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Abstract

The notional defined contribution pension scheme combines pay-as-you-go financing and a defined contribution pension formula. The return on contributions is based on an index set by law, such as the growth rate of GDP, average wages, or contribution payments. The volatility of this return compromises the system's pension adequacy and therefore guarantees may be needed. Here we provide a minimum return guarantee to the pension contributions. The price is calculated in a utility indifference framework. We obtain a closed-form solution for a general dependence structure with exponential preferences and in presence of stochastic short interest rates.

Keywords: public pension, pay-as-you-go, option pricing, incomplete markets, exponential utility

1 Introduction

A notional defined contribution (NDC) model is a pay-as-you-go (PAYG) financed system that deliberately mimics a financial defined contribution (FDC) pension scheme, that is, the pension depends on both contributions and its returns (Palmer 2006). The contributions earn a notional return that reflects the financial health of the system. The account balance is called notional because it is used only for record keeping, that is the system does not invest funds in financial markets as the system is pay-as-you-go financed. Nevertheless, when an individual reaches retirement age, the virtual accumulated capital is converted into an annuity that considers the life expectancy of the cohort, indexation and technical interest rate. This notional return is based on an index set by law, such as the growth rate of GDP, average wages, or the covered wage bill.

There are two main difference between NDCs and FDCs , first, the internal rate of return depends on the productivity and labour force growth, as well as the covered wage bill and pension expenditures, while FDCs rely on a financial market rate of return. Second, the only financial saving occurs in the form of a buffer fund, while FDCs are completely prefunded (Holzmann and Palmer 2006). Furthermore, the notional return is usually lower than the financial return, especially when the economy is in a dynamically efficient state (Knell 2010), which may increase the attractiveness of FDCs¹. However, NDCs “do not involve the transition cost associated with introducing an FDC scheme where some form of pay-as-you-go arrangement already exists” as stated in Holzmann and Palmer (2006). One of the shortcomings of NDC, which is shared with financial defined contributions, is that individuals are subject to some risks they were not exposed to in a defined benefit pension scheme. In particular, they are vulnerable to the risk that the return on contributions is lower than expected, which will negatively affect their pension adequacy.

Pennacchi (1999) proposes as a solution to offer a minimum return guarantee to each contribution made to the first pillar. He prices these guarantees for a funded public defined contribution scheme, like the ones developed in Chile, Uruguay or Colombia². The underlying asset can be fully hedged and he uses the Black & Scholes formula to price the guarantees (Black and Scholes 1973). His approach is based on the pioneering works on the valuation of equity-linked minimum return guarantees of Brennan and Schwartz (1976) and Boyle and Schwartz (1977) when the short interest rates are deterministic and Merton (1973), Heath et al. (1992) and Amin and Jarrow (1992) when the short interest rate is stochastic.

However, the underlying rate of return in notional defined contribution schemes is based on an index which is not traded. We therefore have to find the price of an option written on a nontraded asset in an incomplete markets setting. There are different ways to price in incomplete markets due to the non-uniqueness of the martingale pricing measure. The academic literature offers different approaches to solve this problem. A first approach is to suppose that all derivatives must have the same market price of risk in order to ensure internal consistency in the model (Björk 2004). The price is then the present value of the expected value under a ‘risk-neutral’ measure that depends on the specific market price of risk, which is usually chosen by the market.

¹See Matsen and Thøgersen (2004), De Menil et al. (2006) and Persson (2002) for a discussion on the benefits of a pay-as-you-go over prefunded pension schemes.

²See Queisser (1995) for a discussion on pension reforms in Latin America.

Another approach is the one presented in Cochrane and Saá-Requejo (2000). They relax the no-arbitrage assumption in order to derive tight bounds on prices based on the assumption that investors prefer assets with high Sharpe ratios. They use this setting to calculate bounds on options written on nontraded assets and find that common prices fit within these bounds. A slightly modified version of this setting is used in Floroiu and Pelsser (2013) to obtain closed-form solutions for options in incomplete markets.

A third approach is to use the utility-indifference pricing framework. It consists on finding the price of risks which can't be hedged by incorporating the investor's or issuer's attitude towards the unhedgeable risk. The idea is to price the option on the untraded asset by using as a proxy a traded asset which is correlated to it. This approach has been extensively used in the literature since its introduction by Hodges and Neuberger (1989). They used this approach to value European calls in the presence of transaction costs.

Musiela and Zariphopoulou (2004) and Zariphopoulou (2001) developed an intuitive framework where they obtain closed-form formulas for prices written on nontraded assets in a market with lognormal dynamics. They do their study when risk preferences are exponential, which ensures that the pricing measure is independent of the risk preferences and that it has the minimal entropy with respect to the historical measure. In the same line, Henderson (2002) uses utility maximization and duality methods and obtains prices for CRRA and exponential preferences. She finds that the prices are not highly dependent on the utility chosen when the risk aversion coefficients match locally but that they show a different behaviour when the risk aversion is close to zero.

Rouge and El Karoui (2000) obtain the price for a claim in the exponential utility case and relate the price, modelled as a quadratic backward SDE, with minimal entropy. However, these papers assume that the short interest rate is deterministic. This hypothesis can be unrealistic when studying claims of a long-term nature. Young (2004) studies these prices when the short interest rate is stochastic and has an affine term structure. She uses this theoretical framework to price catastrophe risk bonds and equity-indexed term life insurance.

Other approaches include finding a super-replicating portfolio (see Cvitanic et al. (1999)) with a pay-off which is equal or higher than the payoff of the derivative in any state of the world or pricing via coherent risk measures (see Artzner et al. (1999)). Here they use risk measures to study market and nonmarket risks without the complete markets assumption.

This paper applies the theory of utility indifference pricing in the particular case of a public pension system where the nontraded asset is driven by two distinct noises. One is associated to the mean wage and the other to the working population. These two risks are correlated to each other, as well as to all risks in the market. In this sense we generalise the setting of Musiela and Zariphopoulou (2004) and Henderson (2002) and apply it to public unfunded pension schemes. Furthermore, we present the obtained price as an intermediary price between an insurance premium and a complete-market option price. When the nontraded asset is totally uncorrelated to the market assets we find that our price is a zero-utility exponential insurance premium (Denuit 1999). When the nontraded asset is completely correlated with the market we obtain the same expression as in Black and Scholes (1973) when the short interest rate is deterministic and Amin and Jarrow (1992) when the short rate is stochastic.

The paper is organized as follows. Section 2 presents the pension system, the financial market and the unhedgeable risks which will be priced. Details are given about the derivation of the pricing measure. In Section 3 the closed-form price for the option sought is developed and different particular cases are presented. Section 4 shows some numerical illustrations comparing our prices to the complete markets setting and the independent case. Section 5 and four appendices conclude the paper.

2 The Model

2.1 The pension system

The pension scheme is described for a representative individual participating in the pension system aged x at time t . She participates in the compulsory public state pension system and pays a fixed proportion $\pi \in (0, 1)$ on her age and time dependent income $W(x, t)$ every year. The total individual contribution at time t is then $C(x, t)$ and is compounded until retirement T . The return is based on a nontraded but observable index such as the growth rate of GDP, average wages, or contribution payments. This observable but unhedgeable rate is commonly known as notional rate in defined contribution pay-as-you-go financed pension systems (Palmer 2006). This index depends on risks such as population's and wage's evolution, or productivity. Here we suppose that the nontraded asset is equal to the covered wage bill, which is the aggregate contributions paid to the pension scheme. The notional return is then the covered wage bill's rate of increase. This rate is known as the 'natural rate' of the NDC scheme (Valdés-Prieto (2000) and Börsch-Supan (2006)) or the 'biological rate' of the economy (Samuelson 1958). The covered wage bill at time t will be denoted by $Y(t)$ ³.

The main shortcoming of defined contribution pension schemes is that they transfer too much risk towards the participants. Pension benefits at retirement depend highly on the return on contributions in absence of guarantees. Therefore, in our model the pension provider (government from now on) provides the individuals with a minimum return guarantee in order to increase their pension adequacy. At time of retirement the value of the individual contribution made at time t will be the maximum of the following expressions:

- $K = (1 + i_G)^{T-t}$
- $\frac{Y(T)}{Y(t)}$

where i_G is the constant guaranteed yearly rate of return and $\frac{Y(T)}{Y(t)}$ is the total accumulated return between time t and time of retirement T . The value at retirement for a contribution made at time t (Brennan and Schwartz 1976), when mortality prior to retirement is not taken into account, is then:

³We use the notation $Y(t)$ instead of $C(t)$ for the total contributions, in order to ease the comparison with the works of Zariphopoulou (2001) and Musiela and Zariphopoulou (2004).

$$V(T, t) = C(x, t) \max \left(K, \frac{Y(T)}{Y(t)} \right) = C(x, t) \left[\frac{Y(T)}{Y(t)} + \underbrace{\left(K - \frac{Y(T)}{Y(t)} \right)^+}_{\text{Put Option}} \right] \quad (2.1)$$

The government, which provides the guarantee, has to protect itself from the potential losses when the guarantee is triggered. Approaches include purchasing a put option on the nontraded asset. However, it is unlikely that this option is traded in the market. Therefore, the government has to support the risk of the guarantee himself. Mathematically, the government has to price the following put option:

$$\left(K - \frac{Y(T)}{Y(t)} \right)^+ \quad (2.2)$$

Each contribution made during the participant's career has a guarantee associated to it. These guarantees have the common feature that they trigger (or not) at the same moment of time, namely at time of retirement T ⁴.

The remaining section presents the financial assets in the market as well as the processes that drive the covered wage bill. The nontraded asset, on which an European option is written, is observable at all times and corresponds to the pay-as-you-go contributions $Y(t)$. Due to the unfunded nature of a pay-as-you-go pension plan we cannot buy or sell shares of the underlying asset. Therefore, we work in an incomplete market setting as we cannot hedge the risk related to the nontraded asset.

2.2 The financial assets

a) The cash asset is represented as follows:

$$dS_0(s) = r(s)S_0(s)ds, s \geq t \quad (2.3)$$

where $r(s)$ is the short-term risk free rate of return which can be deterministic or stochastic.

b) The short-term stochastic interest rate is driven by the Vasicek term structure (Vasicek 1977) and has the following expression under the real probability measure \mathcal{P} :

$$dr(s) = a(b - r(s))ds + \sigma_r dB_r(s), s \geq t \quad (2.4)$$

⁴If the pension scheme is pre-funded and the return is fully hedged then unique price formulas can be used.

where a, b and σ_r are positive constants. The stochastic interest rate has the following expression under the risk-neutral measure \mathcal{Q} :

$$dr(s) = a \left(\frac{b^*}{a} - r(s) \right) ds + \sigma_r d\tilde{B}_r(s), s \geq t \quad (2.5)$$

with $d\tilde{B}_r(s) = dB_r(s) + q_s ds$, where q_s is the market price of interest risk. This parameter is assumed to be constant and is represented by $q_t = \frac{ab-b^*}{\sigma_r} = q$.

c) The zero-coupon bond from time s to maturity T , Vasicek (1977), is represented as follows:

$$ZC(s, T) = \mathbb{E} \left[e^{-\int_s^T r(u) du} \right] = A(s, T) e^{-B(s, T)r(s)}, s \geq t \quad (2.6)$$

where

$$A(s, T) = \exp \left\{ \left(b - \frac{\sigma_r^2}{2a^2} \right) [B(s, T) - T + s] - \frac{\sigma_r^2}{4a} B(s, T)^2 \right\} \quad (2.7)$$

$$B(s, T) = \frac{1 - e^{-a(T-s)}}{a} \quad (2.8)$$

This zero-coupon bond with maturity T satisfies the following stochastic differential equation (SDE) under the \mathcal{P} -dynamics:

$$dZC(s, T) = (r(s) - q\sigma_r B(s, T))ZC(s, T)ds - \sigma_r B(s, T)ZC(s, T)dB_r(s) \quad (2.9)$$

The SDE under the \mathcal{Q} -dynamics becomes:

$$dZC(s, T) = r(s)ZC(s, T)ds - \sigma_r B(s, T)ZC(s, T)d\tilde{B}_r(s) \quad (2.10)$$

where $\sigma_r B(s, T)$ is denoted as $\sigma(s, T)$ from now on for simplicity.

d) The price process of the stock at time s is a solution of the following SDE under the real probability measure \mathcal{P} :

$$dS(s) = (r(s) + \lambda_S \sigma_S)S(s)ds + \sigma_S S(s)dB_S(s) \quad (2.11)$$

where $\lambda_S \sigma_S$ is the risk premium, which is assumed strictly positive to illustrate that the return on stocks is higher than the short interest rate of return. The volatility of the

stock is given by σ_S . Finally, it is assumed that both parameters are constant. The SDE under the \mathcal{Q} -dynamics is represented as follows:

$$dS(s) = r(s)S(s)ds + \sigma_S S(s)d\tilde{B}_S(s) \quad (2.12)$$

with $d\tilde{B}_S(s) = dB_S(s) + \lambda_S ds$.

The vector (B_S, B_r) is a two-dimensional Brownian motion. The first one is associated to the risky asset, while the second one corresponds to the short term interest risk. These two Brownian motions are defined on a probability space (Ω, \mathcal{F}, P) and the filtration \mathcal{F}_s is the one generated by $B_r(u)$ and $B_S(u)$ for $0 \leq u \leq s$. We suppose furthermore that these two Brownian motions are correlated, i.e., $\mathbb{E}[B_r(s)B_S(s)] = \rho_{r,S} \cdot s$.

2.3 The liabilities

The liability side of our problem consists on the contribution $C(x, t)$ paid to the pension scheme and the minimum return guarantee expressed in terms of a put option. The minimum return is based on the index $Y(s)$ that represents the covered wage bill of the pension scheme. It depends on the total working population and the average wage in the economy.

a) The total covered wage bill received by the government is represented as follows:

$$Y(s) = \pi P(s)W(s) \quad (2.13)$$

where

π is the constant contribution rate to the public pension system.

$P(s)$ is the total working population.

$W(s)$ is the mean wage for the working population.

The working population at time t under the \mathcal{P} -dynamics is the solution of the following stochastic differential equation:

$$dP(s) = R \cdot P(s)ds + \sigma_P P(s)dB_P(s) \quad (2.14)$$

While the mean wage of the working population under the \mathcal{P} -dynamics is the solution of:

$$dW(s) = \gamma \cdot W(s)ds + \sigma_W W(s)dB_W(s) \quad (2.15)$$

The parameters R and γ (resp. σ_P and σ_W) are the percentage drift (resp. the volatility) of the working population and mean wage, which are supposed to be constant. The vector

(B_W, B_P) is a two-dimensional Brownian motion. The first one is associated to the wage risk, while the second one corresponds to the population risk. We suppose furthermore that mean wage and population are correlated, i.e., $E[B_W(s)B_P(s)] = \rho_{W,P} \cdot s$. The stochastic differential equation associated to the contribution (2.13) is then given by the Ito's lemma:

$$dY(s) = \underbrace{(\gamma + R + \rho_{W,P}\sigma_W\sigma_P)}_{\mu(Y(s),s)} Y(s) ds + \underbrace{\sigma_W Y(s)}_{\sigma_1(Y(s),s)} dB_W(s) + \underbrace{\sigma_P Y(s)}_{\sigma_2(Y(s),s)} dB_P(s) \quad (2.16)$$

with $Y(t) = y \in \mathcal{R}$ and $s \in [t, T]$. The risks associated to this two-dimensional Brownian motion can't be hedged in the market, which causes the market to be incomplete. However, these Brownian motions are correlated to those presented earlier which are traded. In particular, we have:

$$\begin{aligned} E[B_r(s)B_W(s)] &= \rho_{r,W} \cdot t; E[B_r(s)B_P(s)] = \rho_{r,P} \cdot t \\ E[B_S(s)B_W(s)] &= \rho_{S,W} \cdot t; E[B_S(s)B_P(s)] = \rho_{S,P} \cdot t \end{aligned}$$

with $\rho_{r,W}, \rho_{r,P}, \rho_{S,W}$ and $\rho_{S,P} \in [-1, 1]$.

b) The European claim on the nontraded asset $Y(t)$ is a function of the accumulated return between t , which represents the time when the contribution was made by the individual aged x , and T , which is the time when the individual retires and the maturity of the option. This function is denoted by $G = g\left(\frac{Y(T)}{Y(t)}\right)$ and is exercised at maturity T . For example, in the case of a put option we have $g\left(\frac{Y(T)}{Y(t)}\right) = \left(K - \frac{Y(T)}{Y(t)}\right)^+$.

2.4 The pricing measure

The following proposition presents the expression of the forward measure \mathcal{Q}^T which is used to price the European option G on the untraded asset $Y(t)$. The expression of the covered wage bill under the forward measure is also derived to explicit the drift in the pricing environment. This proposition is needed in the proof of Theorem 2 when the Feynman-Kac representation Theorem is used to obtain the expression of the value function in the presence of the claim.

Proposition 1. *a) The minimal entropy pricing measure \mathcal{Q}^T is given by the arbitrage free forward measure:*

$$\begin{aligned} \eta(T) &= \exp\left(-\frac{1}{2} \int_t^T (q + \sigma_r B(s, T))^2 ds - \frac{1}{2} \int_t^T \left(\frac{\lambda_S - \rho_{r,S}q}{\sqrt{1 - \rho_{r,S}^2}}\right)^2 ds\right) \\ &\times \exp\left(-\int_t^T (q + \sigma_r B(s, T)) dB_r(s) - \int_t^T \left(\frac{\lambda_S - \rho_{r,S}q}{\sqrt{1 - \rho_{r,S}^2}}\right)^2 dZ_2(ss)\right) = \frac{d\mathcal{Q}^T}{d\mathcal{P}} \end{aligned} \quad (2.17)$$

where $Z_2(s)$ is a Brownian motion, independent of $B_r(s)$ resulting from the Cholesky decomposition: $dB_S(s) = \rho_{r,S}dB_r(s) + \sqrt{1 - \rho_{r,S}^2}dZ_2(s)$.

b) The expression of the nontraded asset $Y(s)$ (2.16), for $s \in [t, T]$, under the measure \mathcal{Q}^T (2.17) is represented as follows:

$$\begin{aligned} dY(s) = & \left(\mu(Y(s), s) - \sigma(t, T)A_r - \frac{\lambda_S(A_S - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_S)}{1 - \rho_{r,S}^2} \right) dt \\ & + A_r d\tilde{B}_r(s) + \frac{A_S - \rho_{r,S}A_r}{\sqrt{1 - \rho_{r,S}^2}} d\tilde{Z}_2(s) + \sigma_2(Y(s), s)L_{4,4}dZ_4(s) \\ & + (\sigma_1(Y(s), s)L_{3,3} + \sigma_2(Y(s), s)L_{4,3}) dZ_3(s) \end{aligned} \quad (2.18)$$

where $Z_3(s)$ and $Z_4(s)$ are two Brownian motions independent of $B_r(s)$ and $Z_2(s)$ issued from the Cholesky decomposition and $L_{3,3}$, $L_{4,3}$ and $L_{4,4}$ are components of the Cholesky decomposition matrix L . Finally, the expressions of A_r and A_S are given by:

$$\begin{aligned} A_r &= \rho_{r,W}\sigma_1 + \rho_{r,P}\sigma_2 \\ A_S &= \rho_{S,W}\sigma_1 + \rho_{S,P}\sigma_2 \end{aligned} \quad (2.19)$$

Proof. See Appendix A.1. □

2.5 The utility indifference framework

This subsection presents the utility indifference framework in which the European claim G on the nontraded covered wage bill $Y(t)$ is priced. The self-financing market portfolio used in our market setting, as well as the definition of the price of the claim are presented. The financial portfolio is used in this framework for two different things: it is composed by assets that are correlated to the nontraded assets, allowing for partial hedging, and it is a part of the maximization problem in presence and absence of the claim.

In this market we have three financial assets: the riskless cash asset $S_0(t)$, the zero-coupon bond $ZC(t, T)$ as well as the risky asset $S(t)$. The writer has an initial capital x and chooses dynamically his portfolio allocations $\theta_0(s)$, $\theta_B(s)$ and $\theta_S(s)$, respectively for the riskless asset, the zero-coupon bond with fixed maturity T and the risky asset for $t \leq s \leq T$. We suppose that there are no intermediate cash-flows corresponding to consumption or contributions and that the portfolio on the traded assets is self-financing. The SDE of the wealth is represented as follows:

$$\begin{aligned} \frac{dX(s)}{X(s)} &= \theta_0(s) \frac{dS_0(s)}{S_0(s)} + \theta_B(s) \frac{dZC(s, T)}{ZC(s, T)} + \theta_S(s) \frac{dS(s)}{S(s)} \\ &= (r(s) + \theta_S(s)\lambda_S\sigma_S - \theta_B(s)\sigma(s, T))ds + \theta_S(s)\sigma_S dB_S(s) - \theta_B(s)\sigma(s, T)dB_r(s) \end{aligned} \quad (2.20)$$

with $X(t) = x$, $0 \leq t \leq s \leq T$. We assume that $\theta_S(s)$ and $\theta_B(s)$ are \mathcal{F}_t progressively measurable and satisfy the integrability condition $E \left[\int_t^T (X(s)\theta_i(s))^2 ds \right] < \infty$ for $i = P, S$. The set of admissible policies which satisfy this condition is denoted by A .

The government risk preferences are modelled with the exponential utility function:

$$U(x) = -e^{-\varphi x}, \varphi > 0 \quad (2.21)$$

where φ is the risk aversion coefficient of the government. It is not straightforward to conclude whether the government's risk aversion coefficient should be low or high. Pratt (1964) states that "it seems likely that many decision makers would feel they ought to pay less for insurance against a given risk the greater their assets", that is, risk aversion should decrease with wealth (Campbell and Viceira 2002). However, a low risk aversion implies that the government invests higher proportions in risky assets, which is not necessarily the case in practice. For instance, Severinson and Stewart (2012) state that the risk level of the government's investments must be low, and OECD (2013) illustrate that the asset allocation of pension funds and public reserve funds in most OECD countries shows a greater preference for bonds. This is due to the government's will to invest prudentially in order to reduce portfolio's volatility. Section 4 studies the impact of very low and very high risk aversion coefficients on the price of the option.

There are various reasons to use the exponential utility in this framework. First, the price is independent of the initial wealth for exponential preferences (Miao and Wang 2007), which is a desirable feature in the context of public pension schemes provided by the government. Second, the exponential utility minimizes the relative entropy between the historical and the pricing measure (Frittelli 2000). Finally, the exponential utility provides a closed-form formula which is tractable, Henderson (2002), whereas the power utility does not and therefore numerical methods have to be used. Henderson (2002) shows also that the 'exponential price' is very close to the 'power price' when the risk aversion coefficients match locally.

The indifference price of the option is deduced by comparing the expected utility of the wealth at maturity in presence and absence of the option as presented in the following definition from Hodges and Neuberger (1989):

Definition 1. *The indifference price of the European claim $G = g \left(\frac{Y(T)}{Y(t)} \right) = \left(K(t) - \frac{Y(T)}{Y(t)} \right)^+$ written on the covered wage bill $Y(t)$ is defined as the function $p = p(x, y, r, t)$, such that the investor is indifferent between optimizing the expected utility of the wealth represented by the financial portfolio when the derivative is not taken into account and optimizing the wealth when the derivative is taken into account, that is the price at writing time t and the claim payoff at maturity T :*

$$V(x, r, t) = u(x + \phi p(x, y, r, t), y, r, t) \quad (2.22)$$

where

$$V(x, r, t) = \sup_A E_P [U(X(T)) | X(t) = x, r(t) = r] \quad (2.23)$$

is the writer's value function when the derivative is not taken into account and

$$\begin{aligned} u(x + \phi p(x, y, r, t), y, r, t) &= \sup_A \mathbb{E}_P [U(X(T) - \phi G) | X(t) = x, Y(t) = y, r(t) = r] \\ &= \sup_A \mathbb{E}_P \left[U(X(T)) h \left(\frac{Y(T)}{Y(t)} \right) | X(t) = x, Y(t) = y, r(t) = r \right] \end{aligned} \quad (2.24)$$

is the writer's value function when the derivative is taken into account, the function h represents the exponential of the claim $e^{\phi\varphi G}$, and ϕ is the units of claims that are received or sold. A $\phi > 0$ represents that the agent sells the claim, that is, she receives the price and pays the claim at maturity, and $\phi < 0$ means that she receives the claim, that is, the agent pays the price and receives the claim at maturity. The claim is written at time t and no trading of the asset is allowed in the remaining time period up to maturity T (European claim).

Remark 1. Note that the expression of $h \left(\frac{Y(T)}{Y(t)} \right) = e^{\phi\varphi G}$ involves the calculation of the expected value of an exponential function. There have to be extra constraints concerning the function G and ϕ in order to have a finite expected value. The price as stated in Definition 1 is finite for bounded functions, for instance, for put options (for all ϕ) and for call options when $\phi = -1$. If we price a short call option ($\phi = 1$) then h would be an unbounded process and the price would be unbounded too.

The following section presents the expression of the price in this market setting.

3 The price of the guarantee

This section presents the intermediate steps needed to derive the price of the European option. First, the expression of $V(x, r, t)$ (2.23) is calculated in Theorem 1. Then the expression of $u(x, y, r, t)$ (2.24) is developed in Theorem 2. Finally, the condition (2.22) is used to deduce the price p in Proposition 2. The expression of the value function is in both cases calculated by means of the Hamilton-Jacobi-Bellman (HJB) equation methodology. We would like to note that in our HJB equation a term $r(t) \cdot x$ appears which makes it impossible to apply the Lipschitz conditions of the usual verification theorems, as they are both unbounded processes. However, Korn and Kraft (2002) give a suitable verification result in their paper which allows to proceed with the usual three-step procedure of the HJB equation in presence of stochastic interest rates. For a detail of the verification result in presence of stochastic interest rates, as well as the verification of the solution we invite the reader to consult Korn and Kraft (2002).

Theorem 1 (Value function and optimal policy in absence of the derivative). *(a) The writer's value function $V(x, r, t)$ (2.23) when the derivative is not taken into account is given by:*

$$V(x, r, t) = -e^{-\frac{\varphi x}{ZC_Q(t, T)} - \frac{1}{2} \int_t^T \left(\frac{\lambda_S^2 + a^2 - 2\rho_{r, S} \lambda_S a}{1 - \rho_{r, S}^2} + \sigma_r^2 B(s, T)^2 + 2q\sigma_r B(s, T) \right) ds} \quad (3.1)$$

where $B(s, T)$ is given by (2.8) and $ZC_Q(t, T)$ is the expression of the zero-coupon bond under the \mathcal{Q} -dynamics.

(b) The optimal policies $\theta_S^(s)$ and $\theta_B^*(s)$ are given in the feedback for $\theta_i^*(X^*(s), r(s), s)$*

$=\theta_i^*(s)$ for $i = B, S$ and $t \leq s \leq T$:

$$\begin{aligned}\theta_S^*(s) &= \frac{(-\lambda_S + \rho_{r,S}q) V_x}{\sigma_S x V_{xx} (1 - \rho_{r,S}^2)} \\ \theta_B^*(s) &= \frac{(q - \rho_{r,S}\lambda_S) V_x + (1 - \rho_{r,S}^2) \sigma_r V_{xr}}{\sigma(t, T) x V_{xx} (1 - \rho_{r,S}^2)}\end{aligned}$$

where $X^*(s)$ is the wealth process given by (2.20) when the optimal policies are used.

Proof. See Appendix A.2. □

The following theorem develops the value function in the presence of the derivative.

Theorem 2 (Value function and optimal policy in presence of the derivative). *(a) The writer's value function $u(x, y, r, t)$ (2.24) when the derivative is taken into account is given by:*

$$\begin{aligned}u(x, y, r, t) &= -e^{-\varphi \frac{x}{\mathbb{Z}C_Q(t, T)} - \frac{1}{2} \int_t^T \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S}\lambda_S q}{1 - \rho_{r,S}^2} + \sigma_r^2 B(s, T)^2 + 2q\sigma_r B(s, T) \right) ds} \\ &\quad \times \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} \mid Y(t) = y \right] \right)^\delta \\ &= V(x, r, t) \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} \mid Y(t) = y \right] \right)^\delta\end{aligned}\tag{3.2}$$

for $(x, y, r, t) \in \mathcal{R} \times \mathcal{R} \times \mathcal{R} \times [t, T]$ with

$$\mathcal{Q}^T(A) = \mathbb{E}_{\mathcal{P}} [\eta(T) I_A] ; A \in \mathcal{F}_T.$$

where $\eta(T)$ is given by (2.17), $Y(t)$ is given by (2.18) and δ is represented as follows:

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2 - \frac{A_S^2 + A_r^2 - 2\rho_{r,S}A_S A_r}{1 - \rho_{r,S}^2}}\tag{3.3}$$

(b) The optimal policies $\theta_B^(s)$ and $\theta_S^*(s)$ are given in the feedback form $\theta_i^*(X^*(s), Y(s), r(s), s) = \theta_i^*(s)$ for $i = P, S$ and $t \leq s \leq T$:*

$$\begin{aligned}\theta_S^*(s) &= \frac{(-\lambda_S + \rho_{r,S}q) u_x + (\rho_{r,S}A_r - A_S) u_{xy}}{\sigma_S x u_{xx} (1 - \rho_{r,S}^2)} \\ \theta_B^*(s) &= \frac{(q - \rho_{r,S}\lambda_S) u_x + (1 - \rho_{r,S}^2) \sigma_r u_{xr} + (A_r - \rho_{r,S}A_S) u_{xy}}{\sigma(t, T) x u_{xx} (1 - \rho_{r,S}^2)}\end{aligned}$$

where $X^*(s)$ is the wealth process given by (2.20) when the optimal policies are used.

Proof. See Appendix A.3. □

Proposition 2 (Price of the European option). *The price of an European option $G = g\left(\frac{Y(T)}{Y(t)}\right)$ in a market where:*

- the cash asset earns a stochastic short-term interest rate given by (2.4),
- the zero-coupon asset's dynamics are given by (2.6),
- the risky asset's dynamics are given by (2.11),
- and the nontraded asset's dynamics are given by (2.16);

is represented as follows:

$$p(x, y, r, t) = ZC_Q(t, T) \frac{\delta}{\varphi} \log \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right) \quad (3.4)$$

given the government's exponential preferences (2.21).

Proof. The price is issued by searching the $p = p(x, y, r, t)$ which makes the equivalence (2.23)=(2.24) hold:

$$\begin{aligned} V(x, r, t) &= V(x + p, r, t) \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right)^\delta \\ e^{\varphi \frac{p}{ZC_Q(t, T)}} &= \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right)^\delta \end{aligned}$$

The price p is then obtained by applying the logarithm to the previous expression and by doing some calculations:

$$p(x, y, r, t) = ZC_Q(t, T) \frac{\delta}{\varphi} \log \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right)$$

□

Remark 2. Note that the price of the derivative doesn't depend on the initial capital x and only on the nontraded asset under the pricing measure, the utility function, its level of risk aversion as well as the correlation structure, therefore $p(x, y, r, t) = p(y, r, t)$. This is a consequence of using the exponential utility. Rouge and El Karoui (2000) note that the initial capital independence may not be desirable in some cases because it is unlikely that wealthier individuals would give the same price to a claim as poorer agents would, as in the case of stock options.

The corollaries that follow give the price of the option in various particular cases, namely: when the short-term interest rate is deterministic, when the population is not correlated to the wage and financial risks, when the nontraded asset is uncorrelated to the financial risks and when the market is complete.

Corollary 1 (Price with deterministic short interest rate). *The price of an European option $G = g\left(\frac{Y(T)}{Y(t)}\right)$ in a market where:*

- the cash asset earns a deterministic short-term rate,
- the risky asset's dynamics are given by (2.11),
- and the nontraded asset is given by the dynamics (2.16);

given the government's exponential preferences (2.21), is represented as the expected value of the utility of the contingent claim under the risk-neutral measure (A.3):

$$p(x, y, r, t) = e^{-\int_t^T r(s)ds} \frac{\delta}{\varphi} \log \left(\mathbb{E}_{\mathcal{Q}} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right) \quad (3.5)$$

with $Y(t)$ represented as follows:

$$dY(t) = (\mu(Y(t), t) - \lambda_S A_S) Y(t) dt + A_S (dB_S(t) + \lambda_S dt) \\ + \sigma_1(Y(t), t) \sqrt{1 - \rho_{S,W}^2} dB_1(t) + \sigma_2(Y(t), t) \sqrt{1 - \rho_{S,P}^2} dB_2(t)$$

where $B_S(t)$ is independent of $B_1(t)$ and $B_2(t)$ (Cholesky decomposition). In this case δ becomes

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2}{\sigma_1^2(1 - \rho_{S,W}^2) + \sigma_2^2(1 - \rho_{S,P}^2) + 2\sigma_1\sigma_2(\rho_{W,P} - \rho_{S,W}\rho_{S,P})} \quad (3.6)$$

Proof. If the short interest rate is deterministic we have $\sigma_r = 0$, $q = 0$ and $\rho_{r,i} = 0$ for $i = S, W, P$. Therefore the zero-coupon bond is equal to the cash asset (2.3). \square

Remark 3. Note that if $r = 0$ and $\sigma_2 = 0$ the same value as in Musiela and Zariphopoulou (2004) is obtained.

Corollary 2 (Price when the population is independent to the wages' and the financial risk). *The price of an European option $G = g\left(\frac{Y(T)}{Y(t)}\right)$ in the market of Proposition 2 when the population is independent from the financial and wage risks is given by:*

$$p(x, y, r, t) = ZC_{\mathcal{Q}}(t, T) \frac{\delta}{\varphi} \log \left(\mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right) \quad (3.7)$$

with $Y(t)$ given by (2.18) when $\rho_{i,P} = 0$ with $i = r, S, P$ and δ becomes

$$\delta = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \left(1 - \frac{\rho_{S,W}^2 + \rho_{r,W}^2 - 2\rho_{r,S}\rho_{S,W}\rho_{r,W}}{1 - \rho_{r,S}^2} \right) + \sigma_2^2} \quad (3.8)$$

Proof. In this case we have that $\rho_{i,P} = 0$ for $i = r, S, W$ therefore $A_r = \rho_{r,W}\sigma_1$, $A_S = \rho_{S,W}\sigma_1$. \square

Corollary 3 (Price when the nontraded asset is uncorrelated to the financial market). *The price of an European option $G = g\left(\frac{Y(T)}{Y(t)}\right)$ in the market of Proposition 2 is given by the exponential premium under the real measure \mathcal{P} :*

$$p(x, y, r, t) = ZC_{\mathcal{Q}}(t, T) \frac{1}{\varphi} \log \left(\mathbb{E}_{\mathcal{P}} \left[e^{\varphi g\left(\frac{Y(T)}{Y(t)}\right)} | Y(t) = y \right] \right) \quad (3.9)$$

with $Y(t)$ given by its original \mathcal{P} -form (2.16) when $\rho_{i,P} = \rho_{i,W} = 0$ with $i = r, S$.

Proof. In this case we have that $\rho_{i,W} = \rho_{i,P} = 0$ for $i = r, S$, therefore $A_r = 0$, $A_S = 0$. The coefficient δ becomes

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2} = 1 \quad (3.10)$$

\square

Remark 4. Note that the price of the derivative in this case collapses to the zero-utility exponential insurance premium, see Denuit (1999).

Remark 5. Corollary 3 presents a special case of the general formula presented in Proposition 2 when the nontraded asset is uncorrelated to the financial market, which is an unrealistic hypothesis. We are aware that the covered wage bill is far from uncorrelated to the financial markets. For instance, Estrella and Mishkin (1998) show that stock prices are useful to predict recessions and Næs et al. (2011) show that there is a strong relation between stock market liquidity and the business cycle. It is straightforward to note that financial crisis would have a big impact on the covered wage bill. However, we have developed the price when the nontraded asset is not correlated to the financial risks in order to compare it with the complete markets price and the general utility indifferent price in the Numerical section 4.

Corollary 4 (Limit to the complete markets setting). *The price of an European option $G = g\left(\frac{Y(T)}{Y(t)}\right)$ (3.4) when the markets are complete tends to the utility-free expectation of G under the forward measure:*

$$p(x, y, r, t) = ZC_Q(t, T)E_{Q^T} \left[g\left(\frac{Y(T)}{Y(t)}\right) | Y(t) = y \right] \quad (3.11)$$

with $Y(t)$ given by (2.18) when $\rho_{S,W} = \rho_{S,P} = 1$ and $\rho_{r,W} = \rho_{r,P} = \rho_{r,S}$.

Proof. See Appendix A.4. □

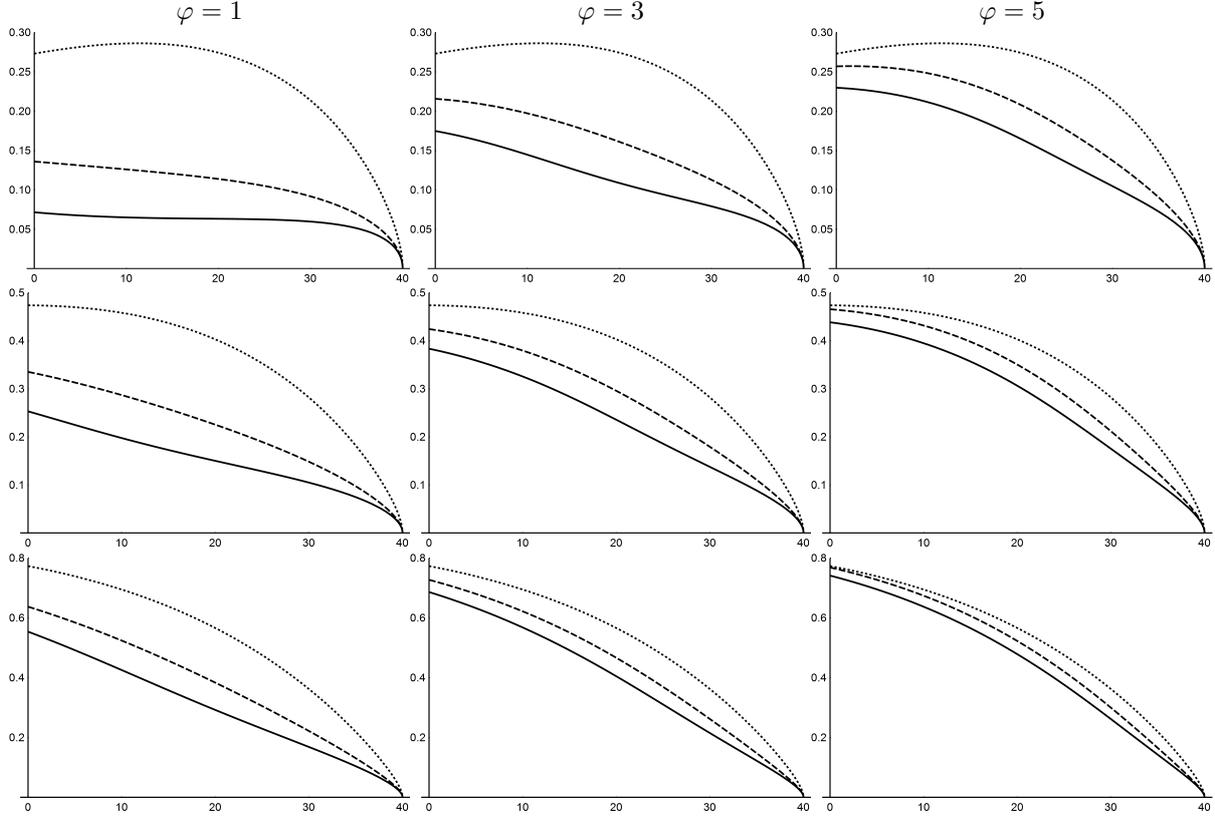
The last corollary shows that when the nontraded asset is fully correlated to the financial risk, hence it is hedgeable, the price that has been obtained in Proposition 2, which is utility and risk aversion dependent, becomes the utility and risk aversion independent price of the option as already presented in Amin and Jarrow (1992) and Brigo and Mercurio (2007).

4 Numerical illustration

This section illustrates the effect of risk aversion, time, guaranteed return and correlation in the price of a put option (2.2). The parameters used are taken from the literature or are consistent with it.

The parameters characterizing the financial assets are taken from Boulier et al. (2001). Therefore the parameters of the short term interest rate and zero-coupon bond are $a = 20\%$, $b = 5\%$, $\sigma_r = 2\%$, $r_0 = 5\%$, $q = 15.28\%$. The risky asset has a market price of risk $\lambda_S = 30\%$ and a volatility of $\sigma_S = 20\%$. The correlation between the risky asset and the short term interest rate is set equal to $\rho_{r,S} = 0.30$. The drift and volatility of the population have been taken from Devolder and Melis (2015) and is $R = 2\%$ and $\sigma_P = 5\%$. We suppose that the wages are less stable than the population and have a higher drift and volatility, i.e., $\gamma = 3\%$ and $\sigma_W = 7\%$. The Wiener processes for wages and population are correlated with $\rho_{W,P} = -0.1$. The negative correlation reflects the fact that cohort size negatively affects earnings (Brunello 2009). The mean one-period notional rate is then

Figure 1: The value of a put option guaranteeing $i_G = 4\%$ (first row), $i_G = 5\%$ (second row) and $i_G = 6\%$ (third row): un-correlated exponential price (continuous line), imperfect correlation exponential price (discontinuous line) and complete markets case (pointed line). *Source:* the authors.



5.054%. We will therefore study the following interest rate guarantees: $i_G = 4\%$, $i_G = 5\%$ and $i_G = 6\%$.

We assume further that the correlation between the population and the markets are $\rho_{r,P} = -0.25$ and $\rho_{S,P} = -0.05$. These correlations are based on the fact that government issued zero-coupon bond are more negatively correlated to demographics than stock returns are (Poterba 2001). Finally we suppose that wage increase is positively correlated to the short term interest rate risk and risky asset risk: $\rho_{r,W} = 0.6$ and $\rho_{S,W} = 0.4$. The correlation matrix (A.4) is thus:

$$\Sigma_B = \begin{pmatrix} 1 & 0.30 & 0.40 & -0.25 \\ 0.30 & 1 & 0.6 & -0.05 \\ 0.40 & 0.6 & 1 & -0.1 \\ -0.25 & -0.05 & -0.1 & 1 \end{pmatrix}$$

Figure 1 compares the prices in three different cases for the same market structure: independent, which we choose to name ‘insurance’ case, totally correlated or hedged, denoted by ‘complete markets’ case, and the case with intermediate correlation structure, denoted by ‘intermediate’ price. The x-axis represents the time when the option is written and the y-axis represents the price for the different cases. Figure 1 shows that the ‘intermediate’ and ‘insurance’ prices are lower than the ‘complete markets’ price for the presented

Figure 2: 3D-Plot showing the evolution of the value of a put option for risk aversions coefficients $\varphi \in (0, 5)$ at different writing times $t \in (0, 40)$ for a guaranteed interest rate of $i_G = 5\%$. *Source:* the authors.

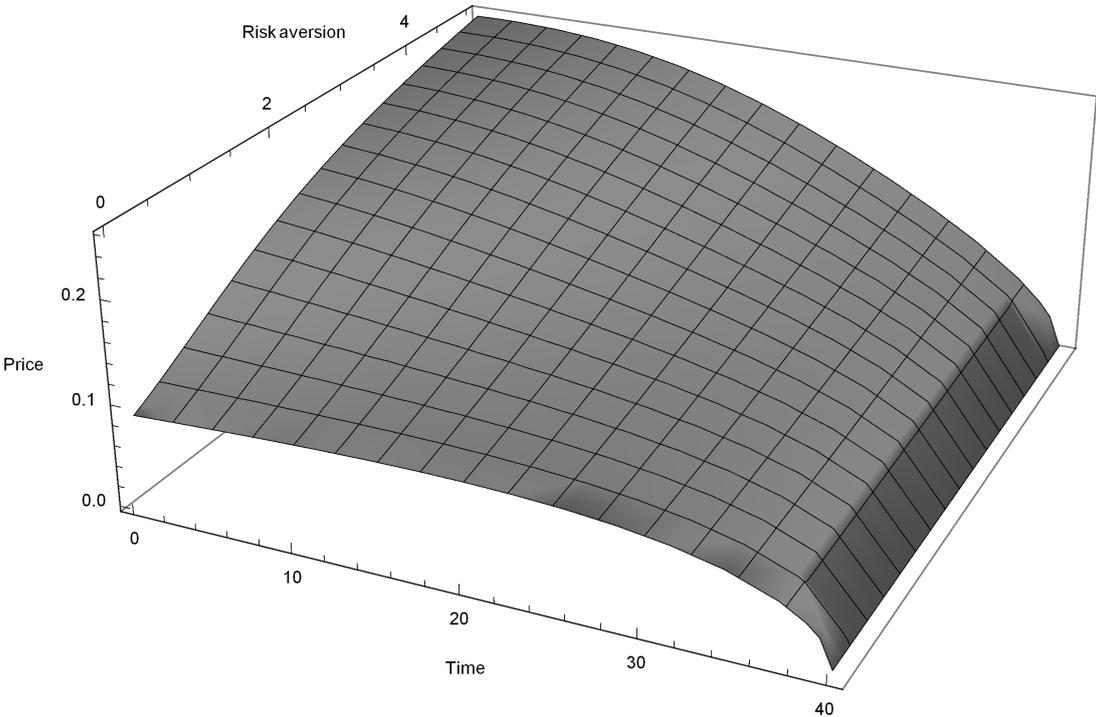


Figure 3: 3D-Plot showing the evolution of the value of a put option for guarantees $i_G \in (3\%, 6\%)$ at different writing times $t \in (0, 40)$ for a risk aversion coefficient $\varphi = 3$. *Source:* the authors.

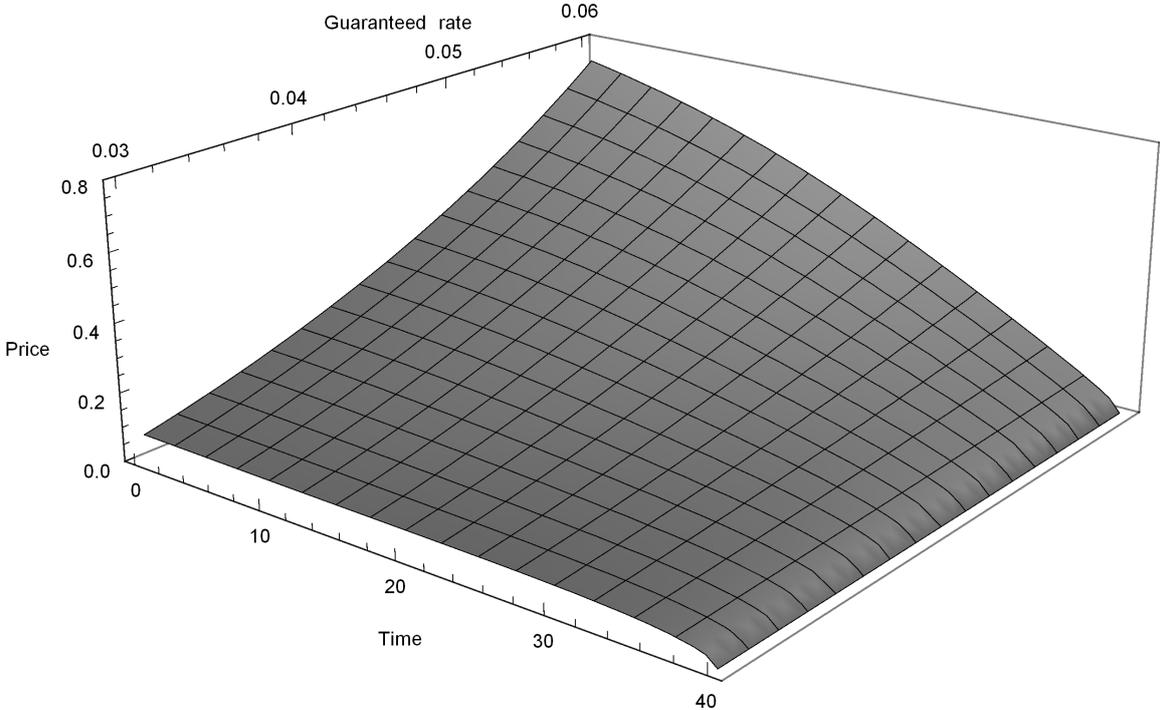


Figure 4: 3D-Plot showing the evolution of the value of a put option for correlation $\rho_{r,W} \in (-1, 1)$, with all other correlations equal to 0, guarantee of $i_G = 4\%$, and with risk aversion coefficient $\varphi = 3$. The second graph shows the evolution of the price according to $\rho_{r,W}$ for options written at $t = 20$ Source: the authors.

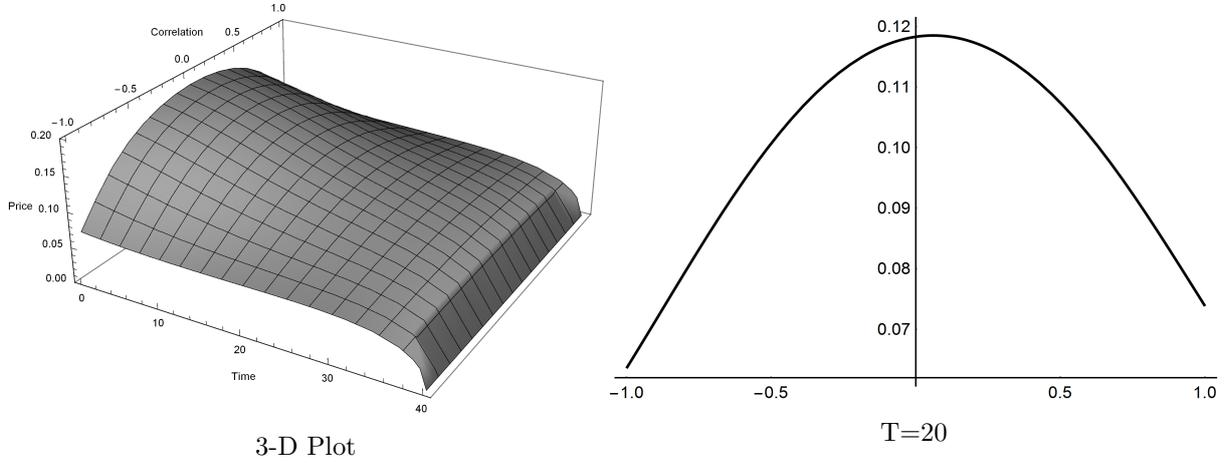


Table 1: Price of the ‘intermediate’ put option for different underwriting times t , risk aversions φ and guaranteed returns i_G . Prices are based on a contribution of 100 €.

	$t = 5$	$t = 15$	$t = 25$	$t = 35$
$i_G = 4\%$				
$\varphi = 0.01$	9.02	9.47	9.12	6.53
$\varphi = 1$	13.12	12.05	10.51	6.96
$\varphi = 3$	20.91	18.09	13.86	7.93
$\varphi = 5$	25.59	23.26	17.60	9.06
$\varphi = 7$	28.59	27.05	21.18	10.32
$\varphi = 10$	31.58	31.10	25.71	12.41
$i_G = 5\%$				
$\varphi = 0.01$	23.05	20.70	16.56	9.14
$\varphi = 1$	31.33	25.78	18.97	9.74
$\varphi = 3$	40.68	34.29	24.09	11.05
$\varphi = 5$	45.39	39.82	28.81	12.52
$\varphi = 7$	48.38	43.65	32.74	14.09
$\varphi = 10$	51.38	47.70	37.38	16.53

correlation structure. It can be stated that the buyer is ready to pay more to be better hedged, which is in line with the literature (Henderson 2002)⁵. We observe as well that price increases with the risk aversion coefficients for the ‘intermediate’ and ‘insurance’ cases and increases with the guaranteed returns for all cases. It is straightforward that the complete markets case doesn’t depend in the risk aversion by construction.

Table 1 shows the ‘intermediate’ prices for a put option with maturity $T = 40$ which is

⁵We have studied other parameters and correlations and obtain similar results. However, in some cases high (resp. low) risk aversion coefficients lead to higher (resp. lower) ‘intermediate’ and ‘insurance’ prices than in the complete markets case, for the same minimum return guarantee i_G .

written at different moments of time t . The prices are studied for two different guaranteed returns (4% and 5%) and different risk aversion coefficients φ . The price is based on a 100 € contribution. For instance, the price for an option written at time $t = 25$ when $\varphi = 3$ and $i_G = 4\%$ is $p_{int} = 13.86$. Therefore, guaranteeing 5% yearly return to a contribution of $C = 100$ € for 15 years costs $C \cdot p_{int} = 13.85$ €.

Figures 2 and 3 show the evolution of the price for different writing times $t \in (0, 40)$, risk aversion coefficients $\varphi \in (0, 5)$ and interest rate guarantees $i_G \in (3\%, 6\%)$. The price increases with the risk aversion coefficient and with the guaranteed return. In other words, issuers who have a higher risk aversion coefficient ask consequently higher prices because they are more averse to the potential losses. In the same line, prices increase with the guaranteed rate because the potential losses are higher for higher guaranteed returns.

Figure 4 shows the evolution of the price when all risks are independent except of the wage risk and the short rate. In this case $\rho_{r,W} \neq 0$ and all others $\rho_{i,j} = 0$. We observe that the correlation has a clear influence on the intermediate price. It seems that the price attains its highest point when it's close to 0, that is, when the nontraded asset is totally uncorrelated to the financial markets.

5 Conclusion

This paper has studied the utility maximization pricing of options on nontraded assets, in the specific case of Notional Defined Contribution (NDC) pension schemes. NDC accounts are designed to be actuarially fairer than defined benefit pension systems, as pension benefits depend on the life expectancy, the contributions made and their return. However, the return risk is borne by the individuals.

We extend the setting of Musiela and Zariphopoulou (2004) and Henderson (2002) when the short-term interest rate is stochastic, the nontraded asset has two distinct noises and the correlations between the financial and the nontraded assets is general. We price minimum return guarantees based on the covered wage bill, in line with the canonical design of NDC accounts (Palmer 2006).

The government provides this minimum return in order to increase the adequacy of the benefits and the attractiveness of the pension scheme. However, providing this guarantee entails a risk that should be priced. The utility indifference framework calculates the price of the guarantee as the one that makes the writer (resp. buyer) of the option indifferent between holding a financial portfolio and holding a financial portfolio plus (minus) the price at time of underwriting, and the portfolio minus (plus) the payoff at maturity.

We obtain a closed-form formula for a general dependence case which is tractable and has sensible properties. The prices increase with risk aversion and with guaranteed returns and are independent of the initial wealth. We have shown also that the obtained price can be seen as an ‘intermediate’ price between a zero-utility exponential premium (Denuit 1999) and a complete markets price (Amin and Jarrow 1992). The zero-utility exponential premium is obtained when the nontraded asset is uncorrelated to the financial markets. On the other hand, the complete markets price is calculated when the nontraded asset is fully correlated to the financial markets.

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A Appendix

A.1 Proof of Proposition 1

a) The forward measure $\frac{dQ^T}{dP}$ is obtained by using the zero-coupon (2.6) as a numeraire:

$$\frac{dQ^T}{dP} = \frac{dQ^T}{dQ} \frac{dQ}{dP} \quad (\text{A.1})$$

with

$$\frac{dQ^T}{dQ} = \frac{\exp\left(-\int_t^T r(u)du\right)}{ZC(0, T)} \quad (\text{A.2})$$

Then, the risk-neutral measure is represented as follows:

$$\frac{dQ}{dP} = \exp\left(-\int_t^T \vartheta^T(s)dB(s) - \frac{1}{2}\int_t^T \|\vartheta(s)\|^2 ds\right) \quad (\text{A.3})$$

The vector $\vartheta(t)$ is given by:

$$\vartheta(t) = \sigma^T(t) \left((\sigma(t)\sigma^T(t))^{-1} (b(t) - r(t)1) \right) = \left(\begin{array}{c} q \\ \frac{\lambda_{S-\rho_{r,S}} q}{\sqrt{1-\rho_{r,S}^2}} \end{array} \right)$$

with

$$B(t) = \begin{pmatrix} B_r(t) \\ Z_2(t) \end{pmatrix}; (b(t) - r(t)1) = \begin{pmatrix} \lambda_S \\ -q\sigma(t, T) \end{pmatrix}$$

$$\sigma(t) = \begin{pmatrix} \rho_{r,S}\sigma_S & \sigma_S\sqrt{1 - \rho_{r,S}^2} \\ -\sigma(t, T) & 0 \end{pmatrix}$$

where $Z_2(t)$ is a Brownian motion independent of $B_r(t)$ which appears in $dS(t)$ (2.11) after a Cholesky decomposition:

$$dS(t) = (r(t) + \lambda_S\sigma_S)S(t)dt + \sigma_S\rho_{r,S}S(t)dB_r(t) + \sigma_S\sqrt{1 - \rho_{r,S}^2}dZ_2(t)$$

We refer the reader to Frittelli (2000) and Miyahara (1996) for the proof of the fact that \mathcal{Q}^T is a martingale measure and minimizes the entropy relative to the historical measure \mathcal{P} .

b) First of all, the detailed Cholesky decomposition of $B_W(t)$ and $B_P(t)$ will be presented. Let $\Sigma_B(t)$ be the variance-covariance matrix of the original Brownian motion vector $B(t) = (B_r(t), B_S(t), B_W(t), B_P(t))$:

$$\Sigma_B = \begin{pmatrix} 1 & \rho_{r,S} & \rho_{r,W} & \rho_{r,P} \\ \rho_{r,S} & 1 & \rho_{S,W} & \rho_{S,P} \\ \rho_{r,W} & \rho_{S,W} & 1 & \rho_{W,P} \\ \rho_{r,P} & \rho_{S,P} & \rho_{W,P} & 1 \end{pmatrix} \quad (\text{A.4})$$

We search lower-triangular matrix L such that $B = L \cdot Z$ where

$$Z(t) = (Z_1(t), Z_2(t), Z_3(t), Z_4(t))$$

is a vector of independent Brownian motions. The matrix L is obtained by means of Cholesky decomposition of Σ_B :

$$L_{j,j} = \sqrt{A_{j,j} - \sum_{k=1}^{j-1} L_{j,k}^2}$$

$$L_{i,j} = \frac{1}{L_{j,j}} \left(A_{i,j} - \sum_{k=1}^{j-1} L_{i,k}L_{j,k} \right)$$

where $A_{i,j}$ is the component of Σ_B of the i^{th} row and j^{th} column. The expression (2.16), omitting the dependence to $Y(t)$ and t for simplicity, becomes then:

$$\begin{aligned}
dY(t) &= \left(\mu - \sigma(t, T)A_r - \frac{\lambda_S(A_S - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_S)}{1 - \rho_{r,S}^2} \right) dt \\
&+ A_r d\tilde{B}_r(t) + \frac{A_S - \rho_{r,S}A_r}{\sqrt{1 - \rho_{r,S}^2}} d\tilde{Z}_2(t) + \sigma_2 L_{4,4} dZ_4(t) + (\sigma_1 L_{3,3} + \sigma_2 L_{4,3}) dZ_3(t)
\end{aligned}$$

where $d\tilde{B}_r(t) = dB_r(t) + (q + \sigma_r B(t, T))dt$ and $d\tilde{Z}_2(t) = dZ_2(t) + \frac{\lambda_S - \rho_{r,S}q}{\sqrt{1 - \rho_{r,S}^2}} dt$ are martingales under \mathcal{Q}^T (2.17). The Brownian motions $dZ_3(t)$ and $dZ_4(t)$, which are orthogonal to the space, are unchanged by law. See Follmer and Schweizer (1991) for details. The expressions for A_r and A_S are those given by (2.19).

A.2 Proof of Theorem 1

The maximum principle applied to the value function $V(x, r, t)$ (2.23) gives the following Hamilton- Jacobi-Bellman (HJB) relation:

$$\begin{aligned}
0 &= V_t + a(b - r(t))V_r + \frac{1}{2}\sigma_r^2 V_{rr} + r(t)xV_x + \max_{\theta(t)} \left\{ \frac{1}{2}\sigma_S^2 x^2 \theta_S(t)^2 V_{xx} \right. \\
&- \theta_B(t)\sigma(t, T)x(qV_x + \sigma_r V_{xr}) + \frac{1}{2}\sigma(t, T)^2 x^2 \theta_B(t)^2 V_{xx} \\
&\left. + \theta_S(t)\sigma_S x (\lambda_S V_x + \rho_{r,S}\sigma_r V_{xr}) - \theta_S(t)\theta_B(t)\rho_{r,S}\sigma_S \sigma(t, T)x^2 \right\}
\end{aligned}$$

where $\theta(t) = (\theta_S(t), \theta_B(t))$ is the vector of optimal proportions. The HJB can be written as the maximum of a functional $\psi(\theta(t))$. At the optimal control $\theta^*(t) = (\theta_P^*(t), \theta_S^*(t))$ we must have simultaneously:

$$\psi(\theta^*(t)) = 0 \tag{A.5}$$

$$\frac{d\psi}{d\theta}(\theta^*(t)) = 0 \tag{A.6}$$

Note that, in presence of $\rho_{r,S}$, the optimal $\theta(t)$ has to be calculated as a system of equations with unknowns $\theta_S(t)$ and $\theta_B(t)$. The second condition (A.6) gives the following system of equations:

$$\begin{cases} \theta_S(t)\sigma_S x V_{xx} - \rho_{r,S}\theta_B(t)\sigma(t, T)xV_{xx} = -(\lambda_S V_x + \rho_{r,S}\sigma_r V_{xr}) \\ -\rho_{r,S}\theta_S(t)\sigma_S x V_{xx} + \theta_B(t)\sigma(t, T)xV_{xx} = qV_x + \sigma_r V_{xr} \end{cases}$$

The optimal investment strategy $\theta^*(t)$ is then:

$$\begin{aligned}
\theta_S^*(t) &= \frac{(-\lambda_S + \rho_{r,S}q) V_x}{\sigma_S x V_{xx} (1 - \rho_{r,S}^2)} \\
\theta_B^*(t) &= \frac{(q - \rho_{r,S}\lambda_S) V_x + (1 - \rho_{r,S}^2) \sigma_r V_{xr}}{\sigma(t, T)xV_{xx} (1 - \rho_{r,S}^2)}
\end{aligned}$$

Putting the expression of the optimal allocations into (A.5), we obtain the following partial differential equation for the value function:

$$0 = V_t + a(b - r(t))V_r + \frac{1}{2}\sigma_r^2 V_{rr} + r(t)xV_x - \frac{1}{2}\sigma_r^2 \frac{V_{xr}^2}{V_{xx}} - q\sigma_r \frac{V_x V_{xr}}{V_{xx}} - \frac{1}{2} \frac{V_x^2}{V_{xx} (1 - \rho_{r,S}^2)} (\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S) \quad (\text{A.7})$$

with limit condition $V(X, r, T) = u(X) = -e^{-\varphi X}$. We try a solution inspired by the expression found in Young (2004):

$$V(x, r, t) = -e^{-\varphi x P(t,r) + K(t)}$$

where $P(t, r)$ and $K(t)$ are independent of $X(t)$ and have limit condition $P(T, r) = 1$ and $K(T) = 0$. Then the partial derivatives corresponding to this transformation are:

$$\begin{aligned} V_r &= -\varphi x P_r V; V_{rr} = (\varphi x P_r^2 - P_{rr})\varphi x V \\ V_x &= -\varphi P V; V_{xx} = \varphi^2 P^2 V \\ V_t &= (-\varphi x P_t + K'(t))V; V_{xr} = (\varphi x P - 1)\varphi P_r V \end{aligned}$$

Substituting into (A.7) and after some calculations:

$$\begin{aligned} 0 = & -\varphi x \left(P_t + r(t)P + a \left(\frac{b^*}{a} - r(t) \right) P_r + \frac{1}{2}\sigma_r^2 P_{rr} - \sigma_r^2 \frac{P_r^2}{P} \right) \\ & + K'(t) - \frac{1}{2} \frac{\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S}{(1 - \rho_{r,S}^2)} - \frac{1}{2}\sigma_r^2 \frac{P_r^2}{P^2} - q\sigma_r \frac{P_r}{P} \end{aligned}$$

We try a solution for this SDE depending on $P(t, r)$ inspired by the expression of a zero-coupon bond as previously used in Korn and Kraft (2002) and Young (2004):

$$P(t, r) = \alpha(t)e^{-\beta(t)r} \quad (\text{A.8})$$

with $\alpha(t)$ and $\beta(t)$ independent of r with limit conditions $\alpha(T) = 1$ and $\beta(T) = 0$. Its partial derivatives are:

$$\begin{aligned} P_t &= \alpha'(t)e^{-\beta(t)r} - P\beta'(t)r \\ P_r &= -\beta(t)P; P_{rr} = \beta^2(t)P \end{aligned}$$

Substituting again in the previous PDE we have:

$$\begin{aligned} 0 = & -\varphi x e^{-\beta(t)r} \left(\alpha'(t) - \alpha(t)r(t) \underbrace{\left(\beta'(t) - a\beta(t) - 1 \right)}_{\text{Eq.A}} - \alpha(t) \left(b^*\beta(t) + \frac{1}{2}\sigma_r^2\beta^2(t) \right) \right) \\ & + \left(\underbrace{K'(t) - \frac{1}{2} \frac{\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S}{1 - \rho_{r,S}^2} - \frac{1}{2}\sigma_r^2 \frac{P_r^2}{P^2} - q\sigma_r \frac{P_r}{P}}_{\text{Eq.B}} \right) \end{aligned} \quad (\text{A.9})$$

This transformation would only make sense if the expression above no longer depends on $r(t)$. In this case the solution to Equation A and B are:

$$\beta(t) = -B(t, T) = \frac{e^{-a(T-t)-1}}{a} \quad (\text{A.10})$$

$$K(t) = -\frac{1}{2} \int_t^T \frac{\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S}{1 - \rho_{r,S}^2} + \sigma_r^2 B(s, T)^2 + 2q\sigma_r B(s, T) ds \quad (\text{A.11})$$

The expression (A.9) becomes:

$$0 = \alpha'(t) - \alpha(t) \left(b^* \beta(t) + \frac{1}{2} \sigma_r^2 \beta^2(t) \right) = \alpha'(t) - \alpha(t) h(t) \quad (\text{A.12})$$

which has the following solution:

$$\alpha(t) = \exp \left\{ \left(-\frac{b^*}{a} + \frac{\sigma_r^2}{2a^2} \right) [B(t, T) - T + t] + \frac{\sigma_r^2}{4a} B(t, T)^2 \right\} = A_Q(t, T)^{-1} \quad (\text{A.13})$$

The expression of $P(t, r)$ is then given by the inverse of the zero-coupon bond under the Q-dynamics $ZC_Q(t, T)$. The value function $V(x, r, t)$ (2.23) becomes (3.1).

A.3 Proof of Theorem 2

In order to ease the notation, the dependence of y and t will not be written and the following functions will be used throughout the proof:

$$\begin{aligned} A_r &= \rho_{r,L}\sigma_1 + \rho_{r,P}\sigma_2 \\ A_S &= \rho_{S,L}\sigma_1 + \rho_{S,P}\sigma_2 \end{aligned}$$

The maximum principle applied to the value function $u(x, y, r, t)$ (2.24) gives the following HJB relation:

$$\begin{aligned} 0 &= u_t + a(b - r)u_r + \\ &+ \max_{\theta(t)} \left\{ \frac{1}{2} \theta_B^2 \sigma^2(t, T) \right. \\ &\left. + \frac{1}{2} \sigma_S^2 x^2 \theta_S^2 u_{xx} - \theta_S(t) \theta_B(t) \rho_{r,S} \sigma_S \sigma(t, T) x^2 u_{xx} + \theta_S \sigma_S x (\lambda_S u_x + \rho_{r,S} \sigma_r u_{xr} + A_S u_{xy}) \right\} \end{aligned}$$

Here we face the same problem as in Proposition 1. Because of the presence of $\rho_{r,S}$ the optimal $\theta(t)$ has to be calculated as a system of equations. The HJB can be written as the maximum of a functional $\psi(\theta(t))$ and at the optimal control $\theta^*(t)$ we must have simultaneously (A.5) and (A.6). The second condition (A.6) gives the following system of equations:

$$\begin{cases} \theta_S(t) \sigma_S x u_{xx} - \rho_{r,S} \theta_B(t) \sigma(t, T) x u_{xx} = -(\lambda_S u_x + \rho_{r,S} \sigma_r u_{xr} + A_S u_{xy}) \\ -\rho_{r,S} \theta_S(t) \sigma_S x u_{xx} + \theta_B(t) \sigma(t, T) x u_{xx} = q u_x + \sigma_r u_{xr} + A_r u_{xy} \end{cases}$$

The optimal investment strategy $\theta^*(t)$ is then:

$$\begin{aligned}\theta_S^*(t) &= \frac{(-\lambda_S + \rho_{r,S}q) u_x + (\rho_{r,S}A_r - A_S) u_{xy}}{\sigma_S x u_{xx} (1 - \rho_{r,S}^2)} \\ \theta_B^*(t) &= \frac{(q - \rho_{r,S}\lambda_S) u_x + (1 - \rho_{r,S}^2) \sigma_r u_{xr} + (A_r - \rho_{r,S}A_S) u_{xy}}{\sigma(t, T) x u_{xx} (1 - \rho_{r,S}^2)}\end{aligned}$$

Putting the optimal θ into (A.5), and after some tedious algebra, we obtain the following partial differential equation for the value function:

$$\begin{aligned}0 &= u_t + a(b - r)u_r + \frac{1}{2}\sigma_r^2 u_{rr} + rxu_x + A_r\sigma_r u_{yr} + \mu u_y + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2)u_{yy} \\ &\quad - q\sigma_r \frac{u_x u_{xr}}{u_{xx}} - A_r\sigma_r \frac{u_{xr} u_{xy}}{u_{xx}} - \frac{u_x u_{xy}}{u_{xx}} \frac{\lambda_S(A_S - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_S)}{1 - \rho_{r,S}^2} \\ &\quad - \frac{1}{2}\sigma_r^2 \frac{u_{xr}^2}{u_{xx}} - \frac{1}{2} \frac{u_x^2}{u_{xx}} \frac{\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S}{1 - \rho_{r,S}^2} - \frac{1}{2} \frac{u_{xy}^2}{u_{xx}} \frac{A_S^2 + A_r^2 - 2\rho_{r,S}A_S A_r}{1 - \rho_{r,S}^2}\end{aligned}\quad (\text{A.14})$$

Due to the good separability properties of the exponential utility we try a solution inspired by the limit condition:

$$u(x, y, t) = -e^{-\varphi x P(t, r)} F(y, t) \quad (\text{A.15})$$

where $F(y, t)$ corresponds to the part of the value function related to the nontraded asset $Y(t)$. The limit condition is $F(Y(T), T) = h\left(\frac{Y(T)}{Y(t)}\right)$. The partial derivatives are then:

$$\begin{aligned}u_t &= -\varphi x u P_t - e^{-\varphi x P} F_t; u_r = -\varphi x P_r u; u_{rr} = (\varphi x P_r^2 - P_{rr}) \varphi x u \\ u_x &= -\varphi P u; u_{xx} = \varphi^2 P^2 u; u_y = -e^{-\varphi x P} F_y; u_{yy} = -e^{-\varphi x P} F_{yy} \\ u_{xr} &= (\varphi x P - 1) \varphi P_r u; u_{xy} = e^{-\varphi x P} \varphi P F_y; u_{yr} = e^{-\varphi x P} \varphi x P_r F_y\end{aligned}$$

Substituting into (A.14) and after some calculations:

$$\begin{aligned}0 &= -\varphi x u \left(P_t + rP + a \left(\frac{b^*}{a} - r \right) P_r + \frac{1}{2} \sigma_r^2 P_{rr} - \sigma_r^2 \frac{P_r^2}{P} \right) \\ &\quad - e^{-\varphi x P} \left(F_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S(A_S - \rho_{r,S}A_r) + q(A_r - \rho_{r,S}A_S)}{1 - \rho_{r,S}^2} \right) F_y \right) \\ &\quad + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P}\sigma_1\sigma_2) F_{yy} - \frac{1}{2} \frac{A_S^2 + A_r^2 - 2\rho_{r,S}A_S A_r}{1 - \rho_{r,S}^2} \frac{F_y^2}{F} \\ &\quad - \frac{1}{2} F \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S}q\lambda_S}{1 - \rho_{r,S}^2} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q\sigma_r \frac{P_r}{P} \right)\end{aligned}\quad (\text{A.16})$$

We now do the following transformation in order to linearize the PDE as in Zariphopoulou (2001):

$$F(y, t) = f(y, t)^\delta \quad (\text{A.17})$$

for a δ which has to be determined. The partial differential equations are then:

$$F_t = \delta f^{\delta-1} f_t; F_y = \delta f^{\delta-1} f_y; F_{yy} = \delta(\delta-1) f^{\delta-2} f_y^2 + \delta f^{\delta-1} f_{yy}$$

Substituting in (A.16)

$$\begin{aligned} 0 = & -\varphi x u \left(P_t + rP + a \left(\frac{b^*}{a} - r \right) P_r + \frac{1}{2} \sigma_r^2 P_{rr} - \sigma_r^2 \frac{P_r^2}{P} \right) \\ & - e^{-\varphi x P} \delta f^{\delta-1} \left(f_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S (A_S - \rho_{r,S} A_r) + q (A_r - \rho_{r,S} A_S)}{1 - \rho_{r,S}^2} \right) f_y \right. \\ & + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P} \sigma_1 \sigma_2) f_{yy} - \frac{1}{2} \frac{f}{\delta} \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{1 - \rho_{r,S}^2} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q \sigma_r \frac{P_r}{P} \right) \\ & \left. - \frac{1}{2} \frac{f_y^2}{f} \left(\delta \frac{A_S^2 + A_r^2 - 2\rho_{r,S} A_S A_r}{1 - \rho_{r,S}^2} - (\delta-1) (\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P} \sigma_1 \sigma_2) \right) \right) \end{aligned} \quad (\text{A.18})$$

If we choose δ as:

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P} \sigma_1 \sigma_2}{\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P} \sigma_1 \sigma_2 - \frac{A_S^2 + A_r^2 - 2\rho_{r,S} A_S A_r}{1 - \rho_{r,S}^2}}$$

then the PDE becomes a linear parabolic differential equation. We perform finally the same transformation for the function $P(t, r)$ as in (A.8) and obtain the same values for $\beta(t)$ (A.10) and $\alpha(t)$ (A.12). The remaining PDE is thus:

$$\begin{cases} f_t + \left(\mu - A_r \sigma_r \frac{P_r}{P} - \frac{\lambda_S (A_S - \rho_{r,S} A_r) + q (A_r - \rho_{r,S} A_S)}{1 - \rho_{r,S}^2} \right) f_y + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + 2\rho_{L,P} \sigma_1 \sigma_2) f_{yy} \\ = \frac{1}{2} \frac{f}{\delta} \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{1 - \rho_{r,S}^2} + \sigma_r^2 \frac{P_r^2}{P^2} + 2q \sigma_r \frac{P_r}{P} \right) \\ f(Y(T), T) = h \left(\frac{Y(T)}{Y(t)} \right) = e^{\frac{\varphi}{\delta} g \left(\frac{Y(T)}{Y(t)} \right)} \end{cases}$$

The drift coincides with the one presented in Proposition 1 under the pricing measure \mathcal{Q}^T . This partial differential equation can be rewritten in terms of expectancy under the measure \mathcal{Q}^T according to the Feynman-Kac representation theorem as follows

$$\begin{aligned} f(y, t) = & e^{-\frac{1}{2} \frac{1}{\delta} \int_t^T \left(\frac{\lambda_S^2 + q^2 - 2\rho_{r,S} q \lambda_S}{1 - \rho_{r,S}^2} + \sigma_r^2 B(s, T)^2 + 2q \sigma_r B(s, T) \right) ds} \\ & \times \mathbb{E}_{\mathcal{Q}^T} \left[e^{\frac{\varphi}{\delta} g \left(\frac{Y(T)}{Y(t)} \right)} | Y(t) = y \right] \end{aligned} \quad (\text{A.19})$$

where $Y(t)$ is given by (2.18). Then the writer's value function when the option is taken into account becomes:

$$u(x, y, r, t) = -e^{-\varphi \frac{x}{P_{\mathcal{Q}}(t, T)}} f^\delta \quad (\text{A.20})$$

with δ given by (3.3).

A.4 Proof of Corollary 4

Let $\rho_{S,W} = 1$ and $\rho_{S,P} = 1$. This means that the wage and population risks are totally correlated with the market risk $B_S(t)$. Furthermore, let the correlation $\rho_{r,W}$ and $\rho_{r,P}$ be equal to $\rho_{r,S}$. In this case we have $A_S = \sigma_1 + \sigma_2$, $A_r = \rho_{r,S}(\sigma_1 + \sigma_2)$ and δ becomes:

$$\delta = \frac{\sigma_1^2 + \sigma_2^2 + 2\rho_{W,P}\sigma_1\sigma_2}{2\sigma_1\sigma_2} \frac{1}{(\rho_{W,P} - 1)} = \frac{A(\rho_{W,P})}{B(\rho_{W,P})} \quad (\text{A.21})$$

When $\rho_{W,P} = 1$ the inverse of δ is zero, that is, $\frac{1}{\delta} = 0$ and the SDE has only one source of risk and mimics the traded asset $S(t)$ completely. Let f :

$$f(G) = \exp\left(\frac{\varphi}{\delta}G\right) = \exp\left(\frac{\varphi B(\rho_{W,P})}{A(\rho_{W,P})}G\right) \quad (\text{A.22})$$

The Taylor series of $f(G)$ around 0 is:

$$f(G) \approx f(0) + f'(0)G + \sum_{n=2}^{\infty} \frac{f^n(0)}{n!} G^n \quad (\text{A.23})$$

with

$$f^i(0) = \frac{\varphi^i B(\rho_{W,P})^i}{A(\rho_{W,P})^i}$$

Replacing (A.23) in (3.4), and simplifying the notation $A(\rho_{W,P}) = A(\rho)$, $B(\rho_{W,P}) = B(\rho)$, $\mathbb{E}_{\mathcal{Q}^T}[G|Y(t) = y] = \mathbb{E}_{\mathcal{Q}^T}[G]$ and $p(x, y, r, t)^* = p^*$:

$$p^* = ZC_{\mathcal{Q}}(t, T) \frac{A(\rho)}{\varphi B(\rho)} \log \left(1 + \frac{\varphi B(\rho)}{A(\rho)} \mathbb{E}_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty} \frac{\varphi^n B(\rho)^n}{n! A(\rho)^n} \mathbb{E}_{\mathcal{Q}^T}[G^n] \right)$$

The limit when $\rho_{W,P}$ (denoted ρ for simplicity) tends to 1 is then:

$$\begin{aligned} \lim_{\rho \rightarrow 1} p^* &= \frac{0}{0} = (\text{L'Hôpital}) \\ &= \lim_{x \rightarrow \infty} ZC_{\mathcal{Q}}(t, T) \frac{1}{\varphi} \left\{ \log \left(1 + \frac{\varphi B(\rho)}{A(\rho)} \mathbb{E}_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty} \frac{\varphi^n B(\rho)^n}{n! A(\rho)^n} \mathbb{E}_{\mathcal{Q}^T}[G^n] \right) \right. \\ &\quad \left. + A(\rho) \frac{\frac{A(\rho) - B(\rho)}{A(\rho)^2} \varphi \mathbb{E}_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty} \frac{\varphi^n n B(\rho)^{n-1} A(\rho)^n - n A(\rho)^{n-1} B(\rho)^n}{A(\rho)^{2n}} \mathbb{E}_{\mathcal{Q}^T}[G^n]}{\left(1 + \frac{\varphi B(\rho)}{A(\rho)} \mathbb{E}_{\mathcal{Q}^T}[G] + \sum_{n=2}^{\infty} \frac{\varphi^n B(\rho)^n}{n! A(\rho)^n} \mathbb{E}_{\mathcal{Q}^T}[G^n] \right)} \right\} \\ &= ZC_{\mathcal{Q}}(t, T) \frac{1}{\varphi} \left(0 + A(\rho) \frac{\varphi \mathbb{E}_{\mathcal{Q}^T}[G]}{A(\rho)} \right) = ZC_{\mathcal{Q}}(t, T) \mathbb{E}_{\mathcal{Q}^T}[G] \end{aligned}$$

This is the price in the complete markets setting with stochastic interest rates (see Amin and Jarrow (1992) or Brigo and Mercurio (2007)) under the forward measure, which is given by:

$$\begin{aligned} \frac{d\mathcal{Q}^T}{d\mathcal{P}} &= \exp \left(-\frac{1}{2} \int_t^T (q + \sigma_r B(s, T))^2 ds - \frac{1}{2} \lambda_S^2 (T - t) \right. \\ &\quad \left. - \int_t^T (q + \sigma_r B(s, T)) dB_r(s) - \int_t^T \lambda_S dB_S(s) \right) \end{aligned}$$

This is the forward measure when the stochastic interest rate has a Vasicek structure independent of the risky asset $S(t)$.