CEPAR Workshop on Longevity and Long-Term Care Risks and Products

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Variable Annuity Products: Latest Developments in Valuation and Hedging Techniques

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Outline of the Presentation

- Motivation
- Variable annuity overview
- Policyholder behaviour
- Valuation of GMMBs with and without surrender
- Valuation of GMIB and GMDB riders
- GMWB valuation
- Hedging Initiatives

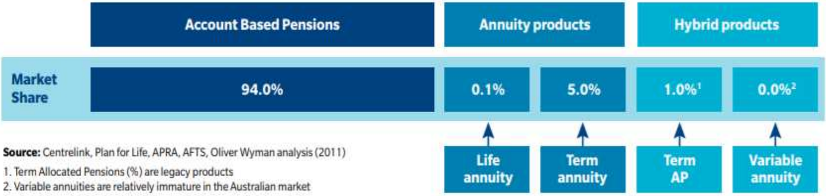
Motivation

- A greater proportion of the **5.5 million** Australian baby boomers born between 1946 and 1965 now entering into retirement.
- These boomers' **life experiences and expectations are profoundly different** from those of previous generations.
- Given the increasing number of Australians **reaching retirement** age and a **maturing superanuation guarantee system**, there has been a new focus on retirement income products to convert superannuation savings into **sustainable, yet flexible, income streams**.
- Such products are required to provide financially secure retirement income streams that finance and insure the **long-term risks** faced by individuals in retirement.
- Murray et al. (2014) recommend such products in order to **improve efficiency,** resilience and fair treatment in the financial system, to support economic growth and to enhance standards of living for current and future generations.

Motivation cont...

- Currently, there exist three pillars of retirement income options in Australia, namely; **account-based pension, life annuities and variable annuities**.
- Majority of retirees elect the account-based pension scheme **to draw down their retirement savings.**
- Account-based schemes elected by at least 94% of retirees provide flexibility for individuals to invest in a range of asset classes including equity, fixed interest and property - **lump sum mentality.**
- However, the main challenge is that they **do not offer longevity protection** and expose individuals to significant investment risk where significant allocations are made to equity investments.
- **Life annuities** are offered by a small number of life insurers but take-up of annuities is minimal in Australia - annuity puzzle.
- **Variable annuities provide longevity protection** through appropriate guarantees embedded in them.

- Variable annuities are still invisible on the Australian market regardless of their many attractive features.



Motivation cont...

- For comfortable retirement, individuals need **flexible choice products** which better insure them against all key risks in retirement such as longevity, investment and inflation.
- Rothman (2012) highlight that under the current status quo, age and service pension payments are projected to rise by 1.2% of GDP, from **2.7% as of 2009-10** financial year to **3.9% of GDP in 2049-50**.
- There is need for default products which can used to **efficiently transform superannuation savings into lifetime income streams**, transferring longevity/investment risk to the market and potentially reducing the call on **government support other than as a safety net**.
- Increasing numbers of retiring Australians have significant implications for both national and individual welfare, especially as budget strains from increased age pension and aged care costs become more pressing.

Variable Annuities

- A variable annuity is a contract between **an insurance company and a policyholder.**
- The insurance company agrees to make periodic payments to the policyholder in future (mainly post retirement).
- The policyholder purchases a variable annuity by paying either a **single premium payment or a series of payments**.
- Unlike traditional mutual funds and life insurance products, variable annuity contracts come with **embedded guarantees which protect the policyholder's savings** against unanticipated outcomes.
- Guarantees can be underwritten for **the accumulation phase, annuity phase or untimely death of the policyholder.**
- These guarantees exhibit financial option-like features, naturally leading to the way they are valued in practise.
- Premiums paid when purchasing variable annuities are usually **invested in various subaccounts** with different characteristics and investment strategies.

Variable Annuities cont...

- Variable annuity subaccounts include **actively managed portfolios, exchange-traded funds, index-linked portfolios, alternative investments and other quantitative-driven strategies.**
- Insurance companies usually **charge proportional fees** on variable annuity contracts as a way of funding the guarantees.
- If the fees are too high relative to the performance of the fund, the policyholder **can choose to surrender** the contract or the guarantee **prior to maturity** in return of a surrender benefit.
- The benefits will be net of surrender/penalty charges enforced as a way of discouraging early termination of the contract.
- Some of the advantages of variable annuities include
  - Tax-deferred earnings,
  - Tax-free transfers across a variety of investment options,
  - Death benefit protection options,
  - Living benefit protection options,
  - Lifetime income options.

Variable Annuities cont...

- Variable Annuities (VAs) were first introduced in the early 1950s.
- Riders embedded in variable annuities can be categorised into two major groups (Ledlie et al. 2008):
    - Guaranteed Minimum Death Benefit introduced in 1980s.
        - GMDB pays beneficiaries a guaranteed sum in the event of the **policyholder's death during** the contract life.
    - Guaranteed Minimum Living Benefits introduced in late 1990s.
        - GMMB/GMAB - minimum guarantee at maturity which guarantees the return of the premium payments or higher stepped-up value **at the end of the accumulation period**
        - GMIB - guarantees **an income stream** when a policyholder **annuitizes the GMMB** regardless of the underlying investment performance.
        - GMWB - guarantees an income stream regardless of the account value. Payments can be guaranteed for a specified period or for the policyholder lifetime.
        - GLWB - allows minimum withdrawals from the invested amount without having to annuitize the investment.

Variable Annuities cont...

- VA industry is large and still expanding:
- US$1.35 trillion in the U.S. as of 2008 (Condron 2008).
- US$1.96 trillion in the U.S. as of third quarter of 2017 (IRI 2017).
- On a year-over-year basis, assets **were up 1.9%,** from US$1.92 trillion at the end of the third quarter of 2016, as **positive market performance outweighed** the impact of lower sales and negative net flows (IRI 2017).
- The riders have varying popularity:
  - 59% elected GLWB, 26% GMIB, 3% GMAB and 2% GMWB as of 2011 (Fung et al. 2014).
  - Death benefits are usually given as an additional rider **'for free'** (Moenig and Bauer 2017).

Variable Annuities Pricing

- The greater part of the literature has focused on the **pricing of riders** embedded in VAs, with a **recent spike of interest in hedging**.
- Pricing in the VA context: Find the **regular** fair fee, as a percentage of the underlying fund, that covers the guarantees.
- The fee is usually paid while the rider is active.
- Main areas of focus have been:
  - Underlying fund dynamics,
  - Policyholder withdrawal and surrender behavior,
  - Computational aspects.

Underlying Fund Dynamics

- Most seminal papers assume that the underlying follows a **Geometric Brownian Motion** (GBM) (Milevsky and Posner 2001; Bauer et al. 2008).
- As a step towards considering a more realistic framework, **regime-switching** (RS) models have been proposed (Hardy 2001).
- However, GBM and RS **do not capture full empirical properties** of asset return distributions such as **heavy tails, skewness and kurtosis.**
- **Levy processes** have been proposed to address the shortcomings of GBM and RS (Chen et al. 2008; Bacinello et al. 2011; Kélani and Quittard-Pinon 2015; Bacinello et al. 2014).
- **Stochastic volatility, or stochastic interest rates** have also been considered too (Peng et al. 2012; Bacinello et al. 2011; Kling et al. 2011; Kang and Ziveyi 2018).

Policyholder behavior and frictions

- Commonly, pricing frameworks assume two main policyholder behavior:
  - **Static**: this is where pre-specified contract characteristics are followed; $\rightarrow$ this has European option-like features
  - **Dynamic**: This is where a policyholder behaves in a way that maximizes the value of the contract (including surrender) $\rightarrow$ this has American option-like features
- In practice, pricing is affected by taxes & management fees too (Moenig and Bauer 2016, 2017)

Policyholder behavior and frictions cont...

- Empirical evidence in the US show that surrender rates for VAs embedded with guaranteed living benefits are very low:

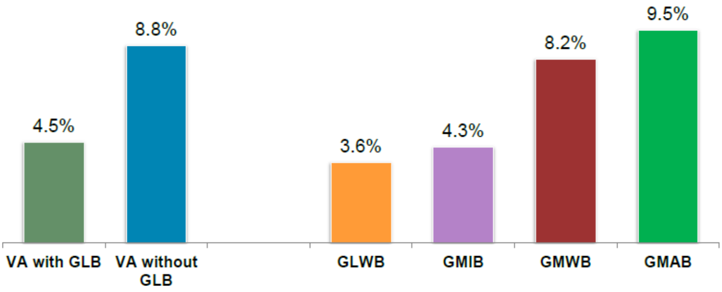


Figure: Surrender rates for VA riders in 2014. Source: Guaranteed Living Benefit Utilization Study-2014 Owners' Experience, LIMRA SRI (2016).

Policy features

To dis-centivize surrender or dynamic behavior, various features are added to the contracts (Moenig and Zhu 2016):

- **surrender schedule**: within a certain number of years, lapsing will incur a surrender fee.
- **roll-up guarantee**: the guaranteed minimum amount increases by a fixed percentage each year.
- **ratchet-type guarantee / automatic annual step-up**: the guarantee is equal to the maximum of the values of the VA account at previous anniversary dates.
- **state-dependent fee**: the fee for the guarantee is only paid if the account value is close to being in the money.
- **enhanced earnings**: an additional earnings feature which provides an additional payout.

Typical Underlying Fund Dynamics - GBM

- The policyholder's premium is normally invested in a fund consisting of units of an underlying asset, $S = (S_t)_{0 \leq t \leq T}$, whose risk-neutral evolution can be modelled by the geometric Brownian motion process

$$\boxed{dS_t = rS_t dt + \sigma S_t dW_t, \qquad (1)}$$

- The fund value at time t is denoted as

$$\boxed{F_t = e^{-ct} S_t, \qquad (2)}$$

where c denotes management fees, hence

$$\boxed{dF_t = (r - c) F_t dt + \sigma F_t dW_t. \qquad (3)}$$

- In the event of the guarantee being terminated early, the resulting benefit fund value for that component is $\mathbf{(1 - \kappa_t) F_t}$ where κ_t is a surrender charge.

Computational aspects

- **Monte Carlo** based methods are commonly used to approach the complex policy features of the contract.
- However, to get the desired accuracy, **high number of scenarios are needed**.
- Recently, there has been increasing focus on computationally efficient methods:
  - **Fast-Fourier Transform** (FFT) (Kélani and Quittard-Pinon 2015; Bacinello et al. 2014).
  - **Fourier Space Time-Stepping** (FST) (Ignatieva et al. 2016).
  - **Fourier-COS method** (Alonso-García et al. 2017).
  - **Grid based approaches** such as Method of lines algorithm (Kang and Ziveyi 2018).
- For simple vanilla payoff functions, these approaches are **at least 50 times faster than Monte Carlo** (Ignatieva et al. 2016).
- Computational efficiency increases **exponentially with payoff complexity**.

Functional forms of Variable Annuity Riders - GMMB/GMAB

- The payoff of a GMMB at maturity can be represented as

$$\vartheta(F_T) = \max(F_T, G_T), \tag{4}$$

where

$$G_T = \begin{cases} G & \text{if the guarantee is fixed} \\ Ge^{\delta T} & \text{if the guarantee is rolled up at a rate of } \delta \\ \left(\prod_{j=0}^{T} F_j\right)^{\frac{1}{T+1}} & \text{if it is a ratchet geometric average guarantee} \\ \frac{1}{T+1}\sum_{j=0}^{T} F_j & \text{if it is a ratchet arithmetic average guarantee,} \end{cases}$$

- Graphically, this can be represented as

GMMB Valuation - Integral Representation

- By letting $x_t = \ln F_t$, the value of a GMMB rider at time $t \in [0, T]$ can be represented as

$$\boxed{\begin{aligned} C_M(t, T, x) &= e^{-r(T-t)}\mathbb{E}_t^Q\left[\vartheta(e^{x_T})|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\int_{-\infty}^{\infty}\vartheta(e^y)f(y|x)dy \qquad (5) \end{aligned}}$$

  where $f(y|x)$ is the transition density function of the underlying process, y given x.
- A variety of numerical integration methods can be employed to solve equation (5).
- In this presentation we will illustrate three techniques namely
  - The Fourier Cosine (COS) method,
  - The fast Fourier transform (FFT) method,
  - The Fourier space time-stepping (FST) method.
- The superiority of all these techniques is tha**t one can as well compute the Greeks** which can be used for hedging purposes.

GMMB Valuation using COS Method – no Surrender

- The COS method takes note of the fact that the transition density function can be expressed as a Fourier transform of its characteristic function, that is

$$\boxed{\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx. \qquad (6)}$$

- Also any function supported in the **real space can be approximated by a cosine expansion.**
- With this in mind, the GMMB value in equation (5) can then be expressed as

$$\boxed{C_M(t, T, x) \approx e^{-r(T-t)} {\sum_{k=0}^{N-1}}{}' \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}}\right\} V_k}$$

GMMB Valuation using COS Method cont...

- The cosine series coefficients of the payoff, V_k are given by

$$\boxed{V_k = \frac{2}{b-a}\int_a^b \vartheta(e^y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \qquad (7)}$$

  with $[a, b]$ being a finite region on which the transition density function if defined.
- These coefficients **can be obtained analytically** for most vanilla type payoff functions.
- Equation (7) can be simplified for **the Lévy and affine type models** so that **many strikes** can be handled simultaneously at no additional computational cost.

GMMB Valuation using FFT Method – no Surrender

- Again, taking advantage of the relationship between the transition density function and its characteristic function and letting $\vartheta(e^x) \equiv h(x)$, the value of a GMMB in equation (5) can be re-expressed as

$$C_M(t, T, F_t) = \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{\infty} \phi(z)\hat{h}(z)dz, \qquad (8)$$

where we note that from (6) we that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixz}\phi(z)dz.$$

- Also, $\hat{h}(z)$ is the Fourier transform of the payoff function, that is

$$\hat{h}(z) = \int_{-\infty}^{\infty} e^{-iyz} h(y)dy. \qquad (9)$$

- Explicit solutions of Eq. (9) for most vanilla type payoff structures can be derived.

GMMB Valuation using FFT Method cont...

- Eq. (10) evaluated with the aid of the FFT algorithm.
- First approximate the integral with a double sum over the lattice

$$\Gamma = \left\{ z(k) = [z(k)] | k = \in \{0, \cdots, N-1\} \right\}, z(k) = -\bar{z} + k\eta.$$

- An approximation of Eq. (10) is then given by

$$C_M(t, T, F_t) \approx \frac{\eta \cdot e^{-r(T-t)}}{2\pi} \sum_{k=0}^{N-1} \phi(z(k) + i\epsilon) \hat{h}(z(k) + i\epsilon), \tag{10}$$

where $\epsilon \in \mathbb{R}$ is a vector such that the Fourier transform of the considered payoff is well defined, and η is the grid spacing.

GMMB Valuation using FST Method – no Surrender

- The valuation problem Eq. (5) is first expressed **in partial differential equation (PDE) form.**
- The PDE is then efficiently solved, bypassing the complexity associated with integral terms.
- This methodology is applicable to **any asset price model which admits a closed form characteristic function.**
- Valuation proceeds by **switching between the real space and the frequency space** with the aid of the Fourier transform as depicted on the diagram on the next slide.
- In so doing, the **PDE is transformed into system of ordinary differential equations (ODEs)** defined in the frequency domain.
- For a GMMB with no surrender features, with the knowledge of the payoff at maturity, the ODE system can easily be solved to find the value at current time.

The Fourier Space Time-Stepping algorithm

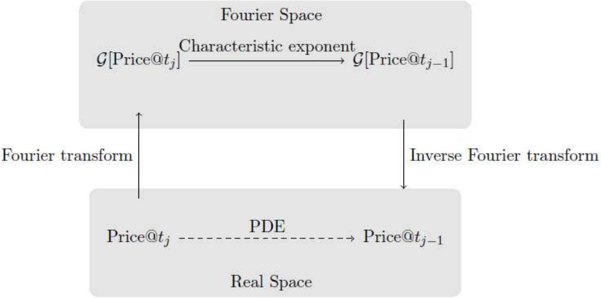


Figure: FST algorithm from t_j to t_{j-1}

- This can be summarised as

$$\boxed{C_M(t,T,x) = \text{FFT}^{-1}[\text{FFT}(\vartheta(e^{x_T}))\phi(x_T)] \qquad (11)}$$

GMMB Valuation - Case with Surrender Features

- For illustrative purposes, we assume an exponentially decreasing surrender fee structure on the guarantee implying that the fund value of the guarantee component is $(1-\kappa_t)F_t = e^{-\kappa(T-t)}F_t$
- The variable annuity contract at anytime prior to maturity can be represented as an optimal stopping problem such that

$$\boxed{C(t,F) = \operatorname*{ess\,sup}_{t\le\tau^*\le T} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{\tau^*} r_s ds} g(\tau^*, F_{\tau^*})|\mathcal{F}_t\right], \qquad (12)}$$

where

$$\boxed{g(t,F_t) = \begin{cases} e^{-\kappa(T-t)}F_t, & t<T \\ \max(F_t, G), & t=T \end{cases}}$$

and the supremum is taken over all stopping times, τ^*.

GMMB Valuation - Case with Surrender Features cont...

- Using similar arguments to those presented in Jacka (1991) and Peskir and Shiryaev (2006), the optimal stopping problem in equation (12) is equivalent to the free boundary problem

$$\boxed{\frac{\partial C}{\partial t} + (r - c)F\frac{\partial C}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC = 0, \qquad (13)}$$

  where $0 < F < b(t)$, with $b(t)$ being the optimal surrender boundary. The PDE (13) is solved subject to boundary and terminal conditions

$$C(T, F) = \max(F, G), \qquad (14)$$

$$C(t, b(t)) = e^{-\kappa(T-t)} b(t), \qquad (15)$$

$$\lim_{F \to b(t)} \frac{\partial C}{\partial F} = e^{-\kappa(T-t)}, \qquad (16)$$

$$C(t, 0) = e^{-r(T-t)} G. \qquad (17)$$

- The PDE (13) can be solving using a variety of techniques **such as the method of lines** (Kang and Ziveyi 2018) or **numerical integration** (Shen et al. 2016).

Guaranteed Minimum Income Benefit - GMIB

- The policyholder is guaranteed a minimum level of income stream, G at periodic intervals as long as he or she stays alive, until maturity T.
- The value of the GIMB can be represented as

$$\boxed{V_I(t, T, S_t) = \sum_{j=t+1}^{T} V_M(t, j, S_t)}$$

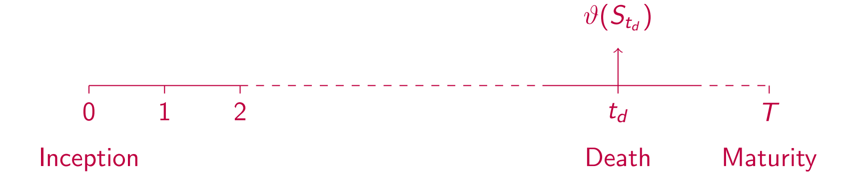
- A stream of benefit payments $\vartheta(S_1), \vartheta(S_2), \ldots, \vartheta(S_T)$ until maturity or death can be expressed as

Guaranteed Minimum Death Benefit - GMDB

- The policyholder's beneficiaries are paid a guaranteed minimum level of benefit in the event of the policyholder's death before the maturity of the contract. Assuming that the benefit is paid immediately upon death, the value of a GMDB rider is given by

$$\boxed{\begin{aligned} C_D(t,F) &= \mathbb{E}_t^{Q}\left[e^{-\int_t^{\tau_x} r ds}\vartheta(F_{\tau_x})\mathbb{1}_{\{t_d \le T-t\}}|\mathcal{F}(t)\right] \\ &= \int_t^T \mathbb{E}_t^{\mathbb{Q}}\left[\mu(x+u)e^{-\int_t^u \mu(x+s)ds}\right] C_M(t,F_u)du, \end{aligned}}$$

  with x being the age of the policyholder at inception of the contract.
- Graphically, this can be represented as



- A GMDB contract is a **byproduct of a GMMB** $\Rightarrow$ usually given as a free benefit to holders of variable annuity contracts Moenig and Bauer (2017).

Guaranteed Minimum Withdrawal Benefit - GMWB

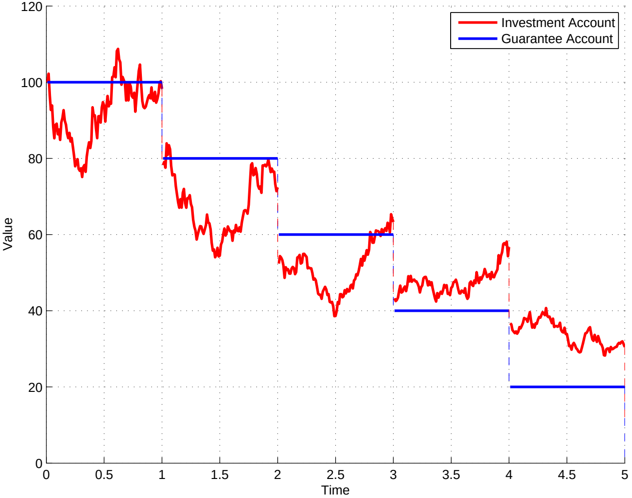
- At inception the policyholder **pays a lump sum to the insurer**, which becomes the initial balance of two accounts forming a VA contract, namely, the investment account, $W(t)$ and the guarantee account, $A(t)$.
- Every time the policyholder withdraws a specified amount, denoted by γ_t, the two account values $(W(t)$ and $A(t))$ **decrease by** γ_t as well.
- γ_t can either be static or dynamic depending on contract specifications.
- The policyholder is able to make withdrawals **as long as the guarantee account value is above zero**, regardless of the performance of the $W(t)$.
- At maturity, the policyholder receives **the larger of the investment account balance and the guarantee account balance**, less any fees.
- At inception of the contract, the two account are equal, that is $W(0) = A(0)$

GMWB cont...

- The balance of the guarantee account at any given time can be represented as
$$\boxed{A(t) = A(0) - \int_0^t \gamma(s)ds, \quad 0 \le \gamma(s) \le G,}$$
with G being the contractually agreed withdrawal rate.
- Excess withdrawals above G attract a penalty fee, which we denote here as κ. The net amount received by the policyholder becomes
$$\boxed{f(\gamma) = \begin{cases} \gamma_t, & 0 \le \gamma_t \le G \\ G + (1-\kappa)(\gamma_t - G), & \gamma_t > G \end{cases}}$$
- The investment account evolves according to
$$\boxed{dW(t) = (r-c)W(t)dt + \sigma W(t)dBt + dA(t), \quad W(t) > 0.}$$
- The value of a VA contract embedded with a GMWB rider can then be represented as
$$\boxed{V(t,W,A) = \sup_{\gamma} \mathbb{E}_t^Q \left[e^{-r(T-t)} \max(W(T), A(T)) + \int_t^T e^{-r(u-t)} f(\gamma_u) du \right].}$$

GMWB cont...

- Example path of the investment and guarantee accounts for a five-year GMWB.



Hedging Initiatives

- In 2008, the total market capilisation of t**he top 10 insurers in the US decreased by 53%** (McKinsey & Company, 2009) with VA losses amounting to **$36 billion**.
- Providers need to be well prepared for **unexpected surrender/lapse** of VA contracts.
- The frameworks developed in literature (eg. Alonso-García et al., 2017 and Kang and Ziveyi, 2017) all consider rational policyholder behaviour.
- Increasing literature on hedging the net liability as presented earlier.
- Need for incorporating realistic surrender behaviour and taxes in the valuation and hedging frameworks!
- Other issues to consider include:
  - **Basis Risk** arising from underlying fund and hedging instruments
  - **Liquidity** of the hedging instruments
  - **Counterparty risk** in cases of OTC contracts.

Efficient Valuation Techniques - Available Literature

- Numerical integration presented in Sherris, Shen and Ziveyi (2016) for valuing a GMMB rider with early surrender features.
- Comprehensive framework for valuing Guaranteed minimum benefits using the Fourier Space Time-Stepping (FST) approach presented in Ignatieva, Song and Ziveyi (2016)
- Method of line approach which is a mesh-based algorithm for solving free-boundary problems like equation (13) as presented in Kang and Ziveyi (2017)
- FST approach for valuing GMWB riders as presented in Ignatieva, Song and Ziveyi (2016)
- Fourier-Cosine approach is presented in Alonso-García, Wood and Ziveyi (2017) for valuing VA with a GMWB rider.
- What if the underlying fund consists of more than one underlying asset? Two asset case presented in Da Fonseca and Ziveyi (2015) who use the Fast Fourier transform (FFT) algorithm.

Questions and Comments?

References I

Alonso-García, J., Wood, O. M., and Ziveyi, J. (2017), "Pricing and hedging guaranteed minimum withdrawal benefits under a general Lévy framework using the COS method," *Quantitative Finance*.

Bacinello, A. R., Millossovich, P., and Montealegre, A. (2014), "The valuation of GMWB variable annuities under alternative fund distributions and policyholder behaviours," *Scandinavian Actuarial Journal*, 2016, 1–20.

Bacinello, A. R., Millossovich, P., Olivieri, A., and Pitacco, E. (2011), "Variable annuities: A unifying valuation approach," *Insurance: Mathematics and Economics*, 49, 285–297.

Bauer, D., Kling, A., and Russ, J. (2008), "A Universal Pricing Framework for Guaranteed Minimum Benefits in Variable Annuities," *Astin Bulletin*, 38, 621–651.

Chen, Z., Vetzal, K., and Forsyth, P. A. (2008), "The effect of modelling parameters on the value of GMWB guarantees," *Insurance: Mathematics and Economics*, 43, 165–173.

References II

Condron, C. M. (2008), "Variable Annuities and the New Retirement Realities," *The Geneva Papers on Risk and Insurance-Issues and Practice*, 33, 12–32.

Fung, M. C., Ignatieva, K., and Sherris, M. (2014), "Systematic mortality risk: An analysis of guaranteed lifetime withdrawal benefits in variable annuities," *Insurance: Mathematics and Economics*, 58, 103–115.

Hardy, M. R. (2001), "A regime-switching model of long-term stock returns," *North American Actuarial Journal*, 5, 41–53.

Ignatieva, K., Song, A., and Ziveyi, J. (2016), "Pricing and Hedging of Guaranteed Minimum Benefits under Regime-Switching and Stochastic Mortality," *Insurance: Mathematics and Economics*, 70, 286–300.

IRI (2017), "Third Quarter 2017 Annuity Sales Report," .

Jacka, S. D. (1991), "Optimal Stopping and American Put," *Mathematical Finance*, 1, 1–14.

Kang, B. and Ziveyi, J. (2018), "Optimal surrender of guaranteed minimum maturity benefits under stochastic volatility and interest rates," *Insurance: Mathematics and Economics*, 79, 43–56.

References III

Kélani, A. and Quittard-Pinon, F. (2015), "Pricing and Hedging Variable Annuities in a Lévy Market: A Risk Management Perspective," *Journal of Risk and Insurance*.

Kling, A., Ruez, F., and Ruß, J. (2011), "The impact of stochastic volatility on pricing, hedging, and hedge efficiency of withdrawal benefit guarantees in variable annuities," *Astin Bulletin*, 41, 511–545.

Ledlie, M. C., Corry, D. P., Finkelstein, G. S., Ritchie, a. J., Su, K., and Wilson, D. C. E. (2008), "Variable Annuities," *British Actuarial Journal*, 14, 327–389.

Milevsky, M. A. and Posner, S. E. (2001), "The Titanic Option: Valuation of the Guaranteed Minimum Death Benefit in Variable Annuities," *Journal of Risk and Insurance*, 93–128.

Moenig, T. and Bauer, D. (2016), "Revisiting the Risk-Neutral Approach to Optimal Policyholder Behavior : A Study of Withdrawal Guarantees in Variable Annuities," *Review of Finance*, 20, 759–794.

— (2017), "Negative Marginal Option Values: The Interaction of Frictions and Option Exercise in Variable Annuities," .

References IV

Moenig, T. and Zhu, N. (2016), "Lapse-and-Reentry in Variable Annuities," *Journal of Risk and Insurance*.

Peng, J., Leung, K. S., and Kwok, Y. K. (2012), "Pricing guaranteed minimum withdrawal benefits under stochastic interest rates," *Quantitative Finance*, 12, 933–941.

Peskir, G. and Shiryaev, A. (2006), *Optimal Stopping and Free Boundary Problems*, Lectures in Mathematics. ETH Zürich, Birkhäuser Basel.

Shen, Y., Sherris, M., and Ziveyi, J. (2016), "Valuation of guaranteed minimum maturity benefits in variable annuities with surrender options ," *Insurance: Mathematics and Economics*, 69, 127–137.