following approximation, obtained by disregarding the first and the second term on the right-hand side of Eq. (2.10):

$$q_x \approx \frac{GH^x}{1+GH^x} \tag{2.13}$$

Approximation (2.13) can be used to calculate actuarial values related to life annuities and pensions, e.g. for $x \geq 65$.

Table 2.1: Parameters of the Heligman-Pollard law

A	B	C	D	E	F	G	H
0.00054	0.01700	0.10100	0.00014	10.72	18.67	2.00532×10^{-6}	1.13025

The parameter values we have adopted in the following examples are shown in Table 2.1. Some corresponding typical values can be found in Table 2.2; in particular:

- the (remaining) life expectancy, $\mathring{e}_x$, at age $x = 0, 40, 65$;
- the *Lexis point*, or modal age at death, that is the age at which the maximum number of deaths in a cohort occurs;
- the one-year probability of dying, q_x, at age $x = 0, 40, 80$.

We note that the life table whose parameters are shown in Table 2.1 is a cohort life table extracted from a *projected life table*, that is, a life table which relies on a forecast of the future mortality trend (that is required to assess actuarial values concerning life annuities).

Table 2.2: Typical values of the life table derived from the Heligman-Pollard law

$\overset{\circ}{e}_0$	$\overset{\circ}{e}_{40}$	$\overset{\circ}{e}_{65}$	Lexis	q_0	q_{40}	q_{80}
85.128	46.133	22.350	90	0.00682	0.00029	0.03475

2.3 Premium calculation: the equivalence principle

Premiums of life insurance and life annuity products are commonly calculated by adopting the *equivalence principle*: the expected value of the profit provided by the product must be equal to zero. It follows that the *single premium* (or *up-front premium*, or *lump-sum premium*) must be equal to the actuarial value of the benefits at the time the policy is issued. In the case of *periodic premiums* the related actuarial value must be equal to the actuarial value of the benefits. The premiums calculated according to the equivalence principle are then named *equivalence premiums*.

The equivalence principle seems to be in contrast with a reasonable profit target. Further, expenses pertaining to the policy, as well as general expenses related to the portfolio, are not accounted for if only the amount of benefits is considered in premium calculation. Actually, premiums paid by the policyholders are *gross premiums*, rather than equivalence premiums. Gross premiums are determined from equivalence premiums by adding:

1. an *expense loading*, meeting various insurer's expenses;
2. a *profit loading* and *contingency margins*, facing the risk that the payout of benefits (and possibly of expenses) is higher than expected.

The items under point 2 can simply be referred to as *profit/safety loading*. Indeed, if actual experience within the portfolio coincides with the related expectation, these items contribute to the portfolio profit. Conversely, in the case of experience worse than expected (that is, annuitants live longer than expected), the items lower the probability and the severity of possible portfolio losses.

Adding the profit/safety loading to the equivalence premium yields the *net premium*. For example, a fixed percentage of the equivalence premium can be added to the equivalence premium itself. Whatever the formula chosen for the loading calculation, an *explicit profit/safety loading* approach is in this case adopted. This approach relies, in the calculation of the equivalence premium, on the use of a technical basis which provides a realistic description of the biometric and financial scenario.

The equivalence principle can also be implemented by adopting, instead of the realistic technical basis, a *prudential technical basis* (or *safe-side technical basis*, or *conservative technical basis*), so that the profit/safety loading is already included in the equivalence premium, then coinciding with the net premium. Hence, this procedure implies an *implicit profit/safety loading* approach. For life annuity and pension business, a prudential technical basis should consist in an expected survivorship longer than that realistically expected within the portfolio, and should include an interest rate lower than the estimated yield from investments.

"Combined" solutions are also feasible: a "weak" prudential basis can be chosen, and an explicit profit/safety loading is then added to determine the net premium.

As we are focussing on the basic actuarial structures, in what follows we will only refer to the calculation of equivalence premiums including an implicit safety loading, thus coinciding with net premiums. Hence, we disregard both expense loadings and explicit profit/safety loadings.

Remark 2.1 The equivalence principle, which plays a prominent role in the calculation of premiums for life insurance and life annuity products (as we will see in the following sections), only relies on expected values, thus excluding e.g. risk measures (variance, standard deviations, etc.), and then takes advantage of the linearity of the expectation operator. It follows that the premium of an insurance product consisting of a package of benefits, which can be formally expressed by a linear "combination" (in particular, a "sum") of benefits, is simply given by the linear combination (the sum, in particular) of the relevant premiums.

2.4 Life annuities: premiums for basic products

As seen in Sect. 2.1, annual benefits provided by a life annuity can be paid either in advance or in arrears. Further, life annuities can be either immediate or deferred. In what follows, y denotes the annuitant's age at annuity commencement, while x denotes the age at the time the accumulation process starts. So, for example, if r denotes the total duration of the accumulation period, then $y = x + r$ is the retirement age.

We now focus on an *immediate life annuity (paid in arrears)*, with flat payment profile. The single premium, Π, is given, according to the equivalence principle, by:

$$\Pi = b\, a_y = b \sum_{h=1}^{\omega-y} (1+i)^{-h}\, {}_h p_y \tag{2.14}$$

(see (2.5)) where b denotes the annual benefit.

As regards the choice of the technical basis, the interest rate i should clearly be lower than that expected as the yield from the investment of premiums. The life table, from which all the probabilities ${}_h p_y$ are derived, must be chosen so that the individual survival probabilities are not underestimated. Hence, a projected life table must be adopted for pricing life annuities. In all the following numerical examples, the Heligman-Pollard law is adopted, with the parameter values given in Table 2.1.

Example 2.1 Table 2.3 shows the single premium of an immediate life annuity (given by formula (2.14)), as a function of the interest rate i. The significant differences are clearly due to the impact of discounting when long durations are involved. ■

Example 2.2 It is interesting to express the benefit b as a percentage of the single premium Π. We find, of course (see Eq. (2.14)):

$$\frac{b}{\Pi} = \frac{1}{a_y}$$

Table 2.3: The single premium of an immediate life annuity; $b = 100$, $y = 65$

$i = 0$	$i = 0.01$	$i = 0.02$	$i = 0.03$
2 185.04	1 923.61	1 706.88	1 525.74

For example, assuming $b = 100$, $y = 65$, and $i = 0.02$, we find (see Table 2.3) that the annuitant will receive at the end of each year, as long as he/she will be alive,

$$\frac{100}{1\,706.88} \approx 5.86\%$$

of his/her premium, that is, a yield much higher than $i = 2\%$. This is due to the relation between the annual benefit and the single premium, which accounts for the limited (albeit unknown) duration of the annuity and, of course, the absence of any benefit on the annuitant's death. ■

The quantities $\frac{1}{a_y}$ and $\frac{1}{\ddot{a}_y}$ are usually called *annuity rates* (or *conversion rates*), and represent the annual benefit financed by a unitary premium. Conversely, the actuarial values, i.e. the quantities a_y and $\ddot{a}_y$, are also named *annuity factors*.

Example 2.3 The life annuity can also be interpreted as a "gamble". With the data specified in Example 2.2, we find that, if the annuitant lived for 18 or more years, than he/she will cash 1 800 or more (disregarding the time value of money) and hence will obtain a gain, otherwise will suffer a loss. ■

The *complete life annuity* (or *apportionable life annuity*) is a life annuity in arrears which provides a pro-rata adjustment on the death of the annuitant, consisting in a final payment proportional to the time elapsed since the last payment date. Assuming that the probability distribution of the time of death is uniform over each year, the single premium can approximately be expressed as follows:

$$\Pi = b\,a_y + \frac{b}{2}\bar{A}_y \tag{2.15}$$

where, according to the usual actuarial notation, $\bar{A}_y$ denotes the (approximate) actuarial value of a unitary amount paid at the time the individual dies.

From (2.5), it follows that the single premium of an *immediate life annuity in advance* is given by:

$$\Pi = b\,\ddot{a}_y = b\,(a_y + 1) \tag{2.16}$$

When dealing with the life annuities we have just described, it is natural to look at the single premium as the result of an accumulation process, in particular carried out during (a part of) the working life of the annuitant.

Life annuity products which also extend over the whole accumulation period can be conceived. This is, typically, the case of the *deferred life annuity* whose deferred period coincides with the accumulation period. In principle, a reasonable premium arrangement should then consist of a sequence of periodic premiums. For example, we can assume that a level annual premium P is payable over the whole deferred period. If the deferred period starts at age x and lasts r years, we have, for a deferred life annuity payable in advance:

$$P = b \, \frac{_{r|}\ddot{a}_x}{\ddot{a}_{x:\overline{r|}}} \tag{2.17}$$

However, we stress that, the longer is the deferred period the higher is the risk borne by the annuity provider, assuming that the pricing basis is, in this case, stated at the policy issue. In particular, an unanticipated improvement in mortality can cause serious technical problems to the annuity provider. These aspects will be addressed in Chap. 4.

2.5 The mathematical reserve

We only address single-premium immediate life annuities. Let y denote the annuitant's age at the annuity commencement.

The (prospective) mathematical reserve at time t, V_t, is defined as the actuarial value of the future benefits, conditional on the annuitant being alive at that time. Hence, for an immediate life annuity with annual benefit b (in arrears) we have:

$$V_t = b \, a_{y+t}; \quad t = 0, 1, 2, \ldots \tag{2.18}$$

We note that the reserve, as defined by Eq. (2.18), refers to a (generic) annuitant or pensioner, and in particular to a policy; for this reason, the expression *policy reserve* is frequently used.

The mathematical reserve can also be expressed in recursive terms; indeed, from Eq. (2.18) we obtain, after some manipulations:

$$V_t = p_{y+t} \, (1+i)^{-1} \, (b + V_{t+1}) \tag{2.19}$$

Recursion (2.19) is self-evident.

Further, as $p_{y+t} = \dfrac{\ell_{y+t+1}}{\ell_{y+t}}$ (see Eq. (2.9)), from (2.19) we obtain the following expression:

$$V_{t+1} = V_t + V_t \, i + V_t \, (1+i) \, \frac{\ell_{y+t} - \ell_{y+t+1}}{\ell_{y+t+1}} - b \tag{2.20}$$

whose interpretation is straightforward if we specifically refer to an individual belonging to a cohort inside an annuity portfolio, and assume that the cohort initially consists of ℓ_y annuitants. Indeed, Eq. (2.20) singles out the three components of the change in the reserve when moving from time t to $t+1$:

- the interest, $V_t \, i$;

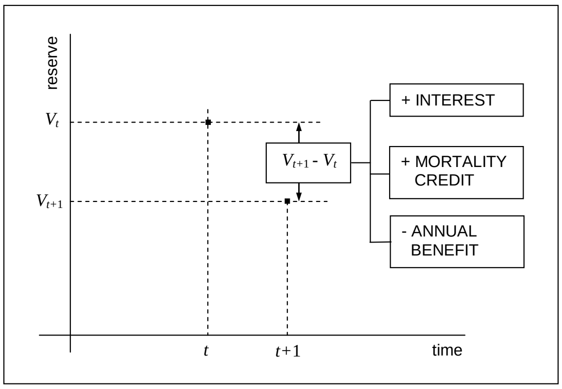

Figure 2.1: Components of the annual variation in the reserve of an immediate life annuity

- the *mortality credit*, $V_t\,(1+i)\,\frac{\ell_{y+t}-\ell_{y+t+1}}{\ell_{y+t+1}}$, that is, the result of sharing, according to the *mutuality mechanism*, the amount $V_t\,(1+i)\,(\ell_{y+t}-\ell_{y+t+1})$, released by the annuitants (belonging to the cohort we are referring to) expected to die between time t and $t+1$, among the ℓ_{y+t+1} annuitants (belonging to the same cohort) expected to be alive at time $t+1$;
- the annual benefit, b, paid at time $t+1$.

The reserve of an immediate life annuity is decreasing throughout the whole policy duration (see Example 2.4). Figure 2.1 shows the causes of the annual decrement in the reserve, as formally expressed by Eq. (2.20).

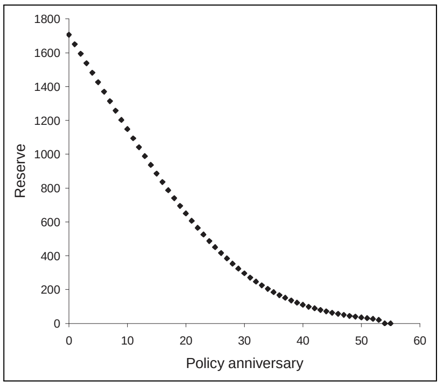

Figure 2.2: Single-premium life annuity

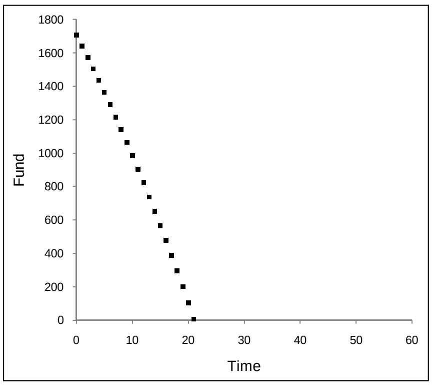

Figure 2.3: Drawdown process

Example 2.4 The reserve of a single-premium immediate life annuity (in arrears) is plotted in Fig. 2.2. Data are as follows: $b=100$, $y=65$, $i=0.02$. Conversely, Fig. 2.3

shows the time profile of a fund, whose initial amount is equal to the single premium of the immediate life annuity, from which the annual amount $b = 100$ is withdrawn; the interest rate is 0.02. The *income drawdown* (or *withdrawal process*) exhausts the fund in 21 years. We note that, of course, no mutuality mechanism works in the income drawdown process which implements a *self-annuitization* choice. Conversely, the presence of the mutuality mechanism in the life annuity explains the substantial difference between the time profiles shown in Figs. 2.2 and 2.3 respectively. ■

Example 2.5 Recursion (2.20) expresses an "equilibrium" situation (in the generic time interval $(t, t+1)$), which is achieved if the numbers of surviving annuitants at the various ages are equal (or proportional) to the numbers $\ell_y, \ell_{y+1}, \ell_{y+2}, \dots$. Assume, conversely, that the actual numbers of survivors (out of the initial number of annuitants ℓ_y) are $\hat{\ell}_{y+1}, \hat{\ell}_{y+2}, \dots$, and, in particular, consider the case in which $\hat{\ell}_{y+1} > \ell_{y+1}$, and hence $\ell_y - \hat{\ell}_{y+1} < \ell_y - \ell_{y+1}$. It follows that the amount actually released by the annuitants who die in the first year is smaller than the amount required to finance the mortality credits. The event $\hat{\ell}_{y+1} > \ell_{y+1}$ might be due to random fluctuations of mortality inside the portfolio. However, a sequence of events, e.g. $\hat{\ell}_{y+t} > \ell_{y+t}$ for $t = 1, 2, \dots$, would constitute a set of systematic deviations from the age-pattern of mortality assumed in the calculations; thus, the aggregate longevity risk would emerge. ■

2.6 Assessing the impact of the mutuality mechanism

We rewrite Eq. (2.20) as follows:

$$V_{t+1} = V_t\,(1+i)\,(1+\theta_{y+t}) - b \tag{2.21}$$

where:

$$\theta_{y+t} = \frac{\ell_{y+t} - \ell_{y+t+1}}{\ell_{y+t+1}} \tag{2.22}$$

Looking at recursion (2.21), we can interpret θ_{y+t} as the "extra-yield" which is required in year $(t, t+1)$ to maintain the decumulation process of the individual reserve. Hence, θ_{y+t} is a measure of the annual contribution provided by the mutuality mechanism.

Example 2.6 In Fig. 2.4 the quantity θ_{y+t} is plotted for $y = 65$ and $t = 0, 1, \dots, 35$. The underlying technical basis is the one adopted in all the previous examples. The (annual) extra-yield provided by the mutuality mechanism is clearly a function of the current age $y+t$. It is interesting to note that, when moderately old ages are involved (say, in the interval 65-75), the values of θ are very small. In such a range of ages, an income drawdown could be preferred to a life annuity, and the retiree could "replace" the mortality credits with a higher yield from investments (provided that riskier investments can be accepted). Conversely, as the age increases, θ reaches very high values, which cannot reasonably be replaced with investment yields. So, when old and very

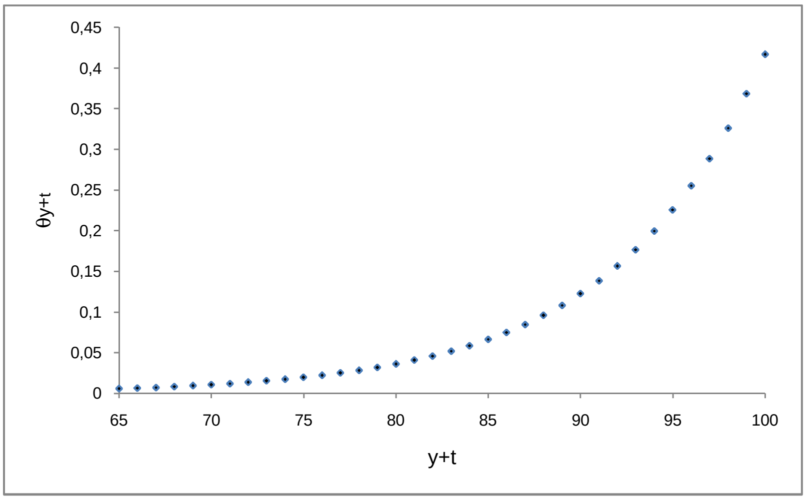

Figure 2.4: The extra-yield provided by the mutuality mechanism

old ages are concerned, the life annuity is the only technical tool which guarantees a lifelong constant income. This issue will specifically be addressed in Sect. 9.1. ■

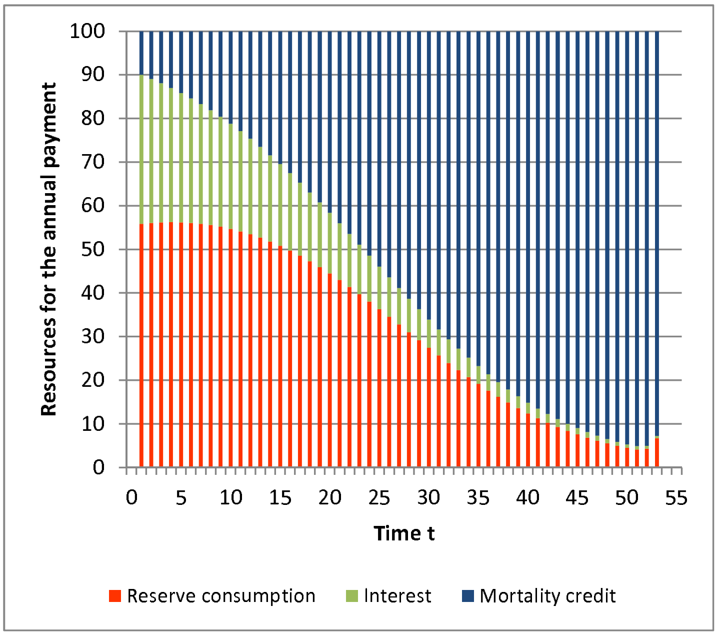

Figure 2.5: Resources financing the annual benefit

From Eq. (2.20) we find:

$$b = \underbrace{\left(V_t - V_{t+1}\right)}_{\text{Reserve consumption}} + \underbrace{V_t\, i}_{\text{Interest}} + \underbrace{\frac{\ell_{y+t} - \ell_{y+t+1}}{\ell_{y+t+1}} V_t\,(1+i)}_{\text{Mortality credit}} \tag{2.23}$$

Equation (2.23) singles out the resources used to finance the benefit b paid at time $t+1$. We note that, although the benefit is constant throughout the whole individual lifetime, the three terms on the right-hand side of (2.23) vary with time.

Example 2.7 The splitting of the annual benefit, according to Eq. (2.23), is shown in Fig. 2.5. Also from this perspective, it clearly appears that the role of mortality credits (that is, the impact of the mutuality mechanism) becomes more and more important as the age increases, because of an increasing mortality among annuitants. ■

Chapter 3

A more general framework

The conventional life annuities we have described in Chap. 2, i.e. the immediate life annuity and the deferred life annuity, constitute particular products within a more general framework. Life annuities can indeed be placed in a large category of insurance and pension products providing living benefits. Some possible generalizations are listed in Sect. 3.1, while a classification scheme is proposed in Sect. 3.2.

3.1 Towards more complex product designs

Starting from the basic products we have described in Chap. 2, it is possible to conceive more complex (and more interesting) product designs by moving in various directions. The following examples illustrate some possible arrangements.

- To obtain a post-retirement income, a (temporary) drawdown process can be followed by a life annuity which commences, say, 5 or 10 years after retirement (see Sect. 4.2 and Chap. 9).
- Accumulation and life annuity products can be designed, in which the presence of guarantees is the result of specific policyholder's options (see Sect. 4.3).
- The time profile of the annuity benefits can be defined in various ways, hence replacing the flat profile so far considered (see Chap. 5).
- Other benefits (e.g. a death benefit) can be added to the life annuity product (see Chap. 6).
- The life annuity benefit can be combined with health-related benefits, in particular long-term care benefits (see Chap. 10).

3.2 Life annuities: classification criteria

It is very difficult to conceive an exhaustive classification of the products belonging to this category. Nevertheless, it is possible to single-out some criteria according to which important features of the life annuity products can be addressed.

In Fig. 3.1 four criteria are shown. Each criterion suggests a specific analysis of the life annuity features, and provides some hints for a possible classification according to the criterion itself.

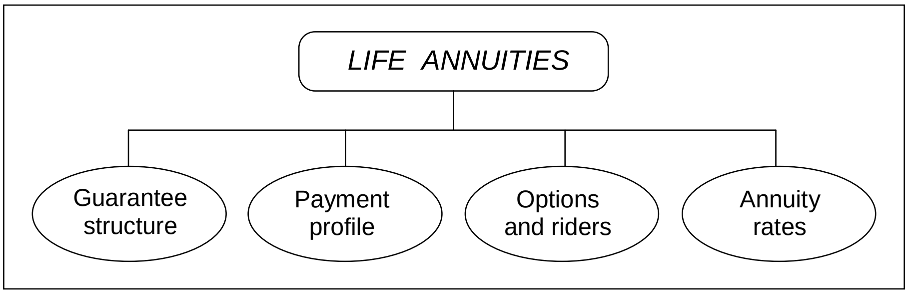

Figure 3.1: Life annuities: towards a taxonomy

The analysis of the *guarantee structure* aims at singling out the risks taken by the annuity provider, in particular the longevity risk, and the possibility of sharing these risks between annuitant and annuity provider. This issue is addressed in Chap. 4.

Various time profiles of the benefit amount can be recognized. To this purpose, the *payment profile* analysis can suggest some classification drivers. See Chap. 5.

Several *options* and *rider benefits* can be added to the conventional life annuity products. Many of them aim at making the annuity product more attractive from the client's perspective, but, at the same time, some of them imply a more significant exposure to risks for the annuity provider. Chapter 6 is devoted to these aspects.

Finally, the choice of appropriate *annuity rates* is a problem in the context of risk classification, that is, recognizing significant risk factors and choosing which risk factors can be adopted as the rating factors in premium calculation. The so called "special-rate life annuities" deserve, in this framework, particular attention. See Chap. 7.

Chapter 4

The guarantee structure

A large number of annuity products have been proposed, involving either the accumulation phase, or the payout phase, or both, and many of them are available on financial and insurance markets, each product having a specific guarantee structure (accumulation plans, conventional life annuities either immediate or deferred, etc.).

In what follows we adopt the guarantee structure classification criterion (see Sect. 3.2), and hence we focus on guarantees provided by each arrangement. Risks taken by the intermediary, in particular the annuity provider (either insurer or pension fund), can immediately be identified looking at the guarantee structure.

In the following figures:

- x denotes the age at which the accumulation phase starts;
- $x+r$ = denotes the age at retirement, at which the accumulation stops and the payout phase starts.

In each figure, the graphical notation shown in Fig. 4.1 is adopted.

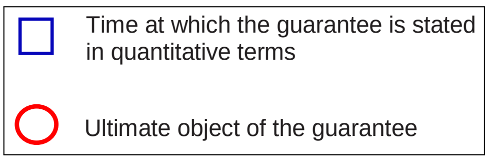

Figure 4.1: Defining the guarantee

As regards the scope of this chapter, we note that no attempt is made to quantify the risks per se, rather than to represent where, how and when they arise in the design of the various life annuity products.

4.1 Some basic structures

Structure 1

Structure 1 only involves the accumulation phase. For any given sequence of (annual) contributions / premiums / savings, $c_0, c_1, \ldots, c_{r-1}$, the amount S is guaranteed; see Fig. 4.2. We consider the following examples.

1. In a financial accumulation product, with guaranteed interest rate i, the guaranteed amount is given by:
$$S = \sum_{h=0}^{r-1} c_h (1+i)^{r-h} \tag{4.1}$$
Hence, the financial risk is taken by the institution which sells the accumulation product and provides the relevant interest guarantee.

2. In a life insurance product, e.g. a pure endowment insurance or an endowment insurance, the sum S at maturity is guaranteed if an interest guarantee is provided, and a longevity guarantee in the case of the pure endowment, or a mortality guarantee in the case of an endowment insurance. Both financial and biometric risks are taken by the insurer over the whole accumulation period.

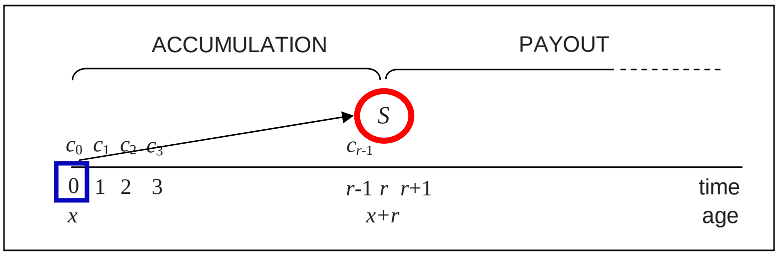

Figure 4.2: Structure 1 - Accumulation phase only

In both the examples, the guarantee on the amount S clearly has the purpose of making the accumulation product more attractive from a client's perspective.

We note that, in accumulation products, a further source of risk is given by surrenders or withdrawals (which can also cause liquidity problems).

Remark 4.1 The amount which will actually be available at time r as the result of the accumulation or the reserving process may be higher than S, thanks to a very good performance of the fund or the assets backing the policy reserve in a participating policy. Hence, S must be interpreted as the *minimum guaranteed amount*. Notwithstanding, according to some arrangements, the possible amount exceeding S is not credited to the individual; for example, this is the case of defined benefits pension schemes, in which the minimum guaranteed amount is often also the maximum guaranteed amount. The same remark also applies to other structures described in this chapter.

Structure 2

Structure 2 involves the payout phase only. For any given amount S (and hence any given single premium $\Pi = S$ to be paid for purchasing the annuity), the annual benefit b (assuming a flat payment profile) is guaranteed. See Fig. 4.3. Two examples follow.

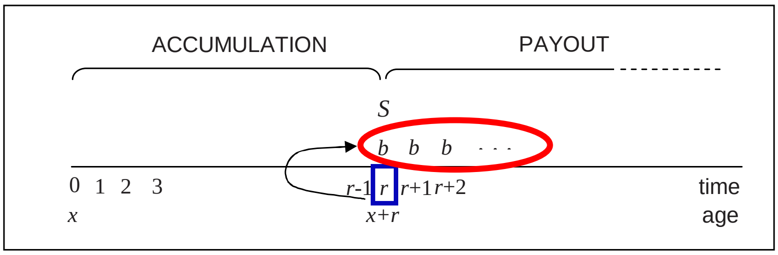

Figure 4.3: Structure 2 - Payout phase only

1. In a financial product, chosen to get the post-retirement income via a drawdown process, the annual benefit b is guaranteed up to (possible) fund exhaustion due to a long lifetime, thanks to an interest rate guarantee (see also Fig. 2.3).
2. In an immediate life annuity, the annual benefit b is guaranteed lifelong thanks to the interest guarantee and the longevity guarantee; the relation between the annuitized amount S and the benefit b is given, in formal terms, by the following relation:
$$b = \frac{1}{\ddot{a}_{x+r}^{[\text{curr}]}} S \tag{4.2}$$
where $\frac{1}{\ddot{a}_{x+r}^{[\text{curr}]}}$ is the *current annuity rate* (*CAR*), i.e. the annuity rate stated at annuitization time r. This life annuity, technically a CAR immediate life annuity, is the conventional single-premium immediate life annuity (in advance), frequently denoted with the acronym *SPIA*. The longevity risk, from time r onwards, is borne by the annuity provider.

In Example 1, the interest guarantee avoids the fund exhaustion because of a poor investment performance, so that the possible exhaustion can only be caused by a long lifetime.

The SPIA, i.e. the life annuity product in Example 2, implies complete avoidance of the fund exhaustion risk. The interest risk and the longevity risk, from time r onwards, are indeed borne by the annuity provider.

The SPIA is currently one of the most common life annuity products, both as a *voluntary* (or *purchased*) *life annuity* and a *pension life annuity* (that is paid from retirement time onwards as a straight consequence of membership of an occupational pension plan). It is worth stressing that in the former case *adverse selection* (or *anti-selection*) constitutes a big issue, as people purchasing a life annuity are usually in very good health conditions, so that their expected lifetime is longer than the average in a population. Conversely, adverse selection is absent in the latter case, as well as if some compulsory purchase mechanism works at the end of the accumulation period. On this aspect, see also Sect. 7.2.

Although the same formula (that is, (4.2)) can be used for both voluntary and pension annuities, a different annuity rate should be adopted in order to account for adverse selection in voluntary life annuities.

Structure 3

Structure 3 involves both the accumulation phase and the payout phase, and combines structure 1 and 2; see Fig. 4.4. We note that the interest guarantee working throughout the accumulation phase is stated at time 0, whereas the guarantee concerning the payout phase is stated at time r, that is, at the beginning of the decumulation.

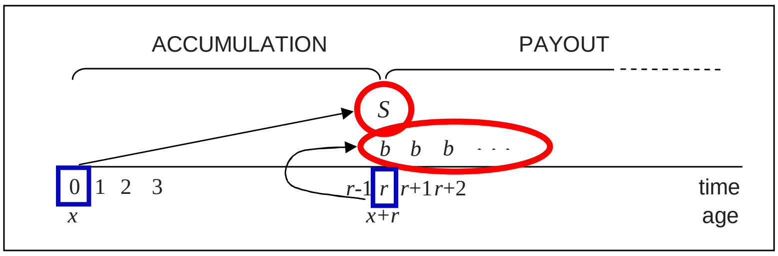

Figure 4.4: Structure 3 - Accumulation phase + Payout phase (1)

An example is given by the following combination:

(a) a financial product or an insurance product provides the guaranteed amount S at time r;

(b) a CAR immediate life annuity (that is, a SPIA) for the payout phase guarantees the lifelong annual benefit b.

As regards the risks, see Structure 1 (Examples 1 and 2) and Structure 2 (Example 2).

Structure 4

Also Structure 4 embraces the accumulation phase and the payout phase. Unlike in Structure 3, all the guarantees are stated at time 0 (a challenge for the annuity provider!); see Fig. 4.5. We consider the following examples.

1. A deferred life annuity, i.e. an annuity with a *guaranteed annuity rate* stated at time 0, briefly a *GAR* annuity, provides, for any given sequence $c_0, c_1, \dots, c_{r-1}$, the corresponding lifelong benefit b. We note that, assuming

$$c_0 = c_1 = \cdots = c_{r-1} = P \tag{4.3}$$

this structure is in particular implied by the classical actuarial formula (see Eq. (2.17)):

$$P = b\,\frac{{}_{r|}\ddot{a}_x^{[\mathrm{guar}]}}{\ddot{a}_{x:\overline{r}|}} \tag{4.4}$$

where ${}_{r|}\ddot{a}_x^{[\mathrm{guar}]}$, which is stated at time 0, denotes the expected present value of a life annuity deferred r years. The amount S represents, in this case, the mathematical reserve at time r, that is,

$$S = V_r = b\,\ddot{a}_{x+r}^{[\mathrm{guar}]} \tag{4.5}$$

2. Combining a financial product with interest guarantee over the accumulation phase and a GAR immediate life annuity for the payout phase yields a similar result.

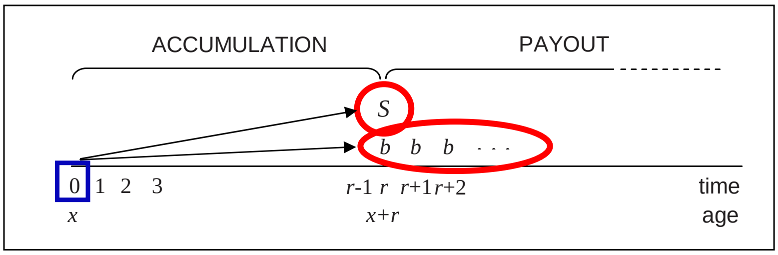

Figure 4.5: Structure 4 - Accumulation phase + Payout phase (2)

The arrangements described in Examples 1 and 2 provide the insured/annuitant with a full protection against financial and longevity risk, over a significant period of time: indeed the protection can extend over, say, 50 or more years. As mentioned, the risks taken by the annuity provider are huge (the longevity risk in particular), so that these arrangements are unlikely to be available in the current scenario which, in particular, is affected by uncertainty in future mortality trends.

It is worth noting that, conversely, the longer the accumulation period the weaker is the adverse selection effect (see Structure 2 as regards the significant presence of adverse selection in a SPIA product).

Structure 5

Structure 5 also involves both the accumulation phase and the payout phase. The annuity rate (that is, the relation between the accumulated amount and the annuity benefit) is stated at time 0. See Fig. 4.6. An example is given by the following combined product:

(a) a financial product for the accumulation phase (possibly providing a guaranteed interest rate);

(b) an immediate life annuity for the payout phase, whose benefit b is determined according to a guaranteed annuity rate (stated at time 0).

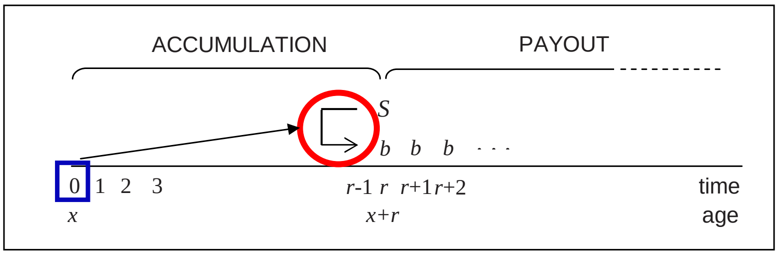

Figure 4.6: Structure 5 - Accumulation phase + Payout phase (3)

In particular, the *GAO* product (where GAO means *guaranteed annuity option*) implies the implementation of the above structure. Actually, the GAO product provides the following options (at retirement), that is the choice among:

- a lump sum;
- the annuitization according to the CAR (with annuity value $\ddot{a}_{x+r}^{[\text{curr}]}$);
- the annuitization according to the GAR (with annuity value $\ddot{a}_{x+r}^{[\text{guar}]}$).

Even if no guarantee is provided as regards the accumulated amount, the risks taken by the annuity provider, in relation to the life annuity, are similar to those originated by Structure 4. As is well known, the GAO product caused the demise of the Equitable Life Assurance Society of London.

Remark 4.2 Assume that the accumulation phase is implemented via an insurance product (e.g. a pure endowment with S as the sum at maturity), and works according to the logic of single recurrent premiums (that is, a particular progressive funding of S). Then, guarantees in both Structure 4 and Structure 5 can be weakened by linking the guarantee specification (the accumulation guarantee and/or the annuity rate) to each single recurrent premium. Thus, the guarantee specified at time 0 only pertains to the first single recurrent premium and the corresponding share of the amount at maturity, i.e. at time r. In general, the guarantee specified at time h ($h = 0, 1, \dots, r-1$) only refers to the single recurrent premium paid at time h and the corresponding share of the amount at maturity.

4.2 Longevity insurance annuities

The basic structures so far described can suggest the design of more specific products. In this section, two relevant examples are proposed.

The most important feature of life annuities, from the point of view of the retiree, is to provide protection against the risk of outliving the assets available at retirement. The products we are going to describe really offer insurance against this risk, at the same time leaving the retiree free to chose how to manage his/her assets during the first years of the retirement period.

Advanced Life Deferred Annuity (ALDA)

The ALDA product was proposed by Milevsky (2005) (with the name *Advanced Life Delayed Annuity*).

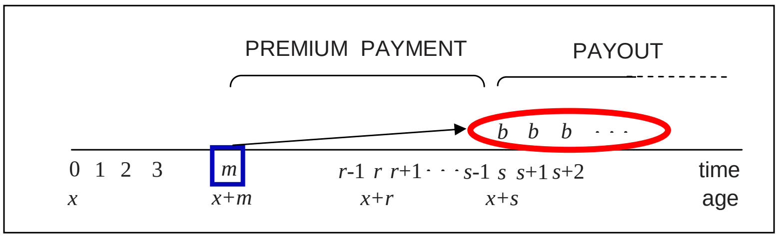

Figure 4.7: The ALDA model

The premium payment period does not necessarily coincide with the usual accumulation phase, being possibly shifted towards older ages. The payout period starts after retirement time (age 80 or 85, say). See Fig. 4.7.

We note that the ALDA product can be interpreted as an implementation of Structure 4, adapted by shifting: $0 \to m$, $r \to s$.

The payout period delayed to time s implies withdrawals from a fund throughout the time interval $(r, s-1)$ in order to get the post-retirement income.

The main purposes of ALDA are the following ones:

- to provide an insurance cover of the longevity risk at old ages only; hence, ALDA results in an insurance product with a deductible, that is the time interval $(r, s-1)$;
- to reduce the premium amount (with respect to conventional life annuities), so to enhance rates of voluntary annuitization.

Ruin Contingent Life Annuity (RCLA)

The RCLA product was proposed by Huang et al. (2009). Also in this product, the premium payment period does not necessarily coincide with the usual accumulation phase, being possibly shifted towards older ages. The payout period starts after the retirement age, at random time T, in the case at that time the fund exhaustion occurs because of an "adverse" scenario, which can result from:

- poor performance of the fund;
- retiree's long lifetime.

According to the features of this product, the post-retirement income is provided by:

(a) withdrawals from a fund from time r onwards, up to (possible) exhaustion of the fund;

(b) possibly, a life annuity paid to the retiree from (random) time T of fund exhaustion.

See Fig. 4.8.

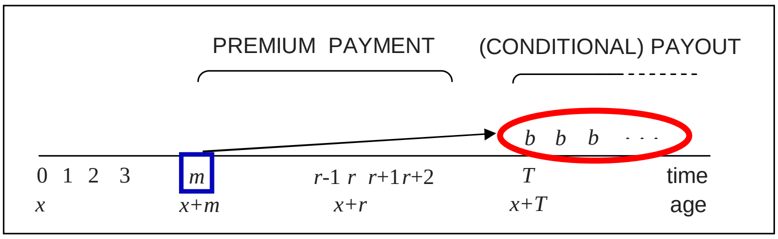

Figure 4.8: The RCLA model

We note that the RCLA can be thought either as an ALDA with random delay $T - r$, where the random time T is defined by a trigger expressing the scenario, or an insurance product which generates a life annuity as a "worst case" scenario.

The pricing procedure for the RCLA product must rely on a properly defined pseudo-index accounting for:

- the behavior of a market performance index, which should represent the performance of the fund used by the retiree during the drawdown phase;
- a set of reasonable benefit amounts (providing the post-retirement income) throughout the drawdown phase.

4.3 Variable Annuities

The basic guarantee structures presented in Sect. 4.1 and the specific products described in Sect. 4.2 are characterized by the presence (and the extension) of embedded biometric and financial guarantees. It is now interesting to move to a class of products in which the possible presence of guarantees is the result of policyholder's options.

The expression *variable annuity* is used to refer to a wide range of life insurance products, whose benefits can be protected against investment and mortality/longevity risks by selecting one or more guarantees out of a broad set of possible arrangements. Originally developed for providing a post-retirement income with some degree of flexibility, nowadays accumulation and death benefits constitute important components of the product design. Indeed, the variable annuity can be shaped so as to offer dynamic investment opportunities with some guarantees, protection in the case of early death and a post-retirement income.

We stress that no guarantee is implicitly embedded in a variable annuity product, whereas one or more guarantees can be chosen by the client and then added to the product. Hence, the guarantee structure is the result of the options exercised by the client (see Fig. 4.9). Guarantees are usually denoted by GMxB, that is Guaranteed

Minimum Benefit of type x. As we will see, including guarantees logically results in structures we have defined in the previous sections.

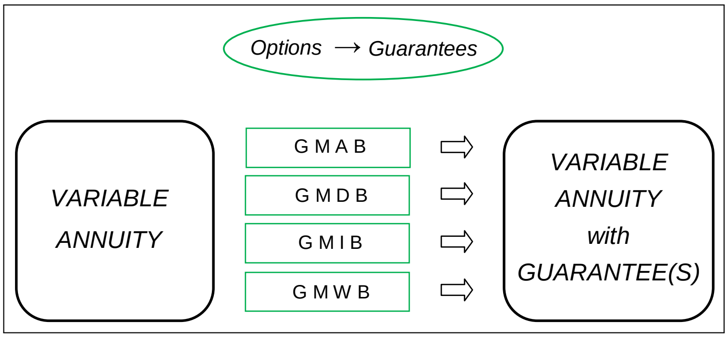

Figure 4.9: Options and related guarantees in Variable Annuity products

In what follows we assume, for simplicity, that the accumulation only relies on the payment of a single premium Π at the time the policy is written, i.e. time 0. Further, we assume that no withdrawals occur prior to retirement time r. Let F_t denote the *policy account value* (or *policy fund value*) at time t, which depends on the evolution of the reference fund in which the single premium is invested.

Figures 4.10 and 4.11 show possible time profiles of the policy account value in two different performance scenarios, that is, in the case of "good" performance (Scenario 1) and in the case of "bad" performance (Scenario 2), throughout both the accumulation and the decumulation phase. We assume absence of guarantees.

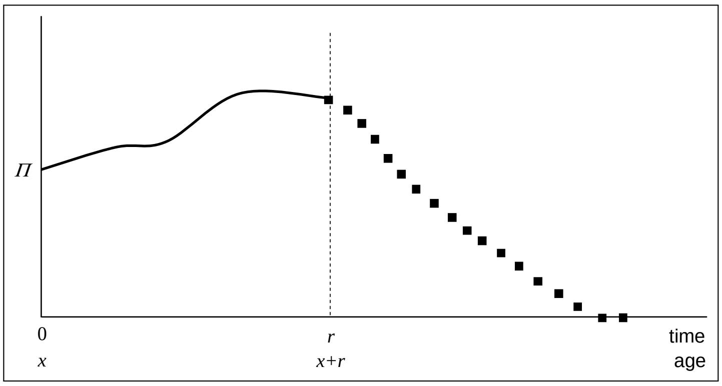

Figure 4.10: Time profile of the policy account value (Scenario1)

In the following subsections we will see how the various GMxB can increase the fund available, in both the phases.

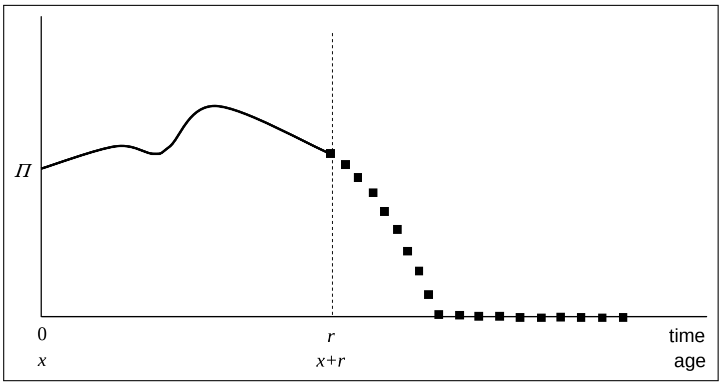

Figure 4.11: Time profile of the policy account value (Scenario2)

The Guaranteed Minimum Accumulation Benefit (GMAB)

The GMAB is usually available prior to retirement. At some specified date, the insured (if alive) is credited the greater between the policy account value and a guaranteed amount. Assuming that the guarantee refers at retirement time r, the guaranteed amount, $G_r^{[A]}$, can be stated according to the following arrangements.

- *Return of premium*
$$G_r^{[A]} = \Pi \tag{4.6}$$

- *Roll-up guarantee*
$$G_r^{[A]} = \Pi\,(1+i)^r \tag{4.7}$$
where i is the guaranteed interest rate.

- *Ratchet guarantee*
$$G_r^{[A]} = \max_{t_h<r}\{F_{t_h}\} \tag{4.8}$$
where $t_h,\ h = 1,2,\ldots,$ are the stated ratchet times; hence, the profits of the reference fund are locked-in at the ratchet times, and then the guaranteed amount never decreases.

We note that, according to guarantees expressed by Eqs. (4.6) and (4.7), the guaranteed amount is fixed and hence known at the time the policy is written, whereas the guarantee expressed by Eq. (4.8) yields an amount which depends on the fund performance and is then unknown at time 0.

In principle, guarantees can be combined; for example:

- *Roll-up & Ratchet guarantee*
$$G_r^{[A]} = \max\left\{\Pi\,(1+i)^r, \max_{t_h<r}\{F_{t_h}\}\right\} \tag{4.9}$$

As the result of any given guarantee mechanism, the amount $B_r^{[A]}$ acknowledged at time r is defined as follows:

$$B_r^{[A]} = \max\{F_r, G_r^{[A]}\} \tag{4.10}$$

We note that $G_r^{[A]}$ corresponds to the amount denoted by S in Structures 1, 3 and 4 defined in Sect. (4.1).

Figures 4.12 and 4.13 illustrate the behavior of the policy account value (the solid line) in the two performance scenarios referred to in Figs. 4.10 and 4.11, as well as the time profile of the return of premium guarantee (see Eq. (4.6)) and the roll-up guarantee (Eq. (4.7)). In the case of good performance (Scenario 1), no guarantee is used, while in the case of bad performance (Scenario 2), if the roll-up guarantee has been chosen by the policyholder, then the consequent amount will be available at maturity.

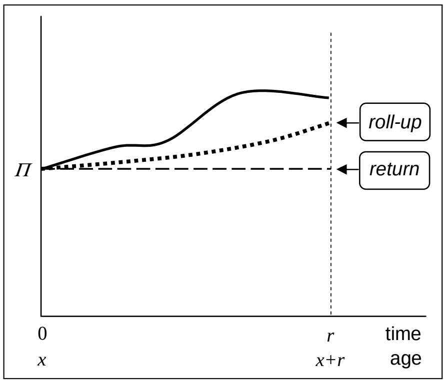

Figure 4.12: GMAB - Return of premium and roll-up guarantees (Scenario 1)

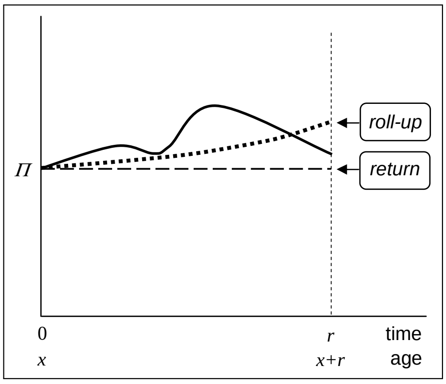

Figure 4.13: GMAB - Return of premium and roll-up guarantees (Scenario 2)

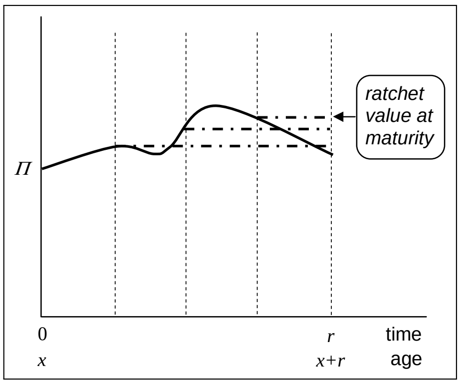

Figure 4.14: GMAB - Ratchet guarantee with “low” frequency (Scenario 2)

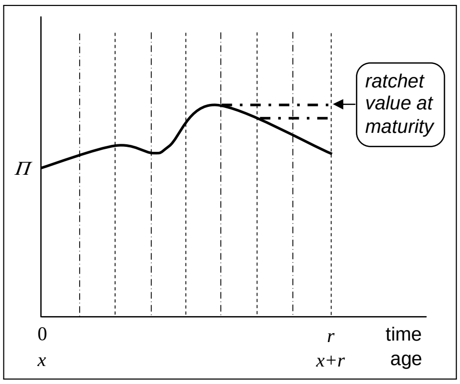

Figure 4.15: GMAB - Ratchet guarantee with “high” frequency (Scenario 2)

Figures 4.14 and 4.15 show the effect (in Scenario 2) of ratchet guarantees (see Eq. (4.8)) with different time intervals.

Remark 4.3 In the presence of a GMAB, a *Guaranteed Minimum Surrender Benefit* (*GMSB*) can be added. The amount of the guaranteed surrender benefit is usually determined consistently with the guaranteed accumulation benefit.

Guaranteed Minimum Death Benefit (GMDB)

Similarly to the GMAB, also the GMDB is available during the accumulation period; however, some insurers are willing to provide a GMDB also after retirement, up to some maximum age (say, 75 years). The structure of the guarantee is similar to the GMAB: in case of death at time t, prior to the stated term of the guarantee, the insurer will pay the greater between the policy account value and a stated amount $G_t^{[\mathrm{D}]}$. Hence, the death benefit at time t is given by:

$$B_t^{[\mathrm{D}]} = \max\{F_t, G_t^{[\mathrm{D}]}\} \tag{4.11}$$

The guaranteed amount $G_t^{[\mathrm{D}]}$ can be defined according to formulae similar to those adopted for the GMAB.

- *Return of premium*
$$G_t^{[\mathrm{D}]} = \Pi \tag{4.12}$$
- *Roll-up guarantee*
$$G_t^{[\mathrm{D}]} = \Pi\,(1+i)^t \tag{4.13}$$
- *Ratchet guarantee*
$$G_t^{[\mathrm{D}]} = \max_{t_h<t}\{F_{t_h}\} \tag{4.14}$$
where t_h, $h = 1,2,\dots,$ are the stated ratchet times.

Moreover, the following guarantee can be chosen:

- *Reset guarantee*
$$G_t^{[\mathrm{D}]} = F_{\max\{t_j:t_j<t\}} \tag{4.15}$$
where t_j, $j = 1,2,\dots,$ are the stated reset times.

We note that the difference between the ratchet and the reset guarantee within the GMDB stands in the time profile of the guaranteed minimum amount: in the ratchet guarantee the minimum amount never decreases, while a reduction may occur in the reset guarantee, if the policy account value decreases between two reset times.

Also the GMDB can in principle be defined as a combination of guarantees, e.g. Roll-up & Ratchet (see Eq. (4.9), in which the maturity time r must be replaced by a generic time t).

Figures 4.16 and 4.17 show the effects (in Scenario 2) of three GMDB guarantees. In particular, Fig. 4.16 illustrates the return of premium and the roll-up guarantee; in the case of death at time t, if the roll-up guarantee has been chosen by the policyholder, then the amount $\Pi\,(1+i)^t$ will be paid to the beneficiary. The effect of the reset guarantee is finally shown in Fig. 4.17.

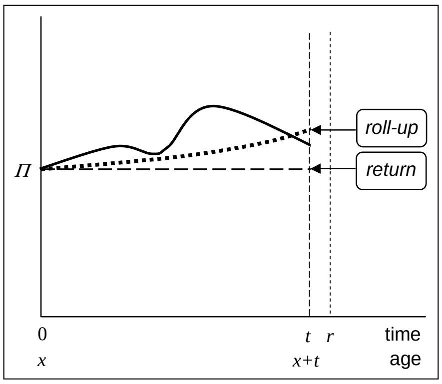

Figure 4.16: GMDB - Return of premium and roll-up guarantees (Scenario 2)

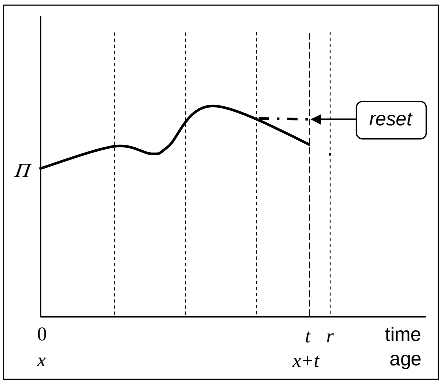

Figure 4.17: GMDB - Reset guarantee (Scenario 2)

Guaranteed Minimum Income Benefit (GMIB)

The GMIB provides the insured with a whole-life annuity from time r on. Let $b^{[\mathrm{I}]}$ denote the guaranteed annual benefit. We focus on the following arrangements.

- *Guarantee on the amount to annuitize*

$$b^{[\mathrm{I}]} = \max\{F_r, G_r^{[\mathrm{I}]}\} \frac{1}{\ddot{a}_{x+r}^{[\mathrm{curr}]}} \tag{4.16}$$

  where $x+r$ is the policyholder's age at time r, and $G_r^{[\mathrm{I}]}$ can be defined as $G_r^{[\mathrm{A}]}$ (see Eqs. (4.6) to (4.8)). We recognize the features of the CAR annuity according to Structure 3 with $S = G_r^{[\mathrm{I}]}$ (see Sect. 4.1). Thus, the annuity rate is stated at time r, on the basis of current market conditions, and hence unknown up to that time.

- *Guarantee on the annuity rate* (stated before time r, in particular at the date the policy is issued)

$$b^{[\mathrm{I}]} = F_r \max\left\{\frac{1}{\ddot{a}_{x+r}^{[\mathrm{curr}]}}, \frac{1}{\ddot{a}_{x+r}^{[\mathrm{guar}]}}\right\} \tag{4.17}$$

  This guarantee is also known as the GAO; see Structure 5.

In principle, the two guarantees can be combined, with the following result (corresponding to Structure 4):

- *Guarantee on the amount & the annuity rate*

$$b^{[\mathrm{I}]} = \max\{F_r, G_r^{[\mathrm{I}]}\} \max\left\{\frac{1}{\ddot{a}_{x+r}^{[\mathrm{curr}]}}, \frac{1}{\ddot{a}_{x+r}^{[\mathrm{guar}]}}\right\}$$

In practice, the resulting variable annuity product would be very expensive, because of the huge risk (which originates from both the fund performance and the longevity dynamics) taken by the insurer.

Guaranteed Minimum Withdrawal Benefit (GMWB)

The GMWB guarantees periodical withdrawals from the policy account, from time r on, even if the policy account value reduces to zero because of:

- poor investment performance;
- insured's long lifetime.

The guarantee affects both:

1. the withdrawal amount;
2. the withdrawal duration, which may be
   (a) fixed, regardless of whether the retiree is alive or not;
   (b) fixed, provided that the retiree is alive;
   (c) lifelong.

In the case of guaranteed durations (a) and (b), if the policy account at the time of retiree's death is positive, then the amount is credited to the retiree's estate.

In the case of guaranteed duration (c), the guarantee is also known as *Guaranteed Lifetime Withdrawal Benefit* (*GLWB*). We note that the logical structure of the RCLA can be recognized in this case (see Sect. 4.2).

The withdrawal amount, $b_t^{[W]}$, is stated as a given percentage, β_t (possibly constant), of a base amount W_t which is usually the policy account value at the date t^* the GMWB is selected. Hence:

$$b_t^{[W]} = \beta_t F_{t^*} \tag{4.18}$$

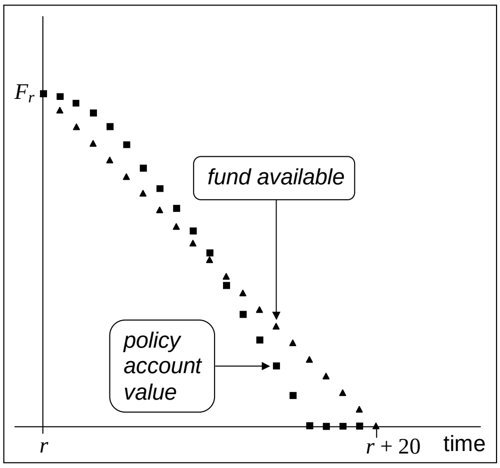

Figure 4.18: GMWB

In Fig. 4.18 the time profile of the fund available under a GMWB of type (a) is displayed, versus the time profile of the policy account value. We assume $t^* = r$, and

$\beta_t = 0.05$ for $t = r, r+1, \dots$. It follows that the guaranteed annual withdrawal is 5% of the fund available at retirement, thus $b_t^{[W]} = 0.05\,F_r$. Hence, withdrawals are guaranteed for 20 years, even in the case of fund depletion because of a poor performance, as shown in the figure.

In some arrangements, at specified dates (e.g., every policy anniversary) the base amount, W_t, may step up to the current value of the policy account, if this is higher. This is a ratchet guarantee, which may be lifetime or limited to some years (10 years, say). According to the ratchet mechanism, we have:

$$b_t^{[W]} = \beta_t\,W_t = \beta_t\,\max\{F_{t^*}, F_t\} \tag{4.19}$$

Remark 4.4 The GMWB is the real novelty of variable annuities in respect of traditional life insurance contracts; it provides a benefit which is similar to an income drawdown, but with guarantees. When comparing a GMIB to a GMWB, three major differences arise:

(i) the duration of the annuity (which is lifetime in the GMIB, while different conditions can be chosen in the GMWB as specified under point 2 above);

(ii) the accessibility to the account value (just for the GMWB);

(iii) the features of the reference fund and the consequent financial structure of the annuity, which usually is unit-linked in the GMWB, but typically participating in the GMIB (according to the terminology that will be defined in Sect. 5.2).

It is also interesting to compare the GLWB, i.e. a GMWB with guaranteed withdrawal duration (c), with a conventional life annuity. On the one hand, both the arrangements pay out lifelong benefits, but, on the other hand, the higher flexibility of the GMWB (consider, in particular, point (ii) above, but also the unit-linked financial structure) implies, under the same market conditions, a higher cost to the insurer and hence, for a given single premium, an (initial) benefit in the GMWB arrangement smaller than that provided by the conventional life annuity.

Chapter 5

The payment profile

According to the *payment profile* criterion (see Sect. 3.2), the following types of life annuities can be recognized:

- annuities with pre-defined benefits;
- annuities with benefits determined according to some linking mechanism.

A classification of life annuities according to the payment profile criterion is sketched in Fig. 5.1. Some details about various time profiles are provided in Sects. 5.1 to 5.3.

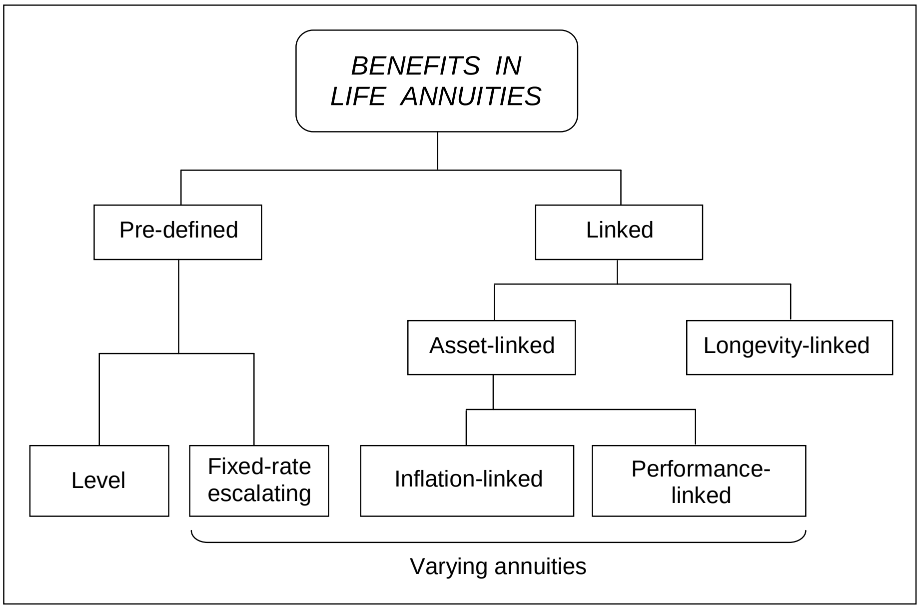

Figure 5.1: Time profiles of life annuity benefits

5.1 Life annuities with pre-defined benefits

Level life annuities (sometimes called *standard life annuities*) provide the annuitant with an annual income, b, which is constant in nominal terms. Thus, the payment profile is flat.

In *fixed-rate escalating life annuities* (or *constant-growth life annuities*) the annual benefit increases according to a fixed annual rate α. In particular:

- in the *arithmetically escalating life annuity* the benefit progression is as follows:

$$b_1,\ b_2 = (1+\alpha)\,b_1,\ b_3 = (1+2\,\alpha)\,b_1,\ \ldots \tag{5.1}$$

- the *geometrically escalating life annuity* provides the following sequence of benefits:

$$b_1,\ b_2 = (1+\alpha)\,b_1,\ b_3 = (1+\alpha)^2\,b_1,\ \ldots \tag{5.2}$$

5.2 Life annuities with asset-linked benefits

A number of mechanisms, which link the annuity benefits to the value of assets backing the reserve, have been proposed and implemented. In particular:

- *inflation-linked life annuities*
- *performance-linked life annuities*
  - *participating life annuities* (i.e. life annuities with investment profit participation mechanisms);
  - *with-profit life annuities* (in the UK);
  - *unit-linked life annuities*;
  - *equity-indexed life annuities*.

According to these linking mechanisms, the progressive changes in the benefit amount, throughout the life annuity duration, are maintained by changes in the policy reserve, which transfers to the benefit the changes in the value of the assets backing the reserve itself. The assets can consist in bonds and other securities for performance-linked annuities, while specific bonds and derivatives must be used for inflation-linked annuities.

In the context of performance-linked life annuities, an important point must be addressed. Participating and with-profit life annuities usually guarantee a minimum benefit level, which is commonly given by the amount b stated at policy issue, and hence depending in particular on the interest rate i adopted in the single premium calculation. Conversely, no guarantee is embedded in unit-linked and equity-indexed life annuities, in which the amount of benefit at time t, b_t, is calculated as follows. The initial payment b_1 is determined by assuming a *hurdle interest rate*, i^* (which represents the assumed investment return), so that, for a given single premium Π, we find:

$$b_1 = \frac{\Pi}{a^*_y} \tag{5.3}$$

where y denotes the age at policy issue (e.g., at retirement time), and the actuarial value a_y^* is calculated by assuming the interest rate i^*. Then, the following annual benefits are given by the recursion:

$$b_t = b_{t-1}\,\frac{1+z_t}{1+i^*}; \quad t = 2,3,\dots \tag{5.4}$$

where z_t denotes the actual return earned by the underlying assets (units or equities) in year $(t-1,t)$. We note what follows:

- for a given i^*, if $z_t > i^*$ we obtain $b_t > b_{t-1}$, otherwise $b_t \le b_{t-1}$, as results from Eq. (5.4);
- the lower is the hurdle rate i^*, the more likely is to obtain an increase in the benefit amount, but, at the same time, the higher is the annuity factor a_y^* and hence the smaller is the initial benefit b_1 (see Eq. (5.3)).

It is worth noting that the expression *varying life annuities* is frequently used to encompass fixed-rate escalating annuities, inflation-linked annuities and performance-linked annuities. Actually, all these types of life annuity share the purpose of providing the annuitant with (partial) protection against the loss of purchasing power because of inflation. However, the following different features should be stressed.

- In fixed-rate escalating life annuities, the premium calculation accounts for the pre-defined benefit profile. For example, for an immediate life annuity in arrears with benefit profile defined by Eq. (5.2), we have:

$$\Pi = b_1 \sum_{h=1}^{\omega-x} (1+i)^{-h}\,(1+\alpha)^{h-1}\,{}_hp_y \tag{5.5}$$

- In asset-linked life annuities, the premium is determined referring to the initial benefit amount, whereas the changes in the benefit amount rely on changes in the policy reserve, as mentioned above.

5.3 Longevity-linked life annuities

Participation mechanisms (see Sect. 5.2) can be extended in order to involve both financial and mortality experience. While a mortality higher than expected can originate mortality profits in a life annuity portfolio or a pension plan, and these can be attributed to some extent to the annuitants, according to conventional life annuity and pension design the risk of mortality lower than expected (thus, the longevity risk) is borne by the annuity provider. Hence, problems may in particular arise from a poor mortality experience because of an unexpected increase in longevity (that is, because of the aggregate longevity risk).

According to alternative product designs, part of the longevity risk can be transferred to the annuitants. This implies the definition of a *longevity-linked life annuity*.

A longevity-linked life annuity involves a benefit adjustment process. The benefit payable at time t is defined as follows:

$$b_t = b_{t-1}\,\alpha_t^{[\mathrm{m}]};\quad t = 2,3,\dots \tag{5.6}$$

where b_1 is the benefit amount initially stated, and $\alpha_t^{[\mathrm{m}]}$ denotes the coefficient of adjustment over the time interval $(t-1,t)$, according to a given mortality trend measure $[\mathrm{m}]$.

We note that, in a longevity-linked life annuity, the annuity provider is in particular entitled to reduce the benefit to all the annuitants in the event of an unanticipated increase in longevity. However, a floor amount should reasonably be stated to keep, at least to some extent, the guarantee characteristics which should feature all life annuity products.

Coefficient $\alpha_t^{[\mathrm{m}]}$ can incorporate investment profit participation, so that the longevity loss can, at least partially, be offset by the investment profit.

Remark 5.1 It is worth stressing that a *tontine scheme* could be recognized in this type of participation mechanism, because of the link between the benefit amount and the number of surviving annuitants. Tontine schemes and tontine annuities will be described in Sect.8.2.

Basic problems in defining the longevity-driven adjustment process are:

1. the choice of the age pattern of mortality referred to;
2. the choice of the link between annual benefits and mortality.

These choices should reasonably aim at sharing the aggregate longevity risk (that is, the systematic component of the longevity risk), while leaving the volatility (the random fluctuation component) with the annuity provider, as the latter can be diversified by risk pooling, viz inside the traditional insurance - reinsurance process.

As regards point 1, the annuity benefit can be adjusted either according to:

(a) the mortality experienced by the specific pool of annuitants (portfolio or pension fund), hence referring to the number of annuitants or to the survival rate in the pool;

or according to:

(b) the mortality experienced by some reference population, thus referring to the number of survivors or to the survival rate in the population;

or according to:

(c) the mortality expressed by a new projected life table.

Adjustment mechanisms of type (a) are named *indemnity-based*, as they (indirectly) refer to the total payout for life annuities, whereas mechanisms (b) and (c) are called *index-based*.

We note that mechanisms (b) and (c) are more “transparent”, since they are based on public data; however, a basis risk arises, as the mortality in the pool can differ from the mortality experienced in the reference population or expressed by the new life table.

Chapter 6

Options and rider benefits

Various options and rider benefits can make the life annuity product more attractive from the point of view of the client. In this chapter, we adopt the relevant classification criterion (see Sect. 3.2).

6.1 Some features of the life annuity

When planning the post-retirement income, some basic features of the life annuity product should carefully be considered. In particular, we recall the following aspects.

1. The (standard) life annuity product relies on the mutuality mechanism. This means that:
   (a) the amounts released by the deceased annuitants are shared, as mortality credits, among the annuitants who are still alive (see Sect. 2.5 and, in particular, Eqs. (2.20) and (2.23));
   (b) on the annuitant's death, his/her estate is not credited with any amount, and hence no bequest is available.
2. A life annuity provides the annuitant with an "inflexible" post-retirement income, in the sense that the annual amounts must be in line with the benefit profile, as stated by the policy conditions (see Chap. 5).
3. Purchasing a life annuity is an irreversible decision: surrendering is generally not allowed to the annuitants (clearly, to avoid adverse selection effects). Hence, the life annuity constitutes an "illiquid" asset in the retiree's estate.

Features 1(b), 2 and 3 can be perceived as disadvantages, and can hence weaken the propensity to immediately annuitize the whole amount available at retirement. We will illustrate how these disadvantages can be mitigated, at least to some extent, either by purchasing life insurance products in which other benefits are packaged (see Sects. 6.2 and 6.3), or adopting specific annuitization strategies (Chap. 9). Options and riders which consist in the possibility of linking the benefit amount and/or the premium amount to health conditions will be described in Sect. 7.3 and in Chap. 10.

6.2 Life annuity with a guarantee period

If the annuitant dies soon after the commencement of the (conventional) life annuity, neither the annuitant nor the annuitant's estate receive much benefit from the purchase of the life annuity. In order to mitigate this risk, it is possible to buy a *life annuity with a guarantee period* (5 or 10 years, say), also named *period-certain life annuity*. This type of product pays the benefit over the guarantee period regardless of whether the annuitant is alive or not.

Let y denote the age at policy issue (e.g., at retirement); the actuarial value, at age y and according to interest rate i, of an immediate life annuity paid in arrears, with a guarantee period of s years and a unitary benefit, is given, according to the traditional notation, by:

$$a_{\overline{y:\overline{s|}}} = a_{\overline{s|}i} + {}_{s|}a_y \tag{6.1}$$

where $a_{\overline{s|}i}$ denotes the present value of a *temporary annuity-certain* (paid in arrears), that is:

$$a_{\overline{s|}i} = \sum_{h=1}^{s} (1+i)^{-h} = \frac{1-(1+i)^{-s}}{i} \tag{6.2}$$

Thus, the annuity product results in a deferred life annuity combined with a temporary annuity-certain. Of course, $a_{\overline{s|}i} > a_{y:\overline{s|}}$, and then:

$$a_{\overline{y:\overline{s|}}} > a_{y:\overline{s|}} + {}_{s|}a_y = a_y \tag{6.3}$$

The single premium which is charged to purchase a life annuity with a guarantee period of s years and benefit b is then given by:

$$\Pi = b\, a_{\overline{y:\overline{s|}}} \tag{6.4}$$

Example 6.1 Table 6.1 shows the single premium Π, given by Eq. (6.4), at age 65 and 70 respectively at policy issue; of course, $s = 0$ denotes the conventional life annuity without any guarantee period. The usual technical basis with $i = 0.02$ has been adopted. It is worth noting the limited cost of this rider benefit, that is, the small increment in the single premium when moving from a conventional life annuity ($s = 0$) to a life annuity with a guarantee period of 5 or 10 years. Actually, the difference $a_{\overline{s|}i} - a_{y:\overline{s|}}$ is very small thanks to the low mortality in the age intervals involved, that is $(y, y+s)$. ■

6.3 Value-protected life annuity

Capital protection represents an interesting feature of some life annuity products, usually called *value-protected life annuities*, or *money-back life annuities*, or *cash-back life annuities*.

Consider, a single-premium immediate life annuity paid in arrears, purchased at age y. In the case of early death of the annuitant (say, between time h and $h+1$), a value-protected life annuity will pay to the annuitant's estate the difference (if positive),

Table 6.1: Single premium Π at age y; $b = 100$

	Guarantee period		
	$s = 0$	$s = 5$	$s = 10$
$y = 65$	1 706.88	1 716.25	1 746.67
$y = 70$	1 426.43	1 443.47	1 497.53

Γ_h, between the single premium Π and the cumulated benefits paid to the annuitant. Usually, capital protection expires at some given age $y+n = \xi$ (75, say), after which nothing is paid even if the difference above mentioned is positive. Hence, we have:

$$\Gamma_h = \max\{\Pi - hb, 0\}; \quad h = 0, 1, \ldots, n-1 \tag{6.5}$$

The single premium is then given by:

$$\Pi = b\,a_y + \left(\Gamma_0\ {}_{0|1}A_y + \Gamma_1\ {}_{1|1}A_y + \cdots + \Gamma_{n-1}\ {}_{n-1|1}A_y\right) \tag{6.6}$$

where ${}_{h|1}A_y$ denotes the actuarial value of a unitary term assurance, deferred h years and temporary 1 year.

Table 6.2: Single premium Π at age y; $b = 100$

	Limit age		
	$\xi = 70$	$\xi = 75$	$\xi = 80$
$y = 65$	1 759.53	1 821.22	1 880.66
$y = 70$	1 426.43	1 506.13	1 593.50

Example 6.2 Table 6.2 shows the single premium Π, given by Eq. (6.6), which is charged to purchase, at age 65 and 70 respectively, a value-protected life annuity with limit age ξ (of course, when $\xi = y$ no capital protection is provided). The usual technical basis with $i = 0.02$ has been adopted. We note that, also for this rider benefit, the increment in the single premium is rather small, even when the protection expires at age $\xi = 80$. Again, this is due to the low mortality in the relevant age intervals. ■

Remark 6.1 From the insurer's perspective, combining a living benefit (the life annuity) with a death benefit (the capital protection) provides, at least in principle, a *natural hedging* of the longevity risk. In practice, however, the low mortality rates in the involved age intervals cause a rather poor impact on the insurer's cash flows, and hence capital protection does not provide an effective hedge against the (aggregate) longevity risk.

6.4 Last-survivor annuities

A *last-survivor annuity* is an annuity payable as long as at least one of two individuals (the annuitants), say (1) and (2), is alive. We use the following notation:

- b = the initial amount of benefit, payable as long as both the individuals are alive;
- b' = the amount of benefit paid to individual (1) if individual (2) dies first;
- b'' = the amount of benefit paid to individual (2) if individual (1) dies first.

Usually, we have $b' \leq b$, $b'' \leq b$. Particular benefit profile arrangements are as follows.

1. It can be stated that the annuity continues with the same annual benefit b until the death of the last survivor, that is, $b' = b'' = b$;
2. In many pension plans, the annual benefit is reduced only if the retiree, say individual (1), dies first; i.e. $b' = b$ while $b'' < b$.

To calculate actuarial values, premiums and reserves, assume that the remaining random lifetimes of the two individuals, initially aged y' and y'' respectively,

(i) can be described by the same life table;

(ii) are stochastically independent.

Remark 6.2 Hypothesis (i) is unrealistic in the case the annuitants have a different gender. Of course, different life tables can be adopted for the two individuals. We note, however, that this hypothesis is mandatory according to the unisex rating principle in the European Union (see Sect. 7.5). Conversely, hypothesis (ii) is rather unrealistic in the case of relatives (because of possible hereditary diseases) or spouses or individuals living together (because of possible accident mortality); if not assumed, joint probability distributions (or marginal distributions and copulas) must be used in the calculations.

Let ${}_hp_{y',y''}$ denote the probability that both the individuals are alive at time h, and ${}_hp_{\overline{y',y''}}$ the probability that one individual at least is alive at time h. According to hypotheses (i) and (ii), we have:

$$ {}_hp_{y',y''} = {}_hp_{y'}\,{}_hp_{y''} \tag{6.7}$$

$$ {}_hp_{\overline{y',y''}} = {}_hp_{y'} + {}_hp_{y''} - {}_hp_{y'}\,{}_hp_{y''} \tag{6.8}$$

The actuarial value of an immediate annuity in arrears with unitary benefit, payable while both the individuals are alive, is given by:

$$ a_{y',y''} = \sum_{h=1}^{+\infty} (1+i)^{-h}\,{}_hp_{y',y''} = \sum_{h=1}^{+\infty} (1+i)^{-h}\,{}_hp_{y'}\,{}_hp_{y''} \tag{6.9}$$

The actuarial value of an annuity in arrears, payable as long as at least one of the two individuals is alive, is given by:

$$\begin{aligned} a_{\overline{y',y''}} &= \sum_{h=1}^{+\infty} (1+i)^{-h} {}_h p_{\overline{y',y''}} \\ &= \sum_{h=1}^{+\infty} (1+i)^{-h} ({}_h p_{y'} + {}_h p_{y''} - {}_h p_{y'} \, {}_h p_{y''}) \\ &= a_{y'} + a_{y''} - a_{y',y''} \end{aligned} \tag{6.10}$$

The single premium of a last-survivor annuity with benefit b is then given by:

$$\Pi = b\, a_{\overline{y',y''}} = b\,(a_{y'} + a_{y''} - a_{y',y''}) \tag{6.11}$$

If the annual benefits of the annuity are b, b', b'', according to the state of the two-lives group, then the single premiums is given by:

$$\begin{aligned} \Pi = {} & b \sum_{h=1}^{+\infty} (1+i)^{-h} {}_h p_{y'} \, {}_h p_{y''} && \text{(benefit paid while both (1) and (2) are alive)} \\ & + b' \sum_{h=1}^{+\infty} (1+i)^{-h} {}_h p_{y'} (1 - {}_h p_{y''}) && \text{(benefit paid if only (1) is alive)} \\ & + b'' \sum_{h=1}^{+\infty} (1+i)^{-h} {}_h p_{y''} (1 - {}_h p_{y'}) && \text{(benefit paid if only (2) is alive)} \\ = {} & b'\, a_{y'} + b''\, a_{y''} + (b - b' - b'')\, a_{y',y''} \end{aligned} \tag{6.12}$$

Example 6.3 The single premiums for two last-survivor annuities are shown in Tables 6.3 and 6.4, respectively. The usual technical basis with $i = 0.02$ has been adopted. Note that: $100 a_{65} = 1\,706.88$. ■

Table 6.3: Single premium of a last survivor annuity, with $b = b' = b'' = 100$

	$y' = 60$	$y' = 65$	$y' = 70$
$y'' = 50$	2 709.10	2 661.02	2 630.35
$y'' = 55$	2 543.34	2 466.87	2 415.43
$y'' = 60$	2 400.20	2 286.98	2 205.33

Table 6.4: Single premium of a last survivor annuity, with $b = b' = 100,\ b'' = 60$

	$y' = 60$	$y' = 65$	$y' = 70$
$y'' = 50$	2458.02	2319.40	2188.82
$y'' = 55$	2358.58	2202.91	2059.87
$y'' = 60$	2272.68	2094.98	1933.81

The mathematical reserve at time t must account for the (actual or hypothetical) state at time t of the two-lives group. Assuming that the benefits are respectively given by b, b', b'', we have:

$$V_t = \begin{cases} b'\, a_{y'+t} + b''\, a_{y''+t} + (b - b' - b'')\, a_{y'+t,y''+t} & \text{if both (1) and (2) are alive at time } t \\ b'\, a_{y'+t} & \text{if only (1) is alive at time } t \\ b''\, a_{y''+t} & \text{if only (2) is alive at time } t \end{cases} \tag{6.13}$$

A last-survivor scheme can also be applied in the case of groups consisting of more than two individuals. Whatever the number of individuals, and whatever the benefit arrangement, the expected duration of a last-survivor annuity is longer than that of a conventional life annuity (that is, with just one annuitant). A higher amount of longevity risk is then taken by the annuity provider (but in the case of longevity-linked annuities).

Chapter 7

The annuity rate

Different annuity rates are adopted to price life annuities. Hence, life annuities can be looked at according to the annuity rate criterion (see Sect. 3.2).

The choice of the annuity rate to be applied when selling a life annuity product can be placed in the area of *risk classification*. Moving from risk factors to rating factors (i.e. those factors which are actually accounted for in the pricing, or "rating", procedure) is, on the one hand, a matter of possibility of detecting the level (or "value") and the impact of risk factors and, on the other, of being allowed by the current legislation to use some risk factors in the pricing procedure.

In the following sections, we first address some general aspects of risk classification and rating procedures in insurance and life annuity business, and then move to specific problems concerning life annuities and pensions.

7.1 Risk factors, rating factors, rating classes

Any given population is affected by some degree of *heterogeneity*, as far as individual mortality is concerned. Heterogeneity is caused by various risk factors which impact on the individual age-pattern of mortality. Heterogeneity in populations should be approached addressing two main issues:

- detecting and modeling observable risk factors (e.g. age, gender, occupation, etc.);
- allowing for unobservable risk factors.

As regards *observable risk factors*, we note that individual mortality depends on:

1. biological and physiological factors, such as age, gender, genotype;
2. features of the living environment; in particular: climate and pollution, nutritional standards (mainly with reference to excesses and deficiencies in diet), population density, hygienic and sanitary conditions;

3. occupation, especially in relation to possible professional disability or exposure to injury, and educational attainment;
4. individual lifestyle, in particular with regard to nutrition, alcohol and drug consumption, smoking, physical activities and pastimes;
5. current health conditions, personal and/or family medical history, civil status, and so on.

Item 2 affects the overall mortality of a population. That is why mortality tables are typically considered specifically for a given geographic area. The remaining items concern the individual and, when dealing with life insurance and life annuities, they can be observed, at least to some extent, at the policy issue. Their assessment, if any, is performed through appropriate questions in the application form and, as to health conditions, possibly through a medical examination.

As already noted, applicants for life annuities are usually in good health, so a medical examination is not necessary to purchase a conventional life annuity; on the contrary, an appropriate investigation is needed for those who apply for special-rate annuities (see Sects. 7.3 and 7.4). Of course, investigations are conversely required for those who purchase insurance products providing death benefits (in particular, with a significant sum assured) given that people in poorer health conditions may be more interested in them and hence more likely to buy such benefits.

The observable risk factors lead to a partitioning of a given population into *risk classes*. However, for various reasons, not all the risk factors are accounted for when pricing an insurance or annuity product. As a straight consequence, some individuals will pay a premium higher and other individuals a premium lower than their "true" premium. Hence, a *solidarity* effect is introduced into the premium system. Risk factors accounted for in the pricing (or "rating") procedure are called *rating factors*. Rating factors yield a less detailed partitioning of the population. Each combination of the possible values taken by the various rating factors results in a premium rate, or an annuity rate. Of course, more combinations can lead to the same rate. The set of different rates constitutes the set of *rating classes*.

When defining a rating procedure, possible *adverse selection* (or *anti-selection*) should be taken into account. This expression denotes a higher propensity to buy insurance in people bearing a "worse" risk profile. Of course, the meaning of "worse" depends on the types of benefits provided by the contract: it is reasonable to assume that people in very good health conditions have a higher propensity to purchase life annuities.

The risk factors considered in a specific rating procedure depend, to some extent, on the types of benefits provided. *Age* is always considered, due to the apparent variability of mortality in this regard. *Gender* should be accounted for, especially when living benefits are involved, given that females on average live longer than males. However in many countries, among which the countries belonging to the European Union, gender is no longer allowed in premium calculation (see Sect. 7.5).

As far as *genetic aspects* are concerned, the evolving knowledge in this area has raised a lively debate (which is still running) on whether it is legitimate for insurance companies to resort to genetic tests for underwriting purposes.

Differences among the individuals can also be attributed to *unobservable risk factors*. Examples of unobservable factors are the individual's attitude towards health, and some congenital personal characteristics.

When modeling the impact of unobservable risk factors, various approaches can be adopted. However, the basic idea is that any life table, or any mortality law, should be interpreted as a mixture of a set of tables or laws, each one expressing a specific level of mortality. We do not deal with these aspects, which are beyond the scope of these Lecture Notes.

Remark 7.1 We note that, although (in principle) we could imagine a rating system structured so that each risk class corresponds to a rating class, some amount of solidarity cannot be removed: indeed, a residual heterogeneity inside each risk class remains, because of the presence of unobservable risk factors.

7.2 Voluntary versus pension life annuities

When deciding what annuity rate should be applied in actuarial calculations (i.e. in pricing and in reserving procedures), we must be aware that a life annuity can arise as the result of different circumstances and/or choices. In particular, we note what follows.

- A *voluntary life annuity* (also called a *purchased life annuity*) is a life annuity bought as a consequence of an individual choice, that is, a choice exercised on a voluntary basis.
- A *pension life annuity* is a life annuity paid to an individual as a direct consequence of his/her membership of an occupational pension plan, or a life annuity bought because a *compulsory purchase* mechanism works.

Although the two kinds of life annuity share the same technical structure, the adverse selection effect is clearly higher in the voluntary annuities, and this should be accounted for when choosing the life table underpinning the annuity rates.

Thus, the type of the life annuity (voluntary annuity versus pension annuity) constitutes a driver in the calculation of the annuity rate, which should be more prudential in the case of voluntary life annuities.

Remark 7.2 As regards the retirement benefits provided by pension plans, it is worth noting that not all pension plans provide the retirees with a life annuity as a straight and compulsory consequence of their membership of the plan itself. In particular, the following aspects should be stressed.

1. Some pension plans only pay the accumulated retirement savings as a lump sum, so that the retiree has to purchase a life annuity on the insurance market, if willing to have a guaranteed lifetime income.
2. Other pension plans offer the choice between a life annuity and a lump sum at the time of retirement.

In both the above cases, possible adverse selection should be accounted for.

7.3 Special-rate annuities: approaches to underwriting

Life annuities are usually sold without any underwriting. As these are attractive mainly for healthy people, in order to expand their business, in recent years some insurers have started offering higher annuity rates to people whose like expectancy is lower than that of likely annuity buyers. *Special-rate life annuity* products have then been designed.

Special-rate life annuities are also called *underwritten life annuities*, because of the ascertainment of higher mortality assumptions via the underwriting requirements. Life annuity underwriting can be implemented in a number of ways, and several classification can be conceived. [1]

It is interesting to focus on:

(1) what risk factors can be chosen as rating factors, besides annuitant's age and gender (if permitted by the local current legislation);

(2) how many rating factors are actually accounted for in the underwriting process of a given special-rate annuity;

(3) how many rating classes, that is, how many different annuity rates, are defined.

As regards (1), we note that higher mortality, and then lower life expectancy, can in particular be due to the following causes.

(1a) The individual health, and in particular the presence of some past or current *disease*, clearly impacts on the mortality pattern.

(1b) The applicant's *lifestyle* (e.g. smoking and drinking habits, sedentary life, etc.) can cause higher mortality.

(1c) The *environment* in which the applicant lives might also impact on his/her mortality, and hence socio-geographical risk factors can be accounted for.

On the one hand, the higher the number of rating factors (see point (2) above), the more complex is the underwriting process; on the other, the higher the number of rating classes (see point (3)) the better is the fitting of the individual risk profile.

As regards the number of rating classes, the following classification reflects alternative pricing schemes that can be adopted in the insurance practice.

(3a) When a *single-class* underwriting scheme is adopted, one or just a few rating factors are accounted for, and the underwriting results in a yes/no answer. If yes, a given annuity rate, higher than the "standard" one, is applied. Examples are provided by the marital status and the smoking habits (see Sect. 7.4).

(3b) The *multi-class* underwriting scheme can be implemented either considering just one rating factor with several possible values, or more rating factors in which case each combination of values yields an annuity rate.

[1] What follows is mainly based on the classifications proposed by Rinke (2002). Differences in terminology can however be detected. In particular, the expression "enhanced annuity" is used, in that paper, as a synonym to underwritten or special-rate life annuity.

(3c) The *individual underwriting* allows to use all the available information about the individual, so that the annuity rate can be tailored on the applicant's characteristics.

The above classifications are sketched in Fig. 7.1.

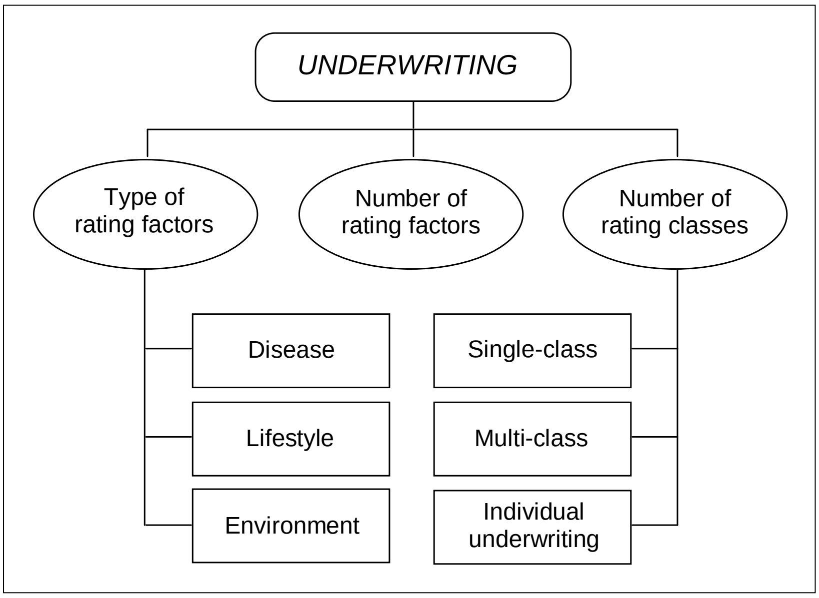

Figure 7.1: Approaches to underwriting for life annuities

7.4 Special-rate annuities: examples

The applicant's health status and, in particular, the presence of past or current diseases (see (1a) in Sect. 7.3) is considered in the following special-rate annuities.

- The *enhanced life annuity* pays out an income to a person with a slightly reduced life expectancy, in particular because of a personal history of medical conditions. Of course, the enhancement in the annuity benefit (compared to a standard-rate life annuity, same premium) comes, in particular for this type of annuity, from the use of a higher mortality assumption.
- The *impaired-life annuity* pays out a higher income than an enhanced life annuity, as a result of medical conditions which significantly shorten the life expectancy of the annuitant (e.g. diabetes, chronic asthma, cancer, etc.).
- Finally, *care annuities* are aimed at individuals, usually beyond age 75, with very serious impairments, or individuals who are already in a senescent-disability (or long-term care) state. See also Sect. 10.1.

Various factors concerning the health status can be accounted for, and this usually leads to a multi-class underwriting. Medical ascertainment is of course required. In particular, as regards the impaired-life life annuity and the care annuity, the underwriting process must result in classifying the applicant as a *substandard risk*. For this reason, these annuities are sometimes named *substandard annuities*.

The underwriting of a *lifestyle annuity* (see (1b) in Sect. 7.3) can take into account one or more rating factors, and can result in a single-class or a multi-class underwriting. Examples are given by smoking and drinking habits, marital status, occupation, and location of housing. These factors might result in a shorter life expectancy. Some specific examples follow.

- *Smoker annuities*: if the applicant has smoked at least a given number of cigarettes for a certain number of years, he/she is eligible for smoker annuities. A single-class underwriting is in this case implemented.
- Mortality differences between married and unmarried individuals underpin the use of special rates in pricing the *unmarried lives annuities*. The observed higher mortality rates of unmarried individuals justify a higher annuity rate. A single-class underwriting is also in this case implemented.

Postcode life annuities constitute an important example of environment-based rating (see (1c) in Sect. 7.3). The postcode can provide a proxy for social class and location of housing, i.e. risk factors which have a significant impact on the life expectancy. Then, its use as a rating factor for pricing life annuities can be justified. Hence, a multi-class underwriting scheme follows.

7.5 Unisex annuity rates

As mentioned in Sect. 7.1, if not all the risk factors are accounted for when pricing an insurance or annuity product, a *solidarity* effect is introduced into the premium system. This is the case of gender which is no longer allowed as a rating factor in many countries (in particular in the European Union), according to the *gender-neutral rating principle* (or *unisex rating principle*).

Nevertheless, gender is a risk factor, as witnessed by the differences between the female and the male age-pattern of mortality, especially in terms of the expected lifetime. Hence, life annuity rates must rely on appropriate assumptions as regards the *gender mix* in the annuity portfolio. These assumptions lead to the construction of a *unisex life table*.

Shares of females and males in the gender mix are estimated relying on past experience and forecasts of future likely trends. Possible deviations of the future actual mix from the estimated one imply situations of non-equilibrium in the portfolio, in the sense that annuity rates which rely on the estimated mix can result either lower or higher than those emerging from the real gender mix.

Chapter 8

Cross-subsidy in life annuity portfolios

Although all insurance transactions can be analyzed at an individual level (e.g. in terms of the equivalence principle), in practice these transactions usually involve a group of insureds (a cohort, in particular) who transfer the same type of risk to an insurer. Of course, this is also the case for life annuity products, and actually these products have been discussed in previous sections also in terms of a cohort of annuitants; see, for example, Sects. 2.5 and 2.6 as regards the mutuality mechanism and its impact on the reserving process.

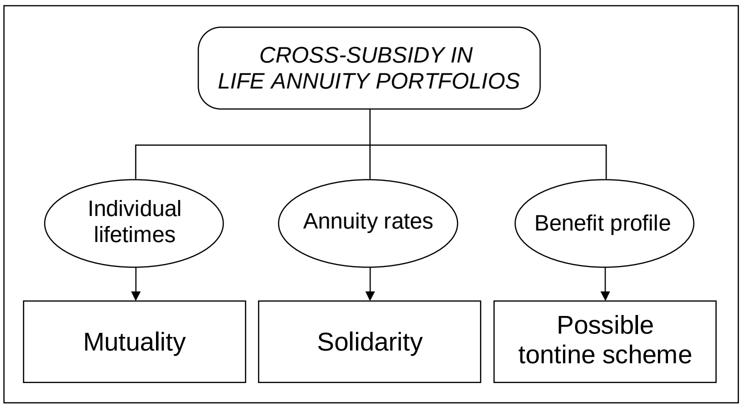

Figure 8.1: Cross-subsidy mechanisms

Thanks to the existence of an insured population (a cohort, in particular), money transfers inside the population itself (i.e. among the policyholders, or the retirees) are possible, causing a *cross-subsidy* among the insureds or annuitants. The term cross-subsidy broadly refers to the effect of some arrangement adopted for sharing among the members of a given population the cost of a set of benefits. However, various types of cross-subsidy can be recognized. While mutuality underpins the management of any insurance or life annuity portfolio, other types of cross-subsidy are not necessarily involved, or are involved just to some extent, for example solidarity. Furthermore, special cross-subsidy structures may occur with particular arrangements; this is the case of *tontine schemes* in the context of lifelong benefits.

In the following parts of this section, we deal with cross-subsidy mechanisms, of course focussing on life annuity portfolios. As sketched in Fig. 8.1, the mutuality mechanism works because of the different lifetimes of the annuitants, while the possible presence of solidarity is caused by the rating system adopted to price life annuity products. According to some time-profile of the benefits, a tontine scheme (in particular a tontine annuity) can be recognized.

8.1 Mutuality and solidarity

The mutuality mechanism underpins the insurance process (whether or not it is run by a "mutual" insurance company or by a proprietary insurance company owned by shareholders, or by a pension plan providing pension annuities), and arises from the pooling of a number of individuals. Moreover, the mutuality effect also works in "mutual associations" of individuals, even without resorting (at least in principle) to an insurance company or a pension plan.

The mutuality mechanism implies money transfers from annuitants who paid premiums greater than the benefits received to annuitants in the opposite situation. Hence, some individuals suffer a loss whereas other individuals obtain a gain (a simple numerical illustration of the possible individual results is given by Example 2.3).

Thanks to the mutuality mechanism the annual equilibrium between assets available and liabilities is achieved on an individual basis (and hence on a cohort basis as well). This equilibrium relies on an asset transfer among annuitants, viz from annuitants dying in the year to annuitants alive at the end of the year. This issue has been formalized, in terms of mathematical reserve and its annual variation, in Sect. 2.5.

The presence of solidarity in a life annuity portfolio has been discussed in Sect. 7.1 and, as specifically regards the unisex rating, in Sect. 7.5. We recall that solidarity transfers arise when one or more risk factors are disregarded by the rating system. The premium rate attributed to a rating class should be an appropriate weighted average of the premiums pertaining to the risk classes grouped into the rating class itself. The weighting should reflect the estimated numbers of (future) annuitants belonging to the various risk classes.

Clearly, such a rating system may cause (or increase) adverse selection, as individuals forced to provide solidarity to other individuals can reject the life annuity policy, moving to other solutions, for example self-annuitization. The severity of this phenomenon depends on how people perceive the solidarity mechanism, as well as on the rating systems adopted by competitors in the life annuity market. In any event, adverse selection can jeopardize the technical equilibrium inside the portfolio, which depends on actual versus expected numbers of annuitants belonging to the various risk classes grouped into a rating class. So, in practice, solidarity mechanisms can work provided that they are compulsory (e.g. imposed by insurance regulation) or they constitute a common market practice.

Finally, it is interesting to stress the implications of mutuality and solidarity transfers. Mutuality affects the benefit payment phase, so that "direction" and "measure" of the actual mutuality effect in a portfolio are definitely known ex-post only, that is, after extinction of the annuitants' cohort. Conversely, solidarity affects the premium income phase, and hence its direction and measure are known ex-ante.

8.2 Tontine annuities

Assume that a financial institution launches a fund raising transaction, and each one of ℓ_x individuals, all aged x at time 0, contributes at that time with the amount c. Against the income $C = \ell_x c$, the financial institution will pay at the end of each year, i.e. at times $t = 1, 2, \dots$, the (total) constant amount B, while at least one of the individuals of the group is alive.

Each year the amount B is divided equally among the survivors. Hence, each individual (out of the initial ℓ_x) alive at time t receives a benefit b_t which depends on the actual number of survivors at that time. Denoting, as usual, with ℓ_{x+t} the estimated number of survivors at time t, an estimate of b_t is given by:

$$b_t = \frac{B}{\ell_{x+t}}; \quad t = 1, 2, \dots \tag{8.1}$$

Clearly,

$$b_1 \le b_2 \le b_3 \le \dots \tag{8.2}$$

The mechanism of dividing B among the survivors is called a *tontine scheme*, whereas the sequence of b_t, as defined by Eq. (8.1), is called a *tontine annuity*. The group (in particular the cohort) consisting of initial ℓ_x individuals constitutes a *tontine group*.

The relation between C (the total initial income) and B (the total annual payment) can be stated (at least in theory) on the basis of the equivalence principle. To this purpose, first note that the number, K, of annual amounts paid by the financial institution is random, depending on the maximum individual lifetime in the tontine group (precisely, coinciding with the integer part of the maximum remaining lifetime). Hence, the equivalence principle requires:

$$C = B\, \mathbb{E}[a_{\overline{K}|}] \tag{8.3}$$

where $\mathbb{E}[a_{\overline{K}|}]$ denotes the expectation of the random present value, $a_{\overline{K}|}$, of an annuity providing unitary annual benefits.

The calculation of $\mathbb{E}[a_{\overline{K}|}]$ is extremely difficult. In practice, a reasonable approximation is provided by $a_{\overline{\omega-x}|}$, that is, the present value of an annuity-certain with $\omega - x$ unitary annual payments. While in general $a_{\overline{\omega-x}|} > \mathbb{E}[a_{\overline{K}|}]$, the larger is ℓ_x the better is this approximation, as there is a higher probability that some individual reaches, or at least approaches, the maximum attainable age ω.

By assuming this approximation, and setting $b_0 = \dfrac{B}{\ell_x}$, from Eq. (8.3) we obtain:

$$c = b_0\, a_{\overline{\omega-x}|} \tag{8.4}$$

Thus, $a_{\omega-x\rceil}$ represents the *tontine factor*. We recall that, conversely, a_x is the life annuity factor (see Sect.2.4), and hence, for a single premium $\Pi = c$, the life annuity benefit b is given by the relation $c = b\,a_x$. As $a_{\omega-x\rceil} > a_x$, it follows that $b_0 < b$.

Further, the life annuity benefit b is likely to be much higher not only than b_0 (the "baseline benefit", which actually will never be paid), but also higher than the initial estimated payments of the tontine annuity, $b_1, b_2, \dots$. Still using the approximation $\mathbb{E}[a_{K\rceil}] \approx a_{\omega-x\rceil}$, from Eqs. (8.1) and (8.4), after a little algebra, we find:

$$b_t = b\,\frac{\ell_x}{\ell_{x+t}}\,\frac{a_x}{a_{\omega-x\rceil}} \tag{8.5}$$

Then, as long as $\frac{\ell_x}{\ell_{x+t}}\,\frac{a_x}{a_{\omega-x\rceil}} < 1$, i.e. $\ell_{x+t} > \ell_x\,\frac{a_x}{a_{\omega-x\rceil}}$, we find $b_t < b$. In other terms, as long as the estimated size of the tontine group, ℓ_{x+t}, remains large, the estimated benefit amount is small. Thus, achieving a "good" amount b_t (when compared with b) in the increasing sequence (8.2) depends on the mortality in the tontine group. Mainly for this reason, tontine annuities were suppressed by many governments, and at present prohibited in most countries.

Table 8.1: Benefits b_t paid by a tontine annuity; $x = 65$

$x+t$	b_t
66	52.28
67	53.17
68	54.15
69	55.21
70	56.37
...	...
83	93.31
84	100.01
85	107.96
...	...
90	182.04
91	209.49
92	244.80
...	...
97	715.68
98	955.32
99	1 315.06
...	...

Example 8.1 Adopting the Heligman-Pollard law with the usual parameters (see Table 2.1) and the interest rate $i = 0.02$, the single premium for an (ordinary) immediate life

annuity in arrears, age at policy issue $x = 65$, providing an annual benefit $b = 100$, is $\Pi = 1\,706.88$ (see Table 2.3). Assume that, conversely, the amount $c = \Pi = 1\,706.88$ is paid as contribution to a tontine scheme. From Eq. (8.4) we find $b_0 = 51.45$, while (8.5) yields the sequence $b_1, b_2, \ldots$; some values are shown in Table 8.1, whereas in Fig. 8.2 the age range has been limited to $x+t = 95$). ■

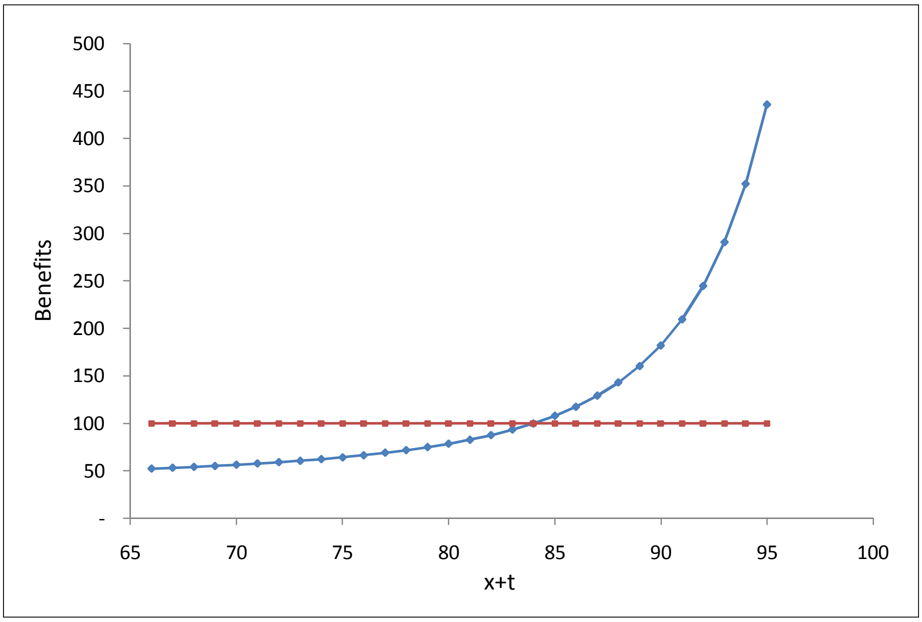

Figure 8.2: Benefits b_t paid by a tontine annuity vs constant benefit b paid by an ordinary life annuity; $x = 65$

Example 8.2 The tontine annuity derives its name from Lorenzo Tonti (a Neapolitan banker living most of his life in Paris) who, around 1650, proposed a plan for raising monies to Cardinal Mazzarino, the Chief Minister of France at the time of King Louis XIV. In this plan, a fund was raised by subscriptions. Let C denote the amount collected by the State. Then, according to the plan, the State had to pay each year the interest on C, at a given annual interest rate i. The constant annual payment, $C\,i$, was to be divided equally among the surviving members of the group and would terminate with the death of the last survivor. Thus, according to our notation, the duration of the annuity is K, and we have $B = C\,i$. Note that $i = \dfrac{1}{a_{\overline{\infty}|}}$ where $a_{\overline{\infty}|}$ is the present value of a perpetuity, given the discount rate i. As (whatever the age x) we have:

$$B = C\,i = \frac{C}{a_{\overline{\infty}|}} < \frac{C}{a_{\overline{\omega-x}|}} < \frac{C}{\mathbb{E}[a_{\overline{K}|}]}$$

(assuming that the same discount rate is used for all the present values), we find that the original Tonti's scheme did not fulfill the equivalence principle according to interest rate i, whilst it was favourable to the issuer (that is, to the State). ■

The following points should finally be stressed.

1. The tontine scheme clearly implies a cross-subsidy among the members of the tontine group; in particular, a specific mutuality effect arises as each dying member releases a share of the amount B, which is divided among the surviving members.

2. Although both tontine annuities and ordinary life annuities provide the annuitant with a hedge against longevity risk, a basic difference between the two arrangements should be recognized. In an ordinary life annuity, the annual (individual) benefit b is stated and guaranteed, in the sense that the life annuity provider has to pay the amount b to the annuitant for his/her whole residual lifetime, whatever the mortality experienced in the portfolio (or pension plan) may be. Conversely, in a tontine annuity the sequence of amounts $b_1, b_2, \dots$ paid to each member of the tontine group depends on the actual number of survivors.

3. When managing an ordinary life annuity portfolio the annuity provider takes the risk of a poor mortality experience in the portfolio, which can cause a total annual amount of benefits higher than expected. On the contrary, in a tontine scheme the only cause of risk is the remaining lifetime K of the last survivor, the annual amount of benefits being B anyhow, as long as at least one member of the tontine group is alive.

4. Although tontine annuities are prohibited in many countries, ideas underlying tontine schemes survive in some mechanisms of profit participation in life annuity portfolios, especially when also mortality profits or longevity losses are involved, as mentioned in Sect. 5.3. Further, it is worth noting that the tontine scheme can be implemented weakening some drawbacks of the original tontine annuities. Recently, implementations of the tontine scheme have been proposed by several Authors; for bibliographic suggestions, the reader is referred to Chap. 11.

Chapter 9

Strategies for the post-retirement income

We describe some *annuitization strategies* which aim at mitigating the disadvantages caused by a full and immediate annuitization of the assets available at the time of retirement.

9.1 Life annuities versus income drawdown

The simplest strategy, to mitigate disadvantages 1(b), 2 and 3 mentioned in Sect. 6.1, is a *partial annuitization*. If S is the amount available at retirement, just a part of S, say S' is converted into an annuity, whereas $S - S'$ constitutes the non-annuitized fund, from which a share of the post-retirement income can be obtained via a drawdown process.

A *temporary drawdown* can also mitigate the disadvantages mentioned above. Assume that the retiree, age y, can choose between the two following alternatives:

1. to purchase an immediate life annuity (that is, a SPIA), with annual benefit b, such that $b\, a_y = S$, which means full immediate annuitization;
2. to leave the whole amount S in a fund (that is, a non-annuitized fund), and then:
   (a) withdraw the amount $b^{(1)}$ at times $t = 1, 2, \dots, k$ (say, with $k = 5$ or $k = 10$); thus, the post-retirement income is obtained via a temporary drawdown process;
   (b) provided he/she is alive, convert at time k the remaining amount R into an immediate life annuity with annual benefit $b^{(2)}$.

Hence, alternative 2 consists of a temporary *self-annuitization*, that is, a financial transaction in which the annuity is managed by the retiree, followed by the purchase of a life annuity. Briefly, this alternative is commonly known as *delayed annuitization*. The two alternatives are sketched in Fig. 9.1.

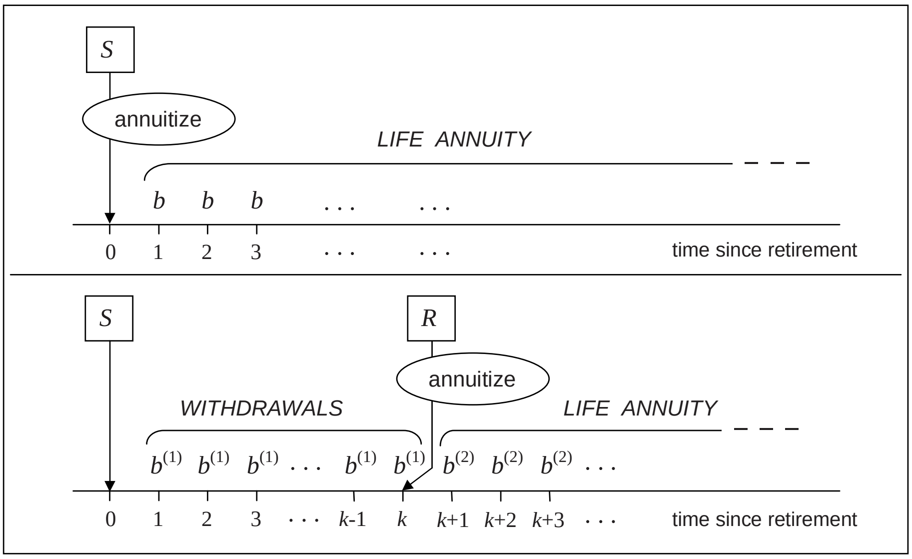

Figure 9.1: Immediate annuitization versus delayed annuitization

If the retiree chooses alternative 2, the amount R available at time k to buy the life annuity depends on the annual withdrawal $b^{(1)}$ and the interest rate, g, credited to the non-annuitized fund. If $g = i$, namely the interest rate assumed in the pricing basis of the life annuity, and $b^{(1)} = b$ then, the amount R is not sufficient to purchase a life annuity with annual benefit $b^{(2)} = b$, because of the absence of mutuality and hence of mortality credits during the drawdown period.

However, the absence of mortality credits can be compensated (at least in principle) by a higher investment yield, namely if $g > i$. In formal terms, we can find relations among the quantities g, i, b, $b^{(1)}$, $b^{(2)}$, and k. In the case the amount S is used to purchase at retirement time a life annuity in arrears (alternative 1), we obviously have:

$$S = b\,a_y \tag{9.1}$$

In the case of k-year delay (alternative 2), the amount R available at time k is given by:

$$R = S\,(1+g)^k - b^{(1)} \sum_{t=1}^{k} (1+g)^{k-t} \tag{9.2}$$

and the resulting annuity benefit $b^{(2)}$ fulfills the following equation:

$$R = b^{(2)}\,a_{y+k} \tag{9.3}$$

in which it is assumed that the underlying technical basis coincides with the one adopted in Eq. (9.1) (see below for comments in this regard).

From Eqs. (9.2) and (9.3), we find:

$$S\,(1+g)^k - b^{(1)} \sum_{t=1}^{k} (1+g)^{k-t} = b^{(2)}\,a_{y+k} \tag{9.4}$$

Several results can be obtained by using Eq. (9.4). For example, given S, i, b, k, and

- given g and $b^{(1)} = b$, calculate $b^{(2)}$;
- given $b^{(1)} = b^{(2)} = b$, calculate the interest rate g which, thanks to the spread $g - i$, compensates the absence of mortality credits; as this interest rate obviously depends on both the age at retirement and the delay, it will be denoted by $g(y,k)$; the interest rate $g(y,k)$ is often called the *Implied Longevity Yield* (*ILY*)[1].

Remark 9.1 The interest rate $g(y,k)$ clearly reflects the impact of the mutuality mechanism that works in life annuities, summarizing the annual extra-yields θ_{y+t} for $t = 0,1,\dots,k-1$ (see Sect. 2.6).

Example 9.1 Assume that the amount $S = 1\,706.88$ is available at age $y = 65$. Adopt the usual technical basis, with $i = 0.02$. Hence, an immediate life annuity with annual benefit $b = 100$ could be bought, as it results from Table 2.3. As an alternative to the immediate annuitization of S, assume that the annual amount $b^{(1)} = b$ is withdrawn from a fund (whose initial value is S). Table 9.1 displays the annuity benefit $b^{(2)}$ as a function of the delay k, and the interest rate g credited to the fund throughout the delay period. We note that the technical basis with $i = 0.02$ and the same life table is adopted, whatever the delay k. If $g = i = 0.02$, then we have, of course, $b^{(2)} < b$; further, $b^{(2)}$ decreases as the delay k increases. If $g > i$, we can have situations in which the higher yield during the delay period implies $b^{(2)} > b$, that is, a higher annuity benefit. ■

Table 9.1: Life annuity benefit $b^{(2)}$ after the delay period; $b^{(1)} = b = 100$

k	$g = 0.02$	$g = 0.025$	$g = 0.03$	$g = 0.035$
5	95.63	98.54	101.50	104.53
10	85.79	92.65	99.87	107.45
15	64.09	76.61	90.21	104.96
20	16.40	37.29	60.88	87.42

Example 9.2 Table 9.2 shows, for various delays k (and still assuming $b^{(1)} = b$), the yield $g(65,k)$ which is required to have $b^{(2)} = b$, hence exactly compensating the absence of mortality credits during the drawdown period. ■

Remark 9.2 In a more realistic setting, we can assume that the investment yield during the delay period does not follow a flat profile (viz because of variability in the estimated interest rates, or because of the term structure of interest rates). According to this setting, the ILY can be

[1] Registered trademarks and property of CANNEX Financial Exchanges.

Table 9.2: Investment yields compensating the mortality credits; $b^{(1)} = b$

k	$g(65,k)$
5	0.02748
10	0.03009
15	0.03336
20	0.03718

interpreted as follows. Assume $b^{(1)} = b^{(1)} = b$ in Eq. (9.4). Then, $g(y,k)$ represents the *internal rate of return* of the transaction which consists in exchanging the lump sum $S = b\,a_y$ at time 0 against the sequence of k annual withdrawals, each one equal to b, at times $1,2,\dots,k$, and the lump sum $b\,a_{y+k}$ at time k.

The delayed annuitization has some advantages. In particular:

- in the case of death before time k, the fund available constitutes a bequest (which is not provided by a life annuity purchased at time 0, because of the mutuality effect);
- more flexibility is gained, as the annuitant may change the income profile modifying the drawdown sequence (however, with a possible change in the fund available at time k).

We also note that, the lower the mortality, the lower is the required yield $g(y,k)$. It follows that, thanks to mortality improvements over time, the delayed annuitization can become more and more attractive.

Conversely, a disadvantage is due to the risk of a shift to a different mortality assumption in the pricing basis of life annuities, which might imply an annuity rate at time k less favorable to the life annuity purchaser than that adopted at time 0. Further, if k is high, it may be difficult to gain the required investment yield (in particular, avoiding too risky investments) to cover the absence of mortality credits.

Remark 9.3 The delayed annuitization strategy can be compared to the post-retirement arrangement involved by an ALDA product (see Sect. 4.2). In both the arrangements, we recognize the initial existence of a non-annuitized fund, and hence the need for income drawdown and, at the same time, the accessibility to the fund itself. However, the amount of the annuity benefit provided by an ALDA should be based on an annuity rate stated at the time the ALDA product is purchased, whereas in the delayed annuitization the annuity rate is defined at the time of annuitization.

The ideas underlying the partial annuitization and the delayed annuitization can be combined, leading to the so-called *staggered annuitization*. As shown in Fig. 9.2, the staggered annuitization can be defined as a process according to which:

- no life annuity is purchased at retirement time (time 0), so that an income drawdown process starts at that time;

- a first life annuity is purchased at time k', by using part of the remaining amount R';
- a second life annuity is purchased at time k'', by using part of the remaining amount R'';
- ... ...

The staggered annuitization implies that (after time k') a share of the post-retirement income still consists of withdrawals, whereas the remaining share is provided by a (set of) life annuities. Advantages and disadvantages of this arrangement can be easily understood looking at what noted above in relation to the delayed annuitization.

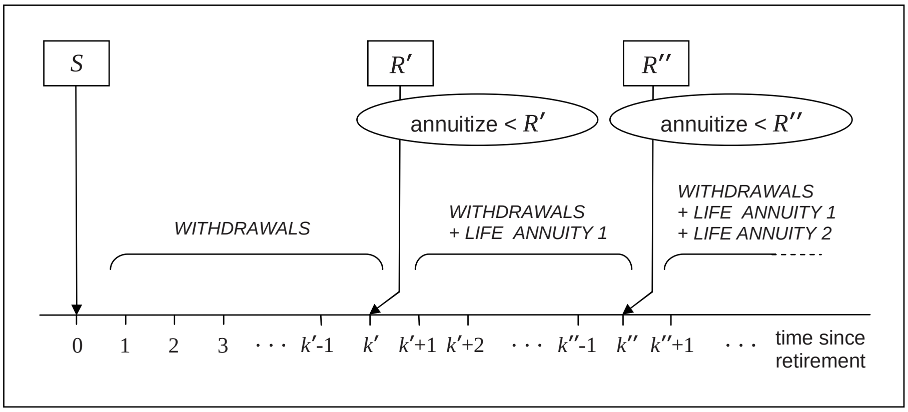

Figure 9.2: Staggered annuitization

Remark 9.4 Delayed annuitization and staggered annuitization imply a tradeoff between mortality risk and financial risk (and longevity risk as well). Indeed, on the one hand the non-annuitizing strategy leaves, at the time of death, an amount available as a bequest, then facing to some extent the impact of the mortality risk; on the other hand, a financial risk is taken because of the need for a higher investment yield to recover mortality credits (and a longevity risk because of possible unfavorable change in the annuity rate).

9.2 Phased retirement

Several employment arrangements allow an employee to gradually move from the working period to the retirement period. Such a progressive shift from full-time work to full-time retirement is usually denoted as *phased retirement*.

The phased retirement can be implemented in several ways (according to possible constraints imposed by the current local legislation). For example:

1. an employee who is approaching the retirement age y continues working with a reduced working load, until the transition to full-time retirement;

2. an employee who reaches the retirement age y stops his/her previous working activity, and starts a similar activity, anyway with a limited working load.

We focus on solution 2, which in particular allows to maintain a higher income than that received, as the post-retirement income, if the employee quits work entirely.

We assume that the employee chooses to obtain his/her income via an immediate life annuity. However, thanks to partial retirement, an annual benefit is chosen, lower than that needed in the case of total retirement. Hence, only a part of the available amount S is annuitized at age y, namely at the beginning of the partial retirement phase (see the partial annuitization strategy). Let $b^{(A)}$ denote the annual benefit which is paid from the beginning of this phase onwards. Clearly $b^{(A)} < b$, where b denotes the annual benefit provided by the full annuitization of S (see Eq. (9.1)). The amount required to purchase a whole life annuity with benefit $b^{(A)}$ is given by $b^{(A)}\,a_y$. Assume that the total duration of the partial retirement phase is m years. At time m the following amount, R, will be available:

$$R = (S - b^{(A)}\,a_y)\,(1+g)^m \tag{9.5}$$

where g denotes the interest rate credited on the non-annuitized fund throughout the partial retirement phase. The amount R can be annuitized to obtain a further life annuity with annual benefit $b^{(B)}$, determined by the following relation:

$$R = b^{(B)}\,a_{y+m} \tag{9.6}$$

Hence, during the total retirement phase, the retiree will cash the annual benefit $b^{(A)} + b^{(B)}$, which clearly depends on the interest rate g and the duration m of the partial retirement phase. Figure 9.3 shows the annuitization process related to the phased retirement.

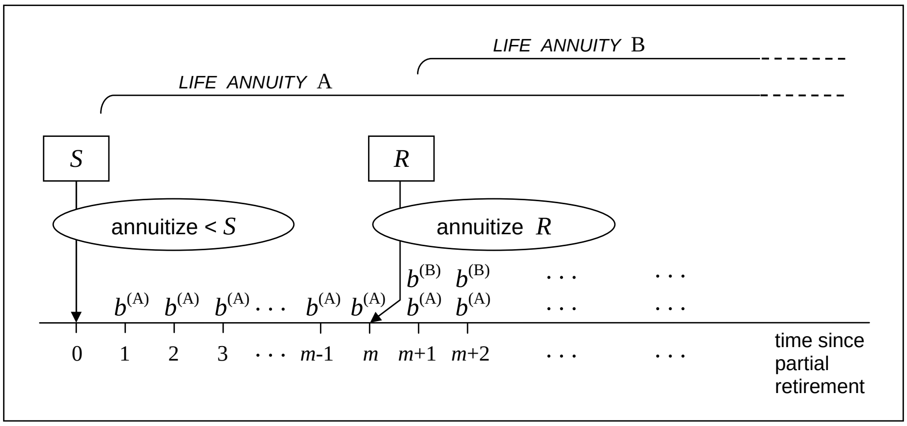

Figure 9.3: Annuitization in phased retirement

Note that, as in the delayed annuitization and in the staggered annuitization processes, the individual bears the risk of an unfavorable change in the technical basis adopted at time m to determine the benefit $b^{(\mathrm{B})}$ (while keeping access to the non-annuitized fund over the whole partial retirement period).

The phased retirement process and the related annuitization process can be generalized in several ways. For example:

- more than just one phase of partial retirement can be envisaged, to implement a more gradual shift from full-time work to full-time retirement;
- life annuities and income drawdown can coexist during the various phases (according to arrangements like those described in Sect. 9.1).

9.3 Transferring the longevity risk

We have so far described several types of life annuity, as well as possible annuitization strategies which constitute alternatives to the full immediate annuitization at retirement time. Of course, a main issue in the choice of a life annuity product or an annuitization strategy should be the amount of longevity risk transferred to the annuity provider. Conversely, the longevity risk taken by the annuity provider should constitute a key point in designing the life annuity products. Table 9.3 summarizes this aspect.

Table 9.3: Longevity risk, where?

Solution		**Longevity risk**
Drawdown process (self-annuitization)		borne by the annuitant
Full immediate annuitization (conventional life annuity)		borne by the annuity provider
Combined solutions	partial annuitization delayed annuitization staggered annuitization ALDA RCLA ……	shared between the annuitant and the annuity provider
Longevity-linked life annuities Tontine annuities		

Chapter 10

Life annuity products providing LTC benefits

As seen in the previous chapters, various products are available to construct the post-retirement income. Nevertheless, a weak propensity to annuitize can be observed in many countries (see also Sect. 1.3). To enhance this propensity, the payout phase can be improved, either by including into the life annuity product, via options and riders, also benefits other than the conventional life annuity, or by adding some flexibility to the annuitization process (see Chaps. 6 and 9, respectively). In this chapter we focus on the possibility of combining life annuity benefits and long-term care benefits.

10.1 Long-term care insurance

Long Term Care insurance (LTCI) provides the insured with financial support, while he/she needs nursing and/or medical care because of chronic (or long-lasting) conditions or ailments. LTCI products can be classified as follows:

- products which pay out benefits with a *pre-defined amount* (usually, a lifelong annuity benefit); in particular
  - a *fixed-amount* benefit;
  - a *degree-related* (or *graded*) benefit, i.e. a benefit whose amount is graded according to the degree of disability, that is, the severity of the disability itself;
- products which provide reimbursement (usually partial) of nursery and medical expenses, i.e. *expense-related* benefits;
- *care service* benefits (for example, provided in the U.S. by the Continuing Care Retirement Communities, briefly CCRCs).

A classification of LTCI products which pay out benefits with pre-defined amount is shown in Fig. 10.1.

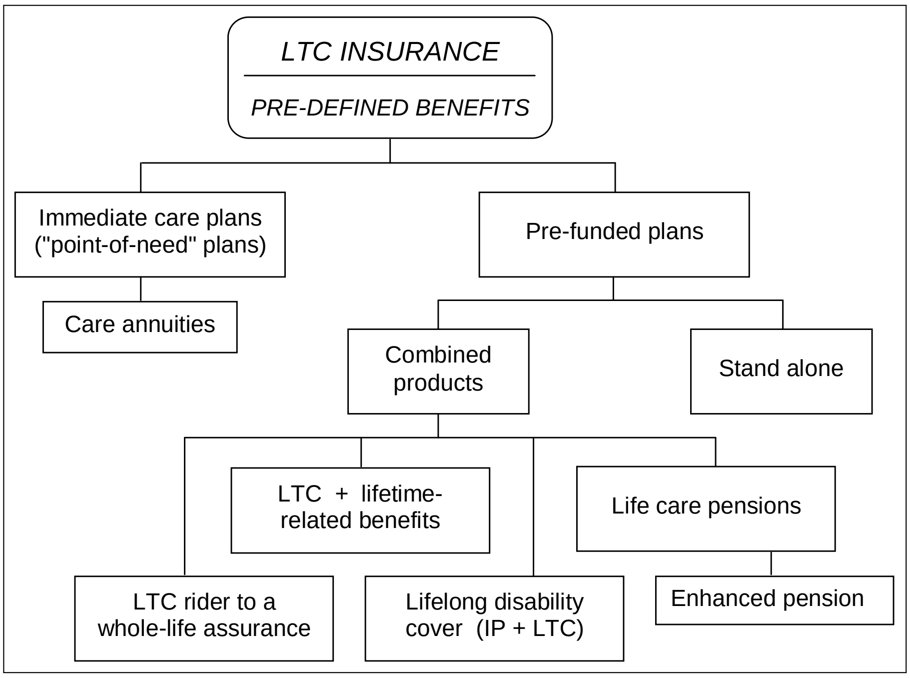

Figure 10.1: A classification of LTCI products providing pre-defined benefits

Immediate care plans, or *care annuities*, relate to individuals already affected by severe disability (that is, in "point of need"), and then consist of:

- the payment of a single premium;
- an immediate life annuity, whose annual benefit may be graded according to the disability severity.

Hence, care annuities are aimed at seriously impaired individuals, in particular persons who have already started to incur long-term care costs. The premium calculation is based on assumptions of short life expectancy. However, the insurer may limit the individual longevity risk by offering a limited term annuity, i.e. a temporary life annuity.

Remark 10.1 Care annuities belong to the class of *special-rate annuities*, also called *underwritten annuities*, because of the ascertainment of higher mortality assumptions via the underwriting requirements. Special-rate annuities sold in several markets have been described in Sects. 7.3 and 7.4.

Pre-funded plans consist of:

- the accumulation phase, during which periodic premiums are paid; the accumulation can degenerate in a single premium;
- the payout period, during which LTC benefits (usually consisting of a life annuity) are paid in the case of LTC need.

Several products belong to the class of pre-funded plans. A *stand-alone LTC cover* provides an annuity benefit, possibly graded according to some severity score. This cover can be financed by a single premium, by temporary periodic premiums, or lifelong periodic premiums. Of course, premiums are waived in the case of an LTC claim. This insurance product only provides a "risk cover", as there is, of course, no certainty in future LTC need and the consequent payment of benefits.

A number of *combined products* have been designed, mainly aiming at reducing the relative weight of the risk component by introducing a saving component, or by adding the LTC benefits to an insurance product with a significant saving component. Some examples follow.

LTC benefits can be added as a *rider benefit to a whole-life assurance* policy. For example, a monthly benefit of, say, 2% of the sum assured is paid in the case of an LTC claim, for 50 months at most. The death benefit is consequently reduced, and disappears if all the 50 monthly benefits are paid. Thus, the (temporary) LTC annuity benefit consists in an *acceleration* of the death benefit. The LTC cover can be complemented by an additional deferred LTC annuity (financed by an appropriate premium increase) which will start immediately after the possible exhaustion of the sum assured (that is, if the LTC claim lasts for more than 50 months) and will terminate at the insured's death.

A *lifelong disability cover* can include:

- an *income protection* cover (briefly, IP) during the working period, that is, during the accumulation period related to LTC benefits; the IP cover provides a periodic income to the individual if he/she is prevented from working, and hence from getting his/her usual income, by sickness or injury; the expression *disability annuity* is frequently used to denote this type of IP benefits.
- an LTC cover during the retirement period.

In the context of combined products, we find some types of benefits that can be packaged with life annuities. We will focus on:

- insurance packages in which LTCI is combined with lifetime-related benefits;
- life care pensions and, in particular, enhanced pensions.

10.2 Combining LTC and lifetime-related benefits

An *insurance package* can merge LTC benefits and lifetime-related benefits. More precisely, the package can consist of:

1. a deferred life annuity starting at time s (e.g. at age 80) with benefit b, while the insured is not in the LTC disability state;
2. a lifelong LTC annuity, with benefit b';
3. a lump sum benefit on death, which can alternatively be given by
    (a) a fixed amount, stated in the policy;

(b) the difference (if positive) between a stated amount and the amount paid as benefit 1 and/or benefit 2.

Three possible individual stories and the consequent outcomes in terms of annuity benefits are shown in Fig. 10.2.

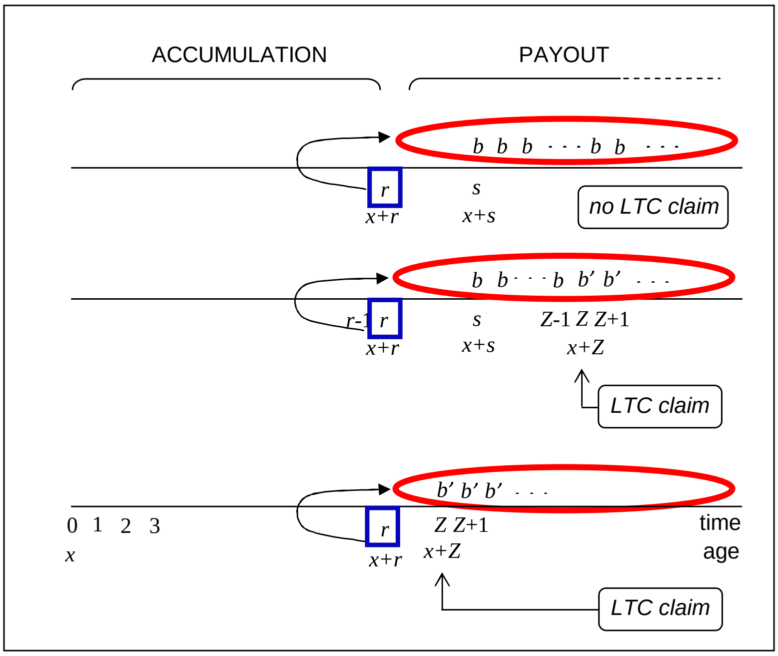

Figure 10.2: Possible annuity outcomes, depending on lifetime and LTC need

This product design clearly aims at a reduction of the prevailing risk feature of the stand-alone LTC annuity.

We note that, apart from the death benefit, this arrangement basically includes the ALDA structure (see Sect. 4.2), as it can provide a deferred life annuity starting at old age (80, say). Conversely, the death benefit defined as in 3(b) aims at capital protection (see Sect. 6.3).

Life care pensions are life annuity products in which the LTC benefit is defined in terms of an uplift with respect to the basic pension. The basic pension b is paid out from retirement onwards, and is replaced by the benefit b' ($b' > b$) in the case of LTC claim. See Fig. 10.3. The uplift can be financed during the whole accumulation period, or during part of this period, by premiums higher than those needed to purchase the basic pension b.

A possible outcome of the annuity payout, according to the life care pension structure, is shown in Fig. 10.4.

Remark 10.2 We note that the life annuity whose benefit is given by $b' - b$ (that is, the amount of the uplift) has the logical structure of the RCLA (see Sect. 4.2); however, the "scenario" is

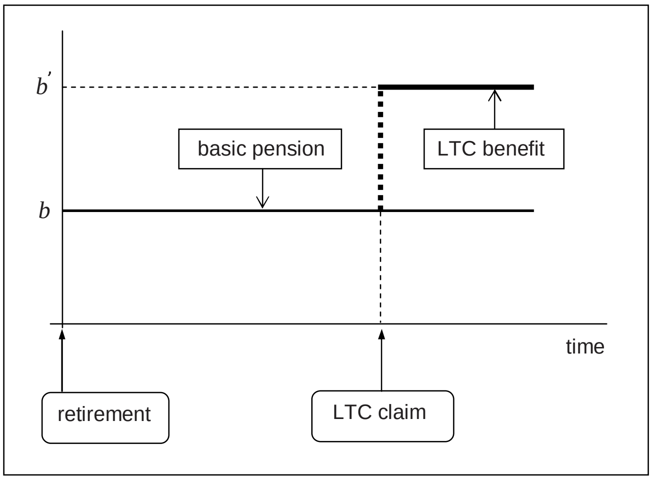

Figure 10.3: The life care pension

now defined by the health conditions of the insured, the trigger being given by the LTC claim. Of course, the different trigger definition is reflected on the set of assumptions needed to assess the benefits.

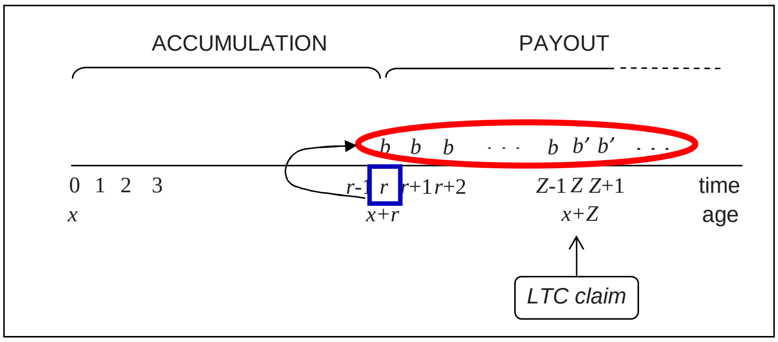

Figure 10.4: The life care pension: a possible outcome

Remark 10.3 The sequence of periodic payments in Fig. 10.4 is similar to the sequence in the middle schema of Fig. 10.2. Nonetheless, it is worth noting what follows.

- The payment of the deferred life annuity in Fig. 10.2 starts at an old age (say 80, or 85), usually well after the retirement age, whereas the basic pension in Fig. 10.4 typically starts at the retirement age.
- The amount b of the pension benefit (Fig. 10.4) is commonly higher than the one in the package, whose main role is to mitigate the prevailing risk features of a stand-alone LTCI cover.

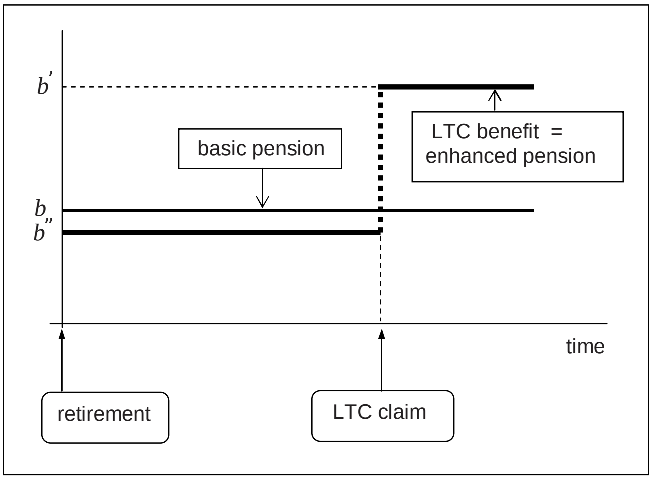

Figure 10.5: The enhanced pension

The *enhanced pension* is a particular life care pension in which the uplift is financed by a reduction (with respect to the basic pension b) of the benefit paid while the policyholder is healthy. Thus, for a given single premium amount, the reduced benefit b'' is paid out as long as the retiree is healthy, while the uplifted benefit b' will be paid in the case of LTC claim (of course, $b'' < b < b'$). See Fig. 10.5.

We note that a life care pension (an enhanced pension, in particular) provides protection against the risk of outliving the retiree's assets, and, at the same time, against the extra-risk originated by senescent disability.

The life care structure, implemented as an enhanced pension, is frequently offered as an option in occupational pension plans and, more generally, in employees benefit packages. It is worth stressing that, in order to limit adverse selection, possible conditions are as follows:

- the option can be exercised not later than a stated date before retirement (say, one year);
- a waiting period, starting from retirement time, can be applied; during this period, the LTC cover is not yet operating.

Remark 10.4 It should be stressed that, when a life care pension or a LTC annuity is involved, a specific type of aggregate longevity risk is taken by the annuity provider, concerning the lifetimes of elderly people claiming for LTC. Various theories concerning the relation between trend in expected total lifetime and trend in expected healthy lifetime have been proposed.

Chapter 11

Suggestions for further reading

An extensive literature deals with technical and financial problems related to life annuities (and life insurance). As regards actuarial issues, see, for example, the following textbooks: Bowers et al. (1997), Dickson et al. (2013), Gerber (1995), Norberg (2002) and Olivieri and Pitacco (2015); an advanced mathematical approach is adopted by Koller (2012). A wide range of life insurance and life annuity products are described by Black and Skipper (2000), where also health insurance (and, in particular, long-term care insurance) is addressed.

Advantages and disadvantages of traditional life annuities are discussed by Milevsky (2005); the idea of longevity insurance with a deductible is then proposed, together with the relevant implementation, which leads to the ALDA (Advanced Life Delayed Annuity) product. On this issue, see also Gong and Webb (2010), and Stephenson (1978). Huang et al. (2009) generalize the idea of longevity insurance suggesting the design of an insurance product, i.e. the RCLA (Ruin Contingent Life Annuity), which generates a life annuity in the case of exhaustion of the (non-annuitized) fund because of poor investment performance or very long lifetime of the retiree. The life annuity as a solution (from the individual perspective) to the longevity risk is widely discussed by Wadsworth et al. (2001) and Swiss RE (2007). A survey of annuity pricing is provided by Cannon and Tonks (2006).

Guarantees and options in life annuities and life insurance products are addressed, in particular, by Gatzert (2009), Hardy (2004) and Pitacco (2012). The paper by Boyle and Hardy (2003) focuses on the impact of the GAO, that is the Guaranteed Annuity Option. Guarantee structures in life annuities constitute the main topic of the paper by Pitacco (2016a), various sections of which have been used in the present Lecture Notes.

Variable annuities are investment products which, thanks to several options that can be exercised by the policyholder, can provide several guarantees involving the accumulation period as well as the post-retirement period. The interested reader can refer, for example, to Kalberer and Ravindran (2009) and Ledlie et al. (2008).

In the more general framework of post-retirement income, we first cite the books by Milevsky (2006, 2013), which provide an in-depth analysis of possible choices for the income construction and the specific role of life annuities in this context; the interested reader should in particular refer to Milevsky (2013) for a detailed literature review. Several post-retirement products are described by Rocha et al. (2011). Shapiro (2010), while discussing post-retirement financial strategies, also provides an extensive literature review.

Annutization strategies and, in particular, the possible delay in annuitization, are focussed by several papers and reports. The reader can refer, for example to Blake et al. (2003), Horneff et al. (2008), Milevsky and Young (2002), and Milevsky (2004). The features of life annuities are compared to those of self-annuitization by Post and Schmeiser (2005).

Life annuities under a historical perspective are addressed by Kopf (1926), Poterba (1997), Milevsky (2013) and, in the framework of actuarial science, by Haberman (1996). The early history of life insurance mathematics is focused by Hald (1987).

As regards the longevity risk originated by the uncertainty in future mortality trend, and the related management issues, the reader can refer to Pitacco et al. (2009) and references therein. Theories about future trends in healthy life and senescent disability are briefly addressed by Pitacco (2014); the interested reader can refer to the bibliography therein.

The longevity guarantee, that is the annuity provider's obligation to pay a lifelong annuity whatever the length of the individual lifetimes may be, is the most significant feature of life annuity products from the annuitant's point of view, and, at the same time, a challenge from the perspective of the annuity provider because of the uncertainty in future mortality trend. This guarantee can be weakened by linking the annuity benefit to some longevity index. Longevity-linked life annuities are widely discussed in the actuarial literature. Although this topic is beyond the scope of the present paper, we cite some recent contributions which should be considered by the interested reader: Denuit et al. (2011), Goldsticker (2007), Maurer et al. (2013), Lüty et al. (2001), Piggott et al. (2005), Richter and Weber (2011), Sherris and Qiao (2013), van de Ven and Weale (2008).

Risk classification in life insurance and life annuities is addressed in many books and papers; a compact review, together with an extensive reference list, is provided by Haberman and Olivieri (2014). The impact of risk classification on the structure of life annuity portfolios is dealt with by Gatzert et al. (2012), Hoermann and Russ (2008) and Olivieri and Pitacco (2016). An extensive literature focuses on the impact of heterogeneity due to unobservable risk factors, usually summarized by the individual "frailty", on the results of a life annuity portfolio. For a detailed bibliography, the reader can refer to Pitacco (2016b), where relations between mortality at high ages and frailty are also addressed.

Underwritten life annuities, or special-rate life annuities, are described in various papers and technical reports: see, in particular Ainslie (2000), Drinkwater et al. (2006), Ridsdale (2012) and Rinke (2002). The article by Edwards (2008) is specifically devoted to life annuity rating based on postcodes. The use of postcodes allows to express the age-pattern of mortality as a function of the annuitant's social class and geographic location of the housing. Socio-geographic variations in mortality are analyzed, for example, by Howse et al. (2011).

The unisex rating principle in insurance products and pensions is addressed by recent papers. See, for example: Chan (2014), Curry and O'Connell (2004), O'Brien

(2013) and Oxera Consulting (2010).

An extensive discussion of the concepts of mutuality and solidarity can be found in Wilkie (1997) (where some terms, however, are used with a meaning different from that adopted in the present Lecture Notes).

Tontine annuities are described, in particular, in the book by Milevsky (2013), where an extensive bibliographic list is also provided. Annuitization decisions, and in particular the choice between a tontine annuity and a conventional life annuity, are addressed by Milevsky (2014), focussing on a fund raising arrangement designed by the English government in the 17th century. Implementations of the tontine scheme have recently been proposed by Chen et al. (2017), Milevsky and Salisbury (2015, 2016), Sabin (2010), and Weinert and Gründl (2016).

Combining a life annuity and a long-term care insurance (LTCI) provides the annuitant with protection against the (individual) longevity risk, i.e. the risk of outliving his/her resources available at retirement, and, at the same time, the extra-risk originated by senescent disability. This subject is extensively dealt with in the book by Warshawsky (2012), where also "progressive" annuitization schemes are considered, i.e. combinations of income drawdown and annuitization. Other interesting contributions on packaging life annuity and LTCI benefits have been provided, for example, by Brown and Warshawsky (2013), Murtaugh et al. (2001), Warshawsky (2007), and Zhou-Richter and Gründl (2011). Merging LTCI and lifetime-related benefits, in particular life annuities, is also discussed in Pitacco (2014). The actuarial structure of LTCI products (both stand-alone as well as including lifetime-related benefits) is described by Haberman and Pitacco (1999). Numerical results of a sensitivity analysis of premiums for LTCI combined products with respect to mortality and disablement assumptions are presented and discussed in Pitacco (2016c).

The low propensity to annuitize the assets available at retirement is in contrast with the well known Yaari's theorem (see Yaari (1965)). The so called "annuity puzzle" has widely been analyzed in the economic literature, in the framework of the life-cycle model of saving and consumption, with special focus on the "worth" of life annuities. See, for example, Benartzi et al. (2011), Brown et al. (2016, 2017), Davidoff et al. (2005), Lockwood (2012), Maurer et al. (2013), Mitchell et al. (1999), Peijnenburg et al. (2016) and Schreiber and Weber (2016). See also Milevsky (2013), and the numerous references therein. The paper by Chen et al. (2016) focusses on the demand for life annuities in the framework of the "cumulative prospect theory".

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Remark

> Where links are provided, they were active as of the time these Notes were completed but may have been updated since then.

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Index

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