Developments in Multi-Factor Continuous Time Mortality Modelling

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Introduction

- Mortality models have attracted research attention over recent years, particularly discrete time mortality models (Lee and Carter (1992), Cairns et al. (2006b), Cairns et al. (2009), Renshaw and Haberman (2006)).
  - focus on improvement trends,
  - impact of uncertainty or volatility of mortality, and
  - cohort effects.
- Continuous time affine cohort mortality models have attracted more recent research
  - single cohort models (Milevsky and Promislow (2001), Dahl and Møller (2006), Biffis (2005), Luciano et al. (2008), Schrager (2006), Cairns et al. (2006a), Blackburn and Sherris (2013))
  - multi-cohort models (Jevtic et al. (2013), Xu et al. (2019a), Chang and Sherris (2018), Zhou et al. (2019)).

Why Continuous Time Multi-factor Models?

- Analytical tractability - closed form survival curves for affine class,
- Consistency between mortality dynamics and functional form of the survival curve,
- Stability of parameter estimates,
- Use of mathematical finance methods for term structure and credit risk models familiar to financial market participants,
- Natural extensions to multi-factor models, capturing differing trends, volatility and correlations by age,
- Arbitrage-free formulation along with real world dynamics to allow calibration of prices of risk.

Coverage of this Session

- Introduce continuous-time multi-factor cohort mortality models - closed-form expressions for survival curves, dynamics of mortality rates, factors of level, slope and curvature for the mortality curve (AFNS mortality models),
- Fitting with age-cohort data,
- Kalman filter and estimation of the models, highlighting how Poisson variation can be incorporated into the model estimation,
- Comparison of fits and prediction using historical US mortality data,
- Multi-cohort models,
- Applications of multi-cohort models to quantifying price of mortality risk using Blackrock CORI indices.

Survival Curve - Continuous Time Affine Mortality Model

- Drawing on term structure of interest rate models - equivalence of average force of mortality rate to yield to maturity for zero coupon bond. Use of similar notation as in yield curve modelling.
- Survival probability $S(x, t, T)$ for single cohort aged x at time t for survival for a duration $(T - t)$ to age $x + (T - t)$, as an affine function of (latent) factors (3 factor case)

$$\begin{aligned} S(x, t, T) &= E[e^{-\int_t^T \mu^i(x,s)ds} | \mathcal{F}_t] \\ &= e^{-\bar{\mu}(t,T)(T-t)} \\ &= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3(t,T)X_3(t)+A(t,T)}, \end{aligned} \tag{1}$$

- $B_j(t, T)$ are factor loadings (functional form derived from mortality dynamics for the latent factors, exponential terms) and $X_j(t)$ are the latent factors (stochastic parameters).

Mortality Rate - Continuous Time Affine Mortality Model

- Average mortality rate - age-period data or age-cohort data

$$\bar{\mu}(t,T) = -\frac{1}{T-t}\log\left[S(t,T)\right] = -\frac{B(t,T)'}{T-t}X_t - \frac{A(t,T)}{T-t}. \quad (2)$$

where vector $B(t,T)$, the factor loadings, and $A(t,T)$ have explicit expressions (derivations similar to term structure models).

- Canonical form for these (Blackburn and Sherris, 2013), where δ_{jj} and σ_{jj} are parameters in the latent factor dynamics (estimated from historical data)

$$B_j(t,T) = -\frac{1-e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1,2,3, \quad (3)$$

$$A(t,T) = \frac{1}{2}\sum_{j=1}^{3}\frac{\sigma_{jj}^2}{\delta_{jj}^3}\left[\frac{1}{2}\left(1-e^{-2\delta_{jj}(T-t)}\right) - 2\left(1-e^{-\delta_{jj}(T-t)}\right) + \delta_{jj}(T-t)\right]. \quad (4)$$

Mortality Rate - AFNS Mortality model

- Mortality model equivalent of the Nelson-Seigel term structure model in arbitrage-free dynamic implementation (Christensen et al., 2011)
- Mortality rate curve has level, slope and curvature factors (independent AFNS mortality model).

$$
\begin{aligned}
B^1(t,T) &= -(T-t), \quad B^2(t,T) = -\frac{1-e^{-\delta(T-t)}}{\delta}, \\
B^3(t,T) &= (T-t)\,e^{-\delta(T-t)} - \frac{1-e^{-\delta(T-t)}}{\delta},
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\frac{A(t,T)}{T-t} = {} & \sigma_{11}^2 \frac{(T-t)}{6} + \sigma_{22}^2 \left[\frac{1}{2\delta^2} - \frac{1}{\delta^3} \frac{1-e^{-\delta(T-t)}}{T-t} + \frac{1}{4\delta^3} \frac{1-e^{-2\delta(T-t)}}{T-t} \right] + \\
& \sigma_{33}^2 \left[\frac{1}{2\delta^2} + \frac{1}{\delta^2} e^{-\delta(T-t)} - \frac{1}{4\delta}(T-t)\,e^{-2\delta(T-t)} - \frac{3}{4\delta^2} e^{-2\delta(T-t)} \right. \\
& \left. - \frac{2}{\delta^3} \frac{1-e^{-\delta(T-t)}}{T-t} + \frac{5}{8\delta^3} \frac{1-e^{-2\delta(T-t)}}{T-t} \right].
\end{aligned}
\tag{6}
$$

Affine Mortality Models - Q Pricing measure

- The price of a longevity zero-coupon bond on a specific cohort currently aged x is

$$\begin{aligned}\bar{P}_x(t,T) &= E^Q\left[\mathrm{e}^{-\int_t^T (r(s)+\mu(x,s))ds}|\mathcal{F}(t)\right] \\ &= E^Q\left[\mathrm{e}^{-\int_t^T r(s)ds}|\mathcal{G}(t)\right] E^Q\left[\mathrm{e}^{-\int_t^T \mu(x,s)ds}|\mathcal{H}(t)\right] \\ &= P(t,T)S^Q(x,t,T), \end{aligned} \tag{7}$$

  where the dynamics of the mortality rates and the dynamics of the interest rates are independent.

- Dynamics of the mortality rate used to derive the (risk-neutral) survival probability $S^Q(x,t,T)$.

- Use the same methodology of term structure modelling applied to (pricing) survival probability.

Affine Mortality Models - Dynamics of Mortality Rates

- The affine dynamics of the latent factors X_t follow a system of stochastic differential equations (SDEs) under the risk-neutral measure Q (Duffie and Kan, 1996; Christensen et al., 2011):

$$dX_t = K^Q \left[\theta^Q - X_t\right] dt + \Sigma D\left(X_t, t\right) dW_t^Q, \tag{8}$$

  - where $K^Q \in \mathbb{R}^{n \times n}$ is the mean reversion matrix,
  - $\theta^Q \in \mathbb{R}^n$ is the long-term mean (usually zero in mortality models),
  - $\Sigma \in \mathbb{R}^{n \times n}$ is the volatility matrix,
  - $W_t^Q \in \mathbb{R}^n$ is a standard Brownian motion, and
  - $D\left(X_t, t\right)$ is a diagonal matrix with the ith diagonal entry as $\sqrt{\alpha^i\left(t\right) + \beta_1^i\left(t\right) x_t^1 + \ldots + \beta_n^i\left(t\right) x_t^n}$. α and β are bounded continuous functions.

Affine Mortality Models - ODEs for Factor Loadings

- Under these dynamics the (risk-neutral) survival probabilities for age x for survival from time t to time T are (see details in Blackburn and Sherris, 2013):

$$S(t,T) = \exp\left(B(t,T)' X_t + A(t,T)\right), \tag{9}$$

- Where $B(t,T)$ and $A(t,T)$ are the solutions to the following system of ordinary differential equations (ODEs):

$$\frac{dB(t,T)}{dt} = \rho_1 + \left(K^Q\right)' B(t,T), \tag{10}$$

$$\frac{dA(t,T)}{dt} = -B(t,T)' K^Q \theta^Q - \frac{1}{2}\sum_{j=1}^{3}\left(\Sigma' B(t,T) B(t,T)' \Sigma\right)_{j,j}, \tag{11}$$

with boundary conditions $B(T,T) = A(T,T) = 0$.

Affine Mortality Models - Gaussian and CIR Models

- Dynamics include Gaussian models (where there is a probability of negative mortality rates).
- These models
  - are readily estimated with (Gaussian) Kalman filter
  - are easily simulated
  - in practice, have very low probabilities of negative mortality rates.
- Dynamics also include square root process dynamics with potential to capture mortality heterogeneity (Cox-Ingersoll-Ross or CIR)
- These models
  - can capture mortality heterogeneity (gamma distributed mortality rates)
  - avoid probabilities of negative mortality rates
  - are more difficult to estimate with the Kalman filter (we use maximum quasi-likelihood).

Affine Mortality Models - Historical Mortality Rates

- To calibrate models to historical mortality rates (P measure) we need a link between the risk neutral dynamics and the historical dynamics - assumption for the price of risk.
- Assuming an essentially affine form for the risk premium (Duffee, 2002):

$$\Lambda_t = \begin{cases} \lambda^0 + \lambda^1 X_t, & \text{Gaussian processes;} \\ D\left(X_t, t\right)\lambda^0, & \text{the CIR model,} \end{cases} \tag{12}$$

where $\Lambda_t \in \mathbb{R}^{n\times 1}$, $\lambda^0 \in \mathbb{R}^{n\times 1}$ and $\lambda^1 \in \mathbb{R}^{n\times n}$.

- The SDEs for factors under the measure P are:

$$dX_t = \begin{cases} K^P\left[\theta^P - X_t\right]dt + \Sigma dW_t^P, & \text{Gaussian processes;} \\ K^P\left[\theta^P - X_t\right]dt + \Sigma D\left(X_t, t\right)dW_t^P, & \text{the CIR model.} \end{cases} \tag{13}$$

Affine Cohort Mortality Models - 3-factor Dynamics

The dynamics of the factors (Canonical and AFNS models) are (we estimate both P and Q measure dynamics):

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{14}$$

- The independent AFNS model (Christensen et al., 2011). The dynamics of the factors under the Q-measure are given by:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{15}$$

- The dependent Blackburn-Sherris model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}. \tag{16}$$

Affine Cohort Mortality Models - 3-factor Dynamics

- The dependent AFNS model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} \delta_{11} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{17}$$

- The CIR model

$$\begin{aligned} \begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} &= -\begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \left[\begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt \\ &+ \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{X_t^1} & 0 & 0 \\ 0 & \sqrt{X_t^2} & 0 \\ 0 & 0 & \sqrt{X_t^3} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \end{aligned} \tag{18}$$

Model Estimation - Kalman Filter

- We model mortality rates but observe deaths - we need a measurement equation capturing the effects of Poisson variation and heterogeneity.
- Mortality rate curve changes stochastically through time, driven by latent factors with trend and uncertainty - we need a state transition equation for the dynamics.
- We then filter the values of latent factors from historical data - deriving means and covariances which are functions of the parameters in the dynamics.
- We can then construct the likelihood (Gaussian) in terms of means and covariance (a function of parameters to be estimated).
- Then numerically select the parameter set that maximises the likelihood using an iterative process.

Model estimation - Measurement equation

- The measurement equation, based on the average force of mortality, for a given current age to different future survival ages is:

$$\bar{\mu}(t,T) = -\frac{B(t,T)'}{T-t}X_t - \frac{A(t,T)}{T-t} + \varepsilon_t, \tag{19}$$

  where the measurement error ε_t is independently and identically distributed noise and X_t are the latent factors.

- We can write this as

$$y_t = BX_t + A + \varepsilon_t. \tag{20}$$

Model estimation - Measurement equation

- For a 3-factor affine mortality model, the measurement equation with N observed average forces of mortality for ages $x+1$ to $x+N$ is:

$$\begin{pmatrix} \bar{\mu}(t,t+1) \\ \bar{\mu}(t,t+2) \\ \vdots \\ \bar{\mu}(t,t+N) \end{pmatrix} = \begin{pmatrix} -B^1(t,t+1) & -B^2(t,t+1) & -B^3(t,t+1) \\ -\frac{B^1(t,t+2)}{2} & -\frac{B^2(t,t+2)}{2} & -\frac{B^3(t,t+2)}{2} \\ \vdots & \vdots & \vdots \\ -\frac{B^1(t,t+N)}{N} & -\frac{B^2(t,t+N)}{N} & -\frac{B^3(t,t+N)}{N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \tag{21}$$

$$+ \begin{pmatrix} -A(t,t+1) \\ -\frac{A(t,t+2)}{2} \\ \vdots \\ -\frac{A(t,t+N)}{N} \end{pmatrix} + \begin{pmatrix} \varepsilon_t(1) \\ \varepsilon_t(2) \\ \vdots \\ \varepsilon_t(N) \end{pmatrix}, \tag{22}$$

Kalman Filter - State Transition Equation

- The state transition equation is a discretized version of the SDE dynamics and is given by:

$$X_t = \exp\left(-K^P\right) X_{t-1} + \eta_t, \tag{23}$$

where η_t is the transition error vector.

- The structure of stochastic error terms is assumed to be:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & H \end{pmatrix} \right], \tag{24}$$

where both the matrix H and matrix R are diagonal, with R being the covariance matrix of the measurement error and H being the covariance matrix of the transition error.

Kalman Filter - Error Assumptions

- The error matrix R for the state transitions, derived from the dynamics of the latent factors, in discrete time form is

$$R = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma \Sigma' e^{-\left(K^P\right)'(t-s)} ds. \tag{25}$$

- Poisson variation is captured in the diagonal of the covariance matrix H, assumed to have exponential form (reflecting exponential increase in mortality rate) given by

$$H(t, T) = \frac{1}{T-t} \sum_{i=1}^{T-t} \left[r_c + r_1 e^{r_2 i} \right], \tag{26}$$

where the values of r_c, r_1 and r_2 are estimated as part of the optimal parameter set.

Kalman Filter - Forecasting step

- Denote the (average) mortality rates at time t by $Y_t = (y_1, \ldots, y_t)$ and the parameters by ψ.
- In the forecasting step we first update the state, X_{t-1}, and its mean square error, Σ_{t-1},

$$X_{t|t-1} = E\left[X_t | Y_{t-1}\right] = \Phi\left(\psi\right) X_{t-1}, \tag{27}$$

$$\Sigma_{t|t-1} = \Phi\left(\psi\right) \Sigma_{t-1} \Phi\left(\psi\right)' + R\left(\psi\right), \tag{28}$$

where $\Phi = \exp\left(-K^P\right)$ and $R = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma \Sigma' e^{-\left(K^P\right)'(t-s)} ds$.

Kalman Filter - Forecasting step

- We then use the historical mortality rate information at time t to update the forecasts to obtain:

$$X_t = E\left[X_t | Y_t\right] = X_{t|t-1} + \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} \nu_t, \tag{29}$$

$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} B\left(\psi\right) \Sigma_{t|t-1}, \tag{30}$$

where

$$\nu_t = y_t - E\left[y_t | Y_{t-1}\right] = y_t - A\left(\psi\right) - B\left(\psi\right) X_{t|t-1}, \tag{31}$$

$$F_t = cov\left(\nu_t\right) = B\left(\psi\right) \Sigma_{t|t-1} B\left(\psi\right)' + H\left(\psi\right). \tag{32}$$

Kalman Filter - Log-likelihood

- The log-likelihood function is then computed as:

$$\log L\left(y_1, \ldots, y_t; \psi\right) = \sum_{t=1}^{T} \left(-\frac{N}{2} \log\left(2\pi\right) - \frac{1}{2} \log\left|F_t\right| - \frac{1}{2} \nu_t^{'} F_t \nu_t \right), \tag{33}$$

  where N is the number of ages with observed average forces of mortality.

- The log-likelihood function is maximized with respect to ψ to obtain the optimal parameter set using an iterative process
- Start with initial values, use Kalman filter to determine likelihood of data, update parameter values and iterate until maximum of likelihood is derived.

Forecasting Survival Curves

- Optimal forecasts, or best-estimate forecasts, are used to project average forces of mortality and survival probabilities fro future cohorts.
- At time t, the one step ahead forecast of the average force of mortality is

$$\bar{\mu}(t+1, T+1) = -\frac{B(t,T)'}{T-t} E[X_{t+1}|X_t] - \frac{A(t,T)}{T-t}, \tag{34}$$

  where $B(t,T)$ and $A(t,T)$ depend on the model.
- The forcasts of survival probabilities are then:

$$S(t+1, T+1) = \exp\left(B(t,T)' E[X_{t+1}|X_t] + A(t,T)\right). \tag{35}$$

Forecasting Survival Curves

- The factor dynamics under measure P in the independent Blackburn-Sherris model and the 3-factor independent AFNS model are the same. The conditional expectation of state variables for these two models are as follows:

$$E\left[X_{t+1}^1|X_t^1\right] = e^{-k_{11}^P}X_t^1, \quad E\left[X_{t+1}^2|X_t^2\right] = e^{-k_{22}^P}X_t^2, \\ E\left[X_{t+1}^3|X_t^3\right] = e^{-k_{33}^P}X_t^3. \tag{36}$$

- For the independent AFNS model, the conditional mean has the same structure but with $X_t = (L_t, S_t, C_t)$.
- The conditional mean of the CIR model for the mortality model is:

$$\begin{aligned} E\left[X_{t+1}^1|X_t^1\right] &= e^{-k_{11}^P}X_t^1 + \theta_1^P\left(1 - e^{-k_{11}^P}\right), \\ E\left[X_{t+1}^2|X_t^2\right] &= e^{-k_{22}^P}X_t^2 + \theta_2^P\left(1 - e^{-k_{22}^P}\right), \\ E\left[X_{t+1}^3|X_t^3\right] &= e^{-k_{33}^P}X_t^3 + \theta_3^P\left(1 - e^{-k_{33}^P}\right). \end{aligned} \tag{37}$$

Mortality Data - Estimating Mortality Models

- Mortality models are usually estimated with age-period historical data (life tables) - US data from 1933 to 2015 at ages from 50 to 100 is shown below.
- Cohort mortality rates are required in practice. Age-period models require forecasting of age-period curves and derivation of cohort mortality rates from the diagonal as the cohort ages.

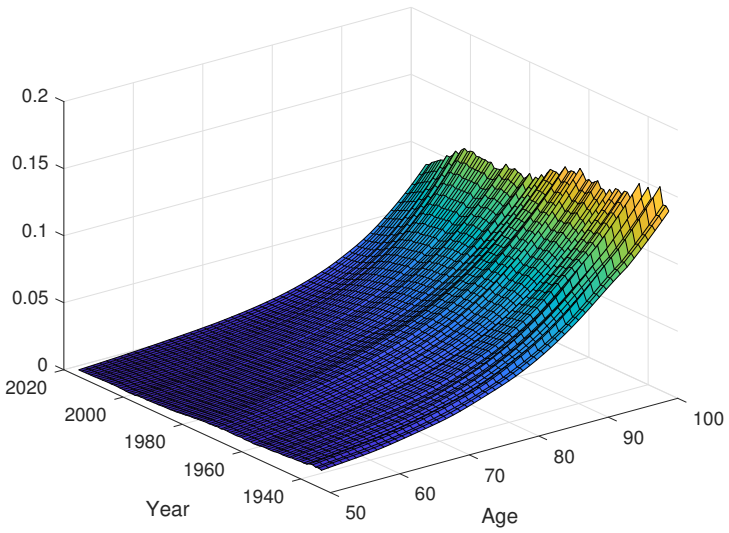


Figure 1: Average force of mortality of US Males Using Age-Period data, from 1933 to 2015

Mortality Data - Estimating Mortality Models

- Age-cohort data allows fitting of age-cohort curves directly but incomplete data for more recent cohorts - see US cohort data below.

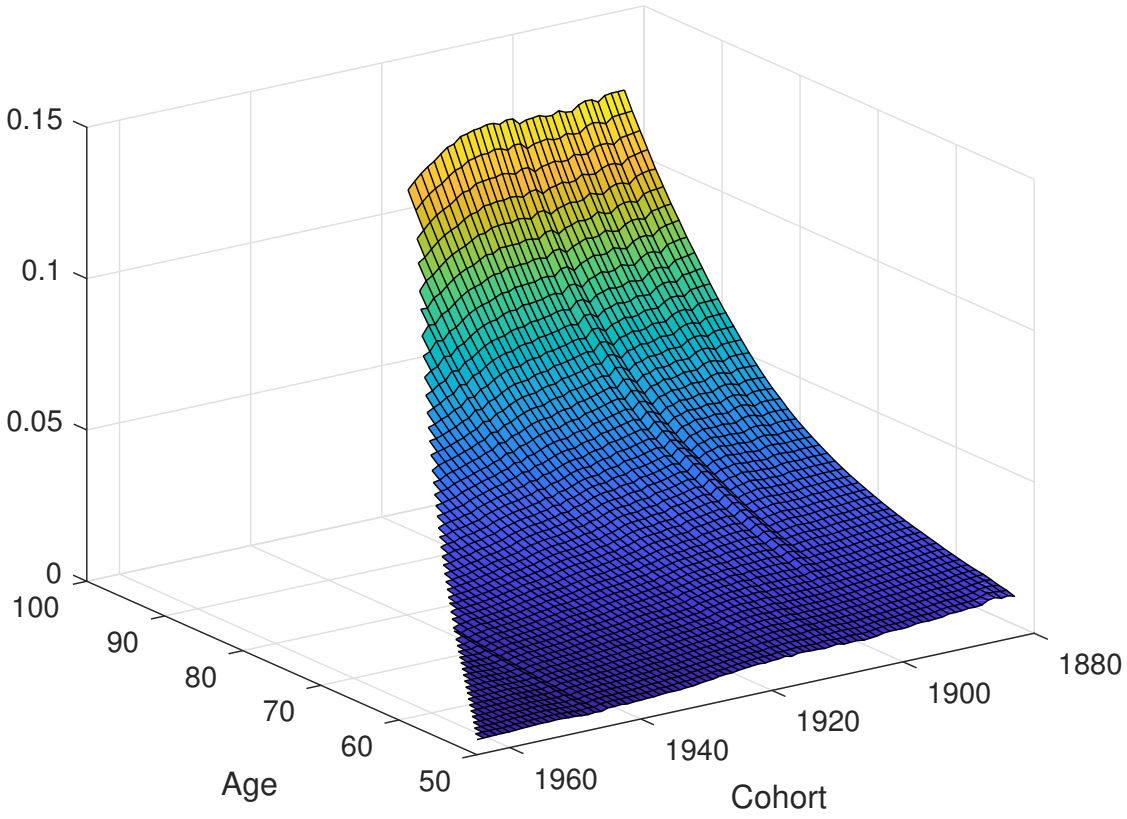


Figure 2: Average Force of Mortality for Males Born from 1883 to 1965

Mortality Data - Estimating Mortality Models

- Complete age-cohort data for cohorts born in earlier years. US complete cohort data below.

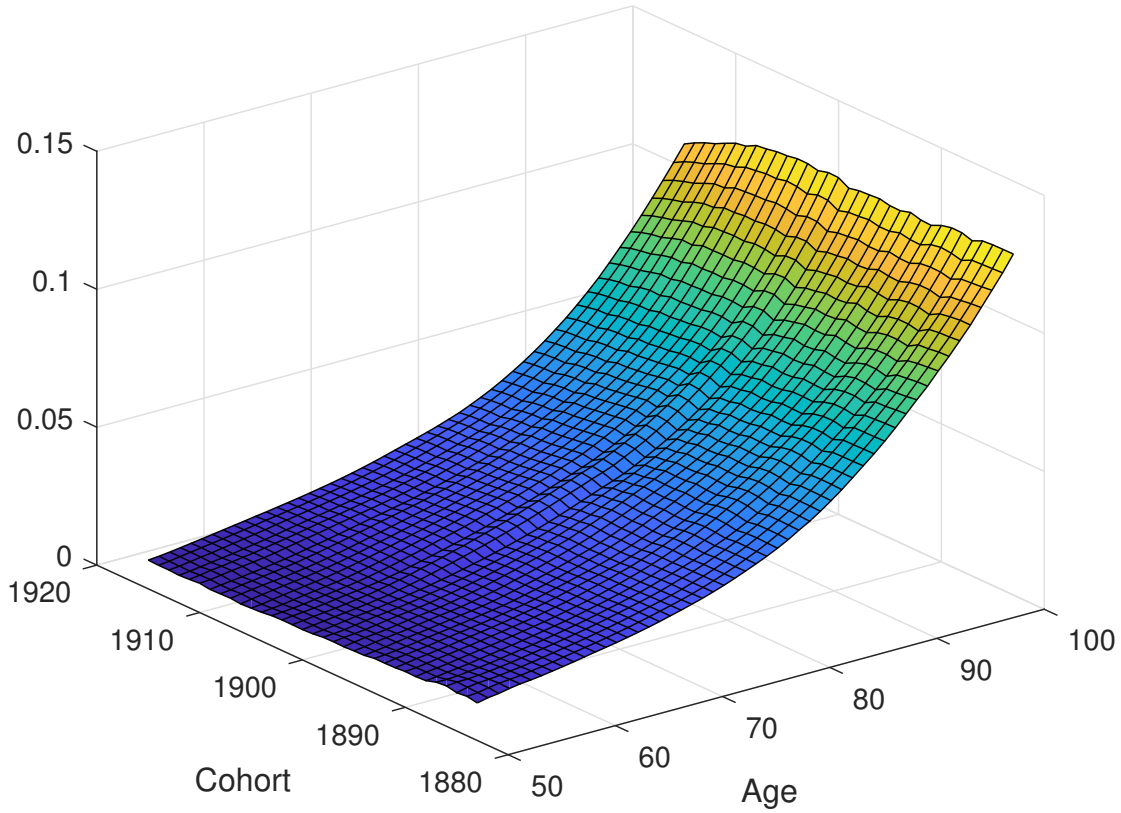


Figure 3: Average Force of Mortality for Males Born from 1883 to 1915

Mortality Data - Calibrating Affine Mortality Models

- US mortality age-cohort data from the Human Mortality Database (2017) (HMD) to calibrate and compare the mortality models.
- Mortality data of males from ages 50 to 100 for the cohorts born from 1883 to 1915.
- Historical survival probability, $S^i(x; t, T)$, and the historical average forces of mortality $\bar{\mu}^i(x; t, T)$ over the period $\tau = T - t$ for each cohort i aged x at time t from the data, using:

$$S^i(x; t, T) = \prod_{s=1}^{T-t} \left[1 - q^i(x + s - 1, t + s - 1)\right], \tag{38}$$

$$\bar{\mu}^i(x; t, T) = -\frac{1}{T - t} \log\left[S^i(x; t, T)\right], \tag{39}$$

where $q^i(x, t)$ is the one year death probability for an individual aged x at time t in cohort i.

Affine Cohort Mortality Models - Goodness of Fit

Table 1: Comparison of Affine Mortality Models

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent-Factor	Dependent-Factor	Independent-Factor	Dependent-Factor	
Log Likelihood	9896.419	9938.696	9665.801	9887.878	10045.70
RMSE	0.00250	7.601e-04	6.856e-04	9.160e-04	5.227e-04
No. of Parameters	12	18	10	13	18
AIC	-19570.837	-19643.392	-19113.602	-19551.757	-19857.40
BIC	-18968.292	-19008.277	-18521.914	-18943.783	-19222.29
Probability of Negative Mortality	0.02700	1.011e-32	1.722e-31	4.34e-14	-

- AFNS model fits historical age-cohort data well. Low negative mortality probabilities. CIR the best fit.

Canonical Age-Period Mortality Curve Factors

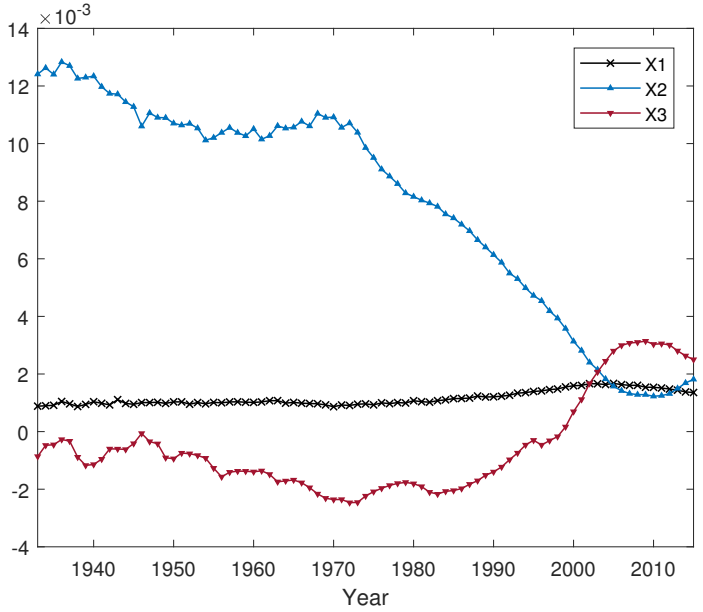
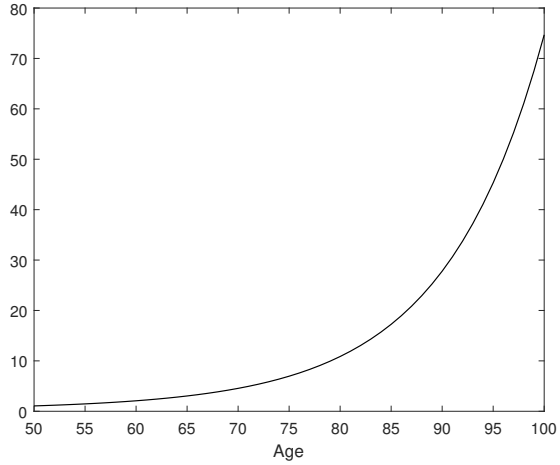


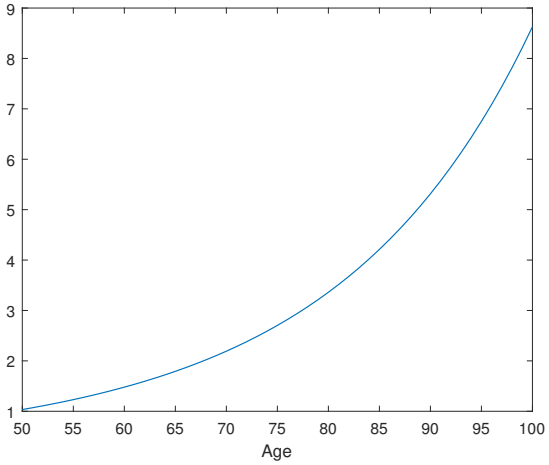
Figure 4: Factors in the Blackburn-Sherris Model with Age-Period Data

- Factor X_2 captures trend change around 1970's.

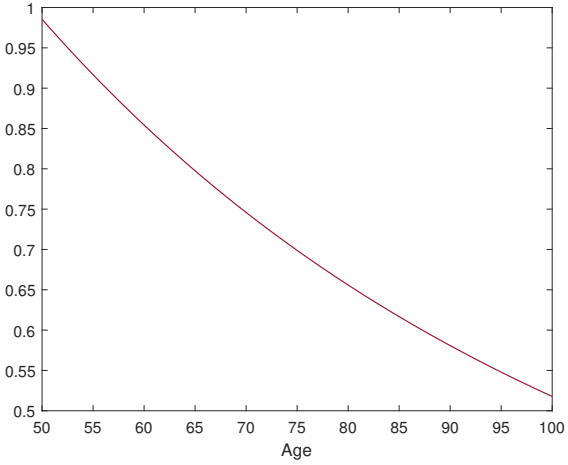
Canonical Age-Period Mortality Curve Factor Loadings



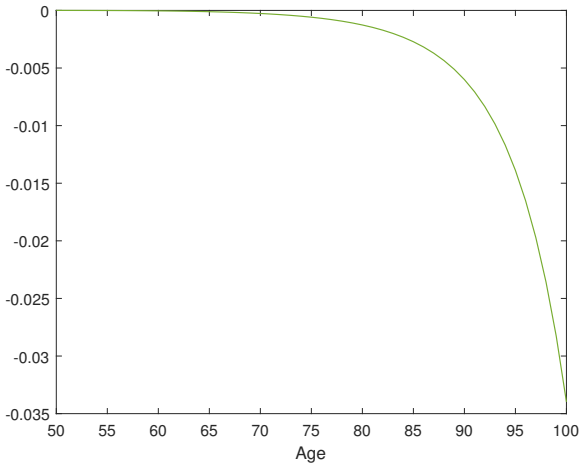
(a) $B_1 = -\frac{B^1(t,T)}{T-t}$



(b) $B_2 = -\frac{B^2(t,T)}{T-t}$



(c) $B_3 = -\frac{B^3(t,T)}{T-t}$



(d) $A = -\frac{A(t,T)}{T-t}$

AFNS Age-Cohort Mortality Curve Factors

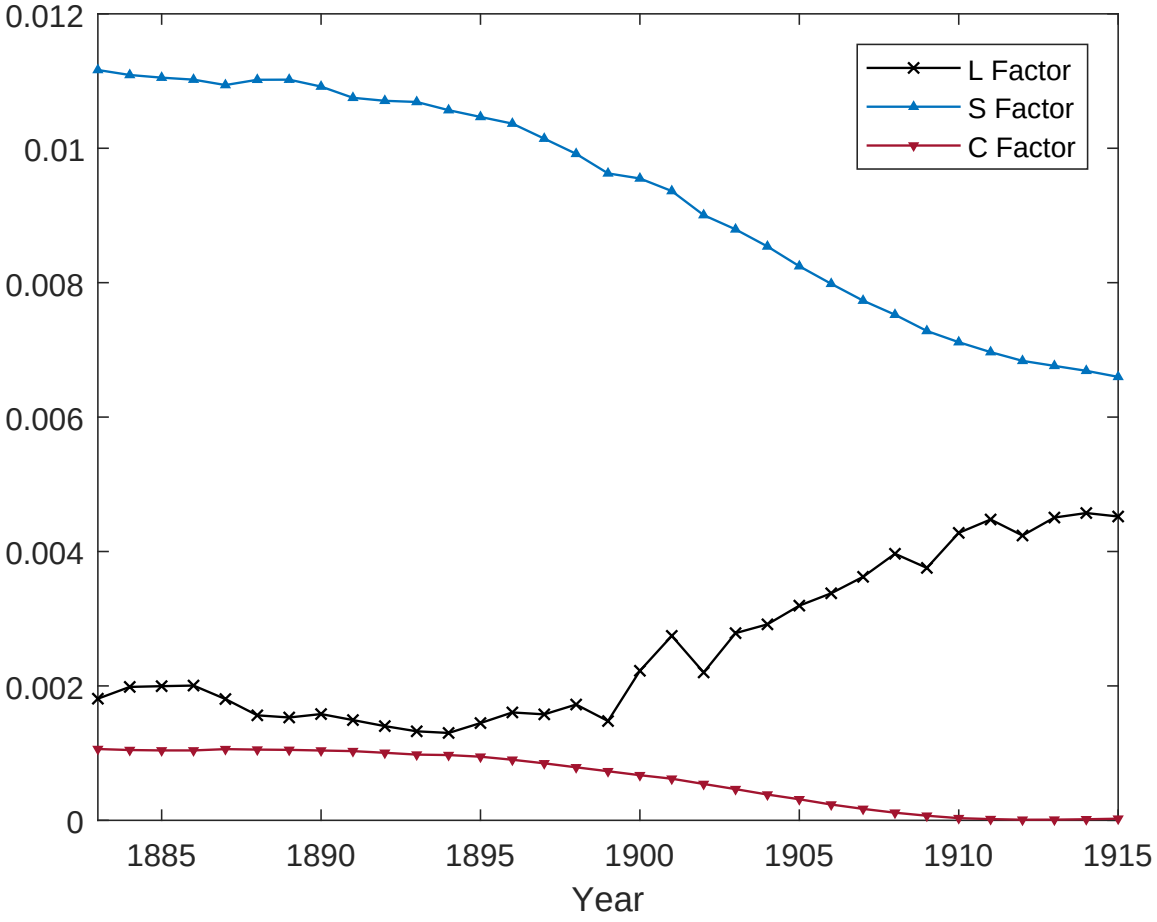
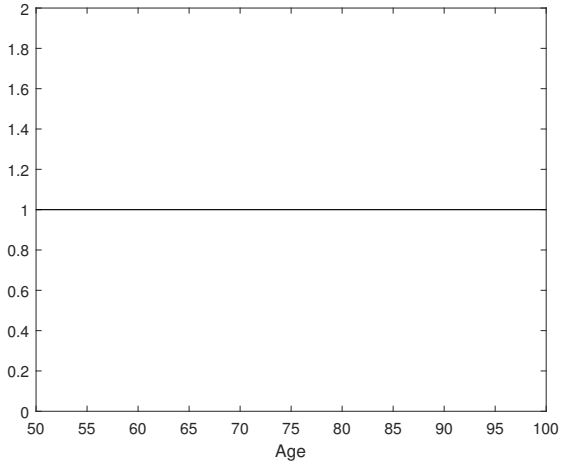
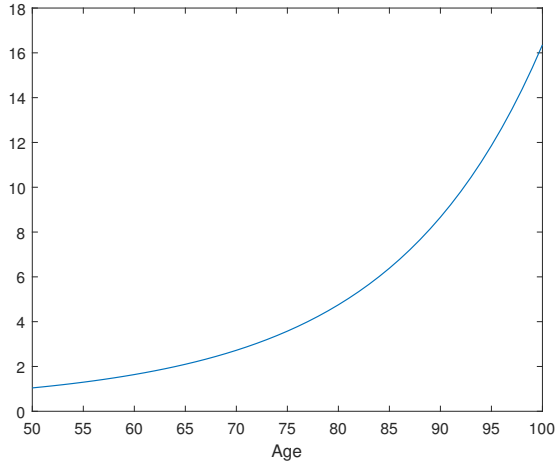


Figure 6: Factors in the Independent AFNS Model

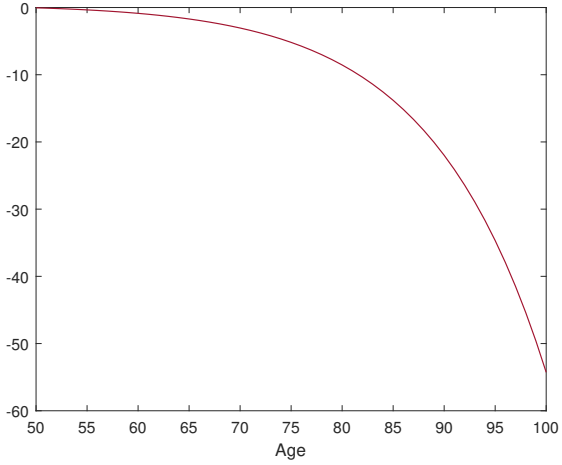
AFNS Age-Cohort Mortality Curve Factor Loadings



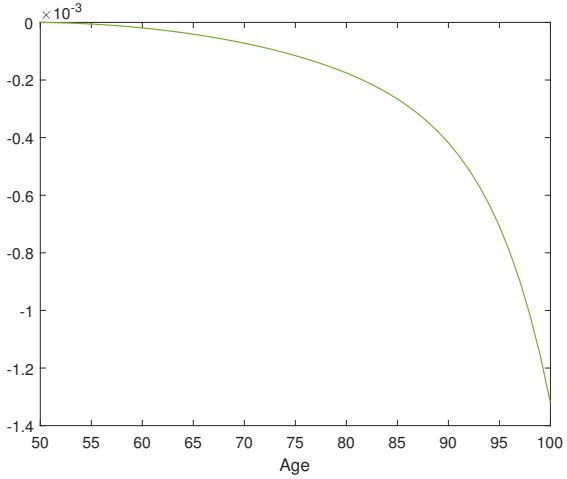
(a) $B_1 = -\frac{B^1(t,T)}{T-t}$



(b) $B_2 = -\frac{B^2(t,T)}{T-t}$

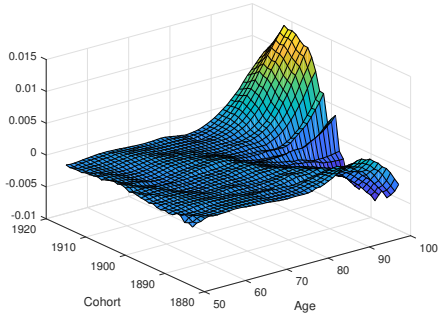


(c) $B_3 = -\frac{B^3(t,T)}{T-t}$

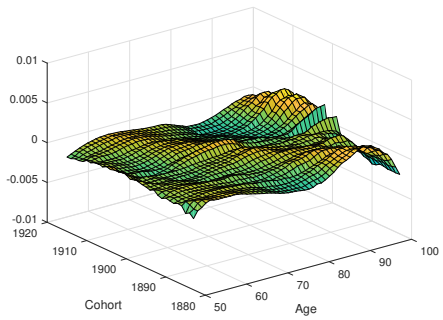


(d) $A = -\frac{A(t,T)}{T-t}$

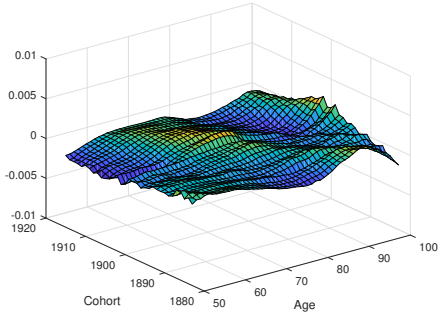
Affine Cohort Mortality Models - Residual Analysis



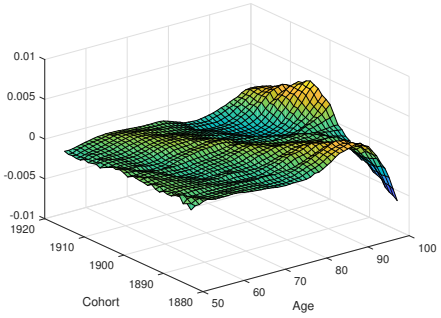
(a) The Independent Blackburn-Sherris Model



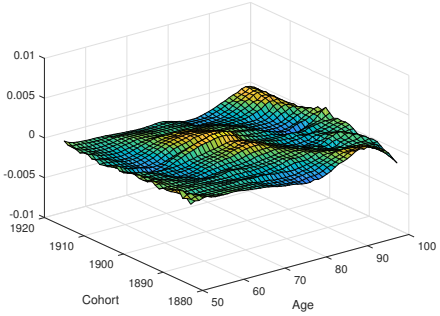
(b) The Dependent Blackburn-Sherris Model



(c) The Independent AFNS Model



(d) The Dependent AFNS Model



(e) The CIR Model

Figure 8: Residuals of Affine Mortality Models

Affine Cohort Mortality Models - MAPE

Mean Absolute Percentage Error (MAPE) for each age, across all cohorts

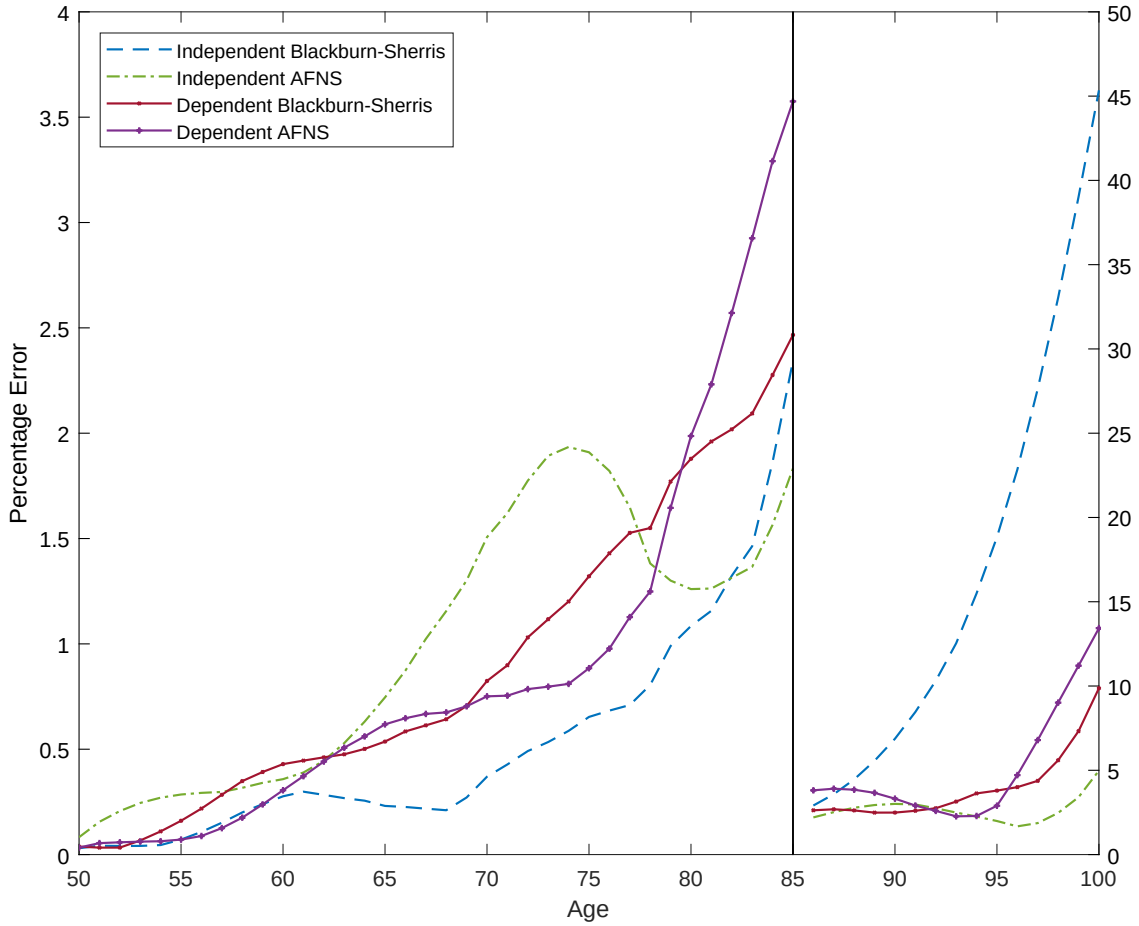


Figure 9: The Models with Gaussian Processes

Affine Cohort Mortality Models - MAPE

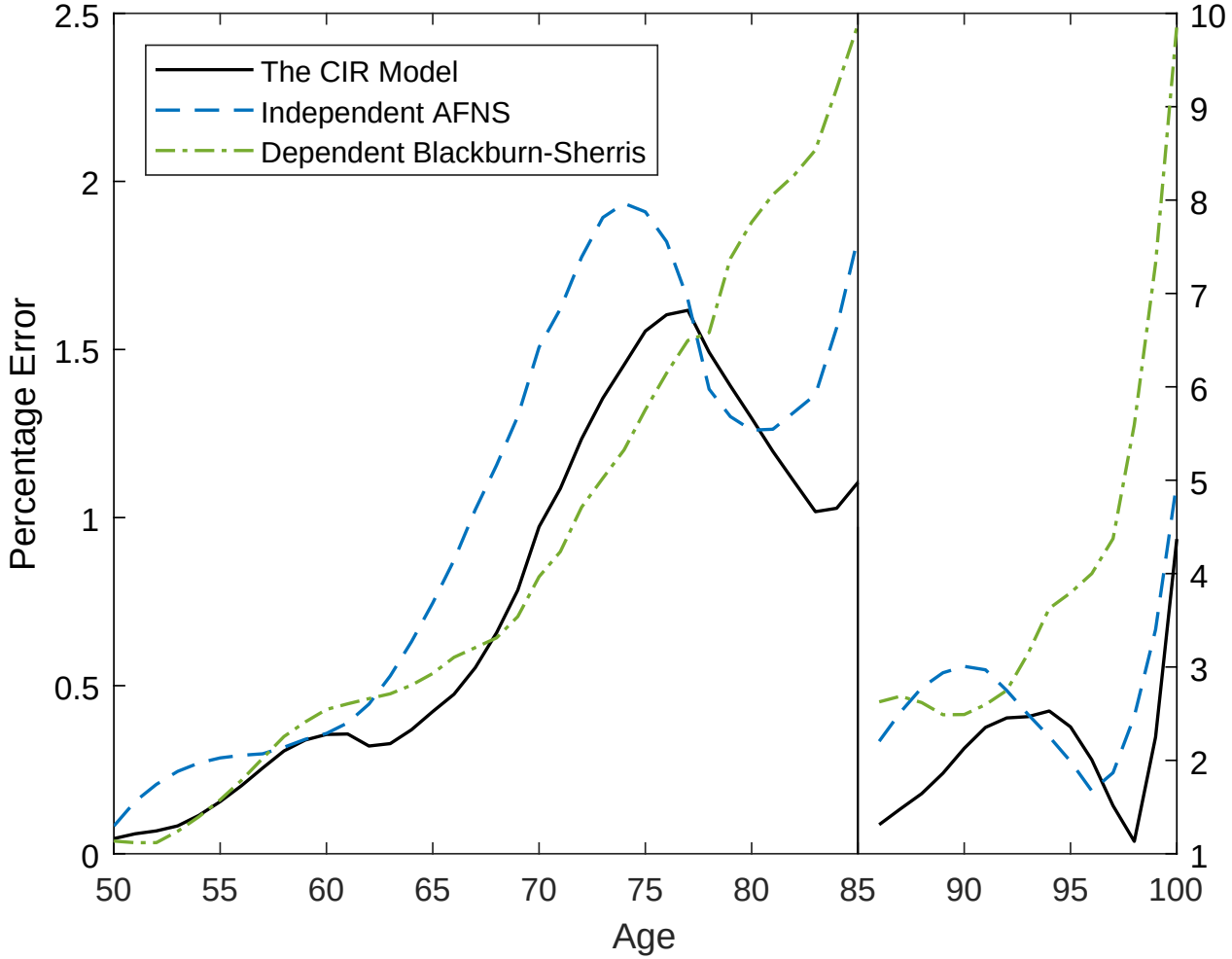


Figure 10: The CIR Model, the Dependent Blackburn-Sherris Model and the Independent AFNS Model

Affine Cohort Mortality Models - Forecast RMSE

Table 2: RMSE by Comparing the Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent	Dependent	Independent	Dependent	
RMSE	0.03197	0.00726	0.00668	0.00754	0.01835

- AFNS model performs well. CIR model has poorer forecasting performance. Forecast for a single cohort.

Affine Cohort Mortality Models - Forecast RMSE

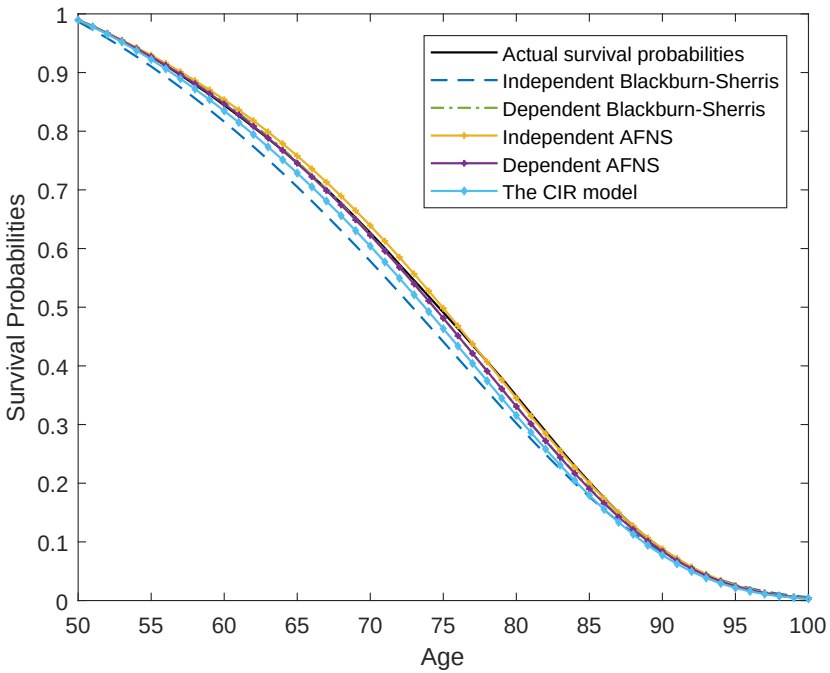


Figure 11: Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

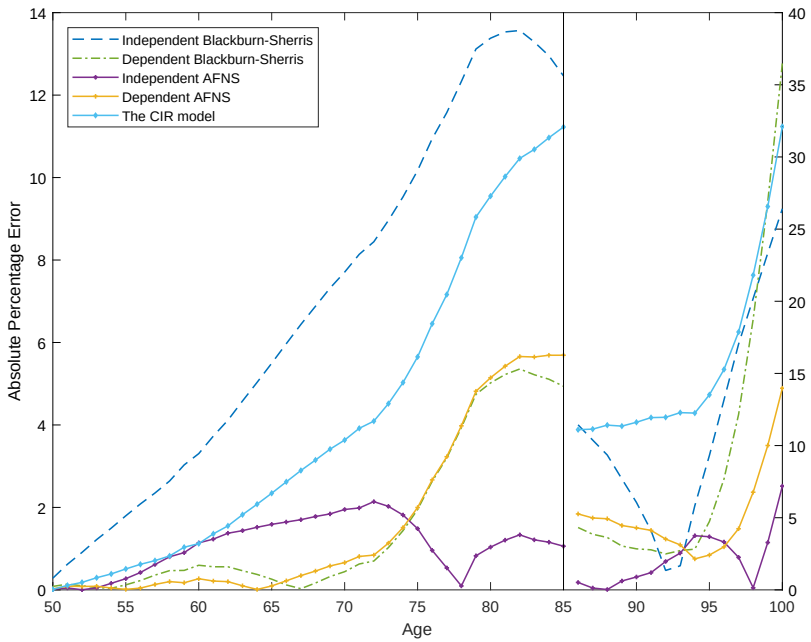


Figure 12: Absolute Percentage Errors between Actual and Best-Estimate Survival Probabilities

Multi-cohort Continuous Time model

- We cover the multi-cohort mortality model in Xu et al. (2019b).
- This is a three-factor affine mortality model, with mortality intensity process for each cohort i aged $x+t$ at time t is

$$\mu^i(x,t) = X_1(t) + X_2(t) + Z^i(t), \tag{40}$$

where $X_1(t)$, $X_2(t)$ are two common factors and $Z^i(t)$ is a cohort specific factor.

- Under best-estimate measure $\bar{Q}$, the state variables $(X_1(t), X_2(t), Z^i(t))$ have the following dynamics

$$dX_j(t) = -\phi_j X_j(t)dt + \sigma_j dW_j^{\bar{Q}}(t), j = 1, 2 \tag{41}$$

$$dZ^i(t) = -\phi_3^i Z^i(t)dt + \sigma_3^i dW_3^{\bar{Q},i}(t) \tag{42}$$

where ϕ_1, ϕ_2, ϕ_3^i, σ_1, σ_2 and σ_3^i are constants, and $W_1^{\bar{Q}}(t)$, $W_2^{\bar{Q}}(t)$ and $W_3^{\bar{Q},i}(t)$ are standard Wiener processes under $\bar{Q}$.

Multi-cohort Survival Probability

- The best-estimate survival probability, $S^{\bar{Q},i}(x,t,T)$ for cohort i aged x at time t over duration $T-t$, has a closed-form solution:

$$\begin{aligned} S^{\bar{Q},i}(x,t,T) &= E^{\bar{Q}}[e^{-\int_t^T \mu^i(x,s)ds}|\mathcal{F}_t] \\ &= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3^i(t,T)Z^i(t)+A^i(t,T)}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} B_1(t,T) &= -\frac{1-e^{-\phi_1(T-t)}}{\phi_1}, \\ B_2(t,T) &= -\frac{1-e^{-\phi_2(T-t)}}{\phi_2}, \\ B_3^i(t,T) &= -\frac{1-e^{-\phi_3^i(T-t)}}{\phi_3^i}, \\ A^i(t,T) &= \frac{1}{2}\sum_{j=1}^{2}\frac{\sigma_j^2}{\phi_j^3}\left[\frac{1}{2}(1-e^{-2\phi_j(T-t)})-2(1-e^{-\phi_j(T-t)})+\phi_j(T-t)\right] \\ &\quad + \frac{1}{2}\frac{(\sigma_3^i)^2}{(\phi_3^i)^3}\left[\frac{1}{2}(1-e^{-2\phi_3^i(T-t)})-2(1-e^{-\phi_3^i(T-t)})+\phi_3^i(T-t)\right]. \end{aligned} \tag{44}$$

Price of Mortality Risk

- Best-estimate measure $\bar{Q}$ and the parameters are estimated using historical age-cohort mortality data.
- The pricing risk-neutral measure Q has affine market price of risk specification in Dai and Singleton (2000) and Duffee (2002).

  $\Lambda^i = (\lambda_{\mu,1}, \lambda_{\mu,2}, \lambda^i_{\mu,3})^T$ is vector of market prices of risk associated with cohort i
- Market prices of longevity risk, $\lambda_{\mu,1}$ and $\lambda_{\mu,2}$ assumed the same across cohorts - common factors, but $\lambda^i_{\mu,3}$ differs by cohort.
- From Girsanov's Theorem

$$dW_j^Q(t) = dW_j^{\bar{Q}}(t) + \lambda_{\mu,j} dt, j = 1, 2 \tag{45}$$

$$dW_3^{Q,i}(t) = dW_3^{\bar{Q},i}(t) + \lambda^i_{\mu,3} dt \tag{46}$$

where $W_1^Q(t)$, $W_2^Q(t)$ and $W_3^{Q,i}(t)$ are standard Wiener processes under the risk-neutral measure Q.

Calibrating Price of Risk - Blackrock CORI

- BlackRock introduced the CoRI Indexes in June 2013 to help investors estimate and track the cost of \$1 of annual lifetime income at retirement.
- The CoRI consists of twenty indexes corresponding to twenty cohorts born from 1941 to 1960 in U.S.
- For cohorts with an age below 65 the index is the discounted cost of purchasing inflation-adjusted lifetime retirement income at age 65, and for other cohorts it is the cost of purchasing inflation-adjusted retirement income for remaining life.
- The CoRI indexes are constructed based on real-time market data, do not include any fees or premium taxes that would be associated with the price of an annuity. Available on NYSE.
- Investors can use the CoRI index as a risk metric directly or invest in the BlackRock CoRI Funds that track the index.

Multi-cohort Mortality Model Estimation

- We use U.S. male mortality data from Human Mortality Database for 1934 to 2013, aged 50 to 100, cohorts born 1884 to 1913.
- Restructured on a cohort basis.
- Sample survival probability for cohort i aged x at time t over duration $T - t$ is

$$\tilde{S}^i(x, t, T) = \prod_{s=1}^{T-t} (1 - \tilde{q}_x^i(t + s - 1)), \tag{47}$$

  where $\tilde{q}_x^i(t)$ is the observed death rate at time t.
- Corresponding sample average force of mortality is

$$\tilde{\mu}^i(x, t, T) = -\frac{1}{T - t} \log \tilde{S}^i(x, t, T). \tag{48}$$

Multi-cohort Mortality Model Estimation

- Figure 13 shows the average force of mortality in U.S. for cohorts born between 1884 and 1913, ages 50 to 100.

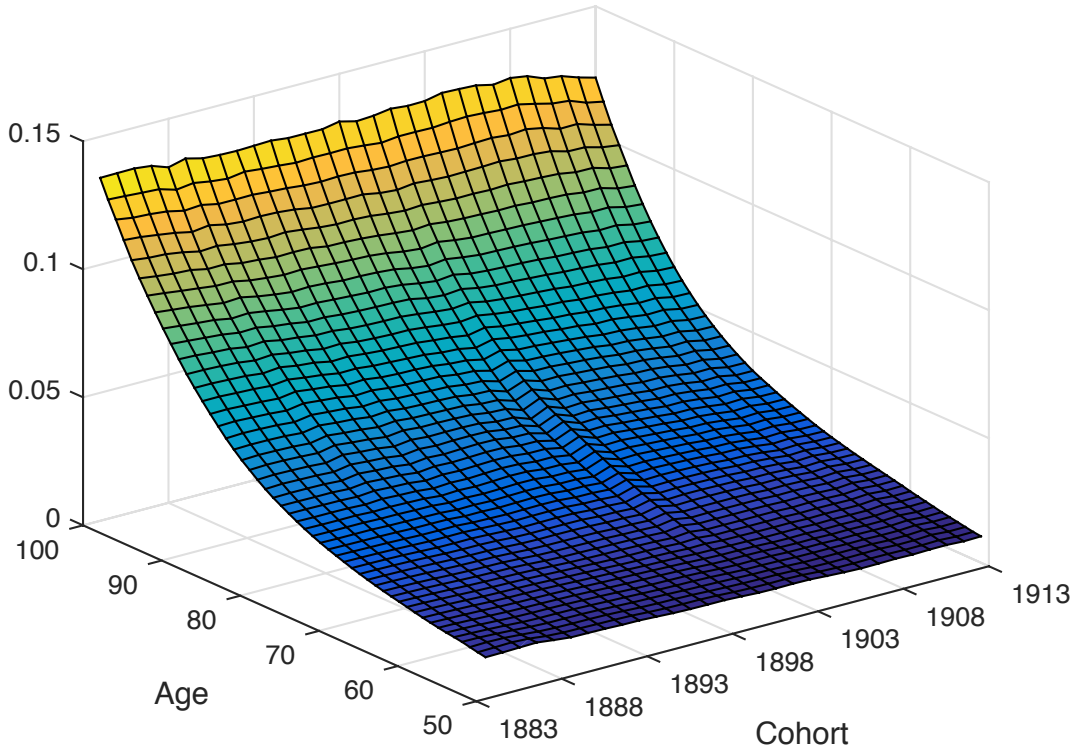


Figure 13: Male average force of mortality in U.S. for cohorts born between 1884 and 1913, ages 50 to 100.

Multi-cohort Mortality Model Estimation

- We use the Kalman filter estimation for the two common factors - parameter estimates shown in Table 3.

ϕ_1	-0.14313
ϕ_2	-0.07904
σ_1	0.00006
σ_2	0.00018
$\varepsilon_1(\times 10^7)$	2.74881
$\varepsilon_2(\times 10^7)$	1.99699
Log likelihood	24440
RMSE	0.00051

Table 3: Kalman filter parameter estimates, log likelihood and RMSE.

Multi-cohort Mortality Model Estimation

- Parameters for the cohort specific factors are estimated by minimising calibration error in the model after including the cohort factor.
- Grouping by 10 cohorts.
- Cohort parameters are shown in Table 4.

i cohort	ϕ_3^i	σ_3^i	Z^i
1884-1893	0.06791	0.00558	0.00163
1894-1903	0.05228	0.00719	0.00106
1904-1913	0.05463	0.00122	-0.00079

Table 4: Estimation results for cohort specific factors with a 10-year interval.

Implied Price of Longevity Risk based on CORI indices

- The calibrated risk premiums are shown in Table 5.

$\hat{\lambda}_{\mu,1}$	$\hat{\lambda}_{\mu,2}$	$\hat{\lambda}^1_{\mu,3}$	$\hat{\lambda}^2_{\mu,3}$
0.3601	0.0892	0.1099	0.0973

Table 5: Calibrated market price of longevity risk.

- Note that all prices of risk are positive
- Prices of risk are consistent with and of similar magnitude to other studies.
- There are similar prices of risk for each cohort risk factor.

Implied Price of Longevity Risk

Table 6 shows model risk-neutral index levels and the values of CoRI indexes published by BlackRock on 31 March 2015.

Cohort	Age	Name	Index level	Risk-neutral index level	Difference
1941	74	CoRI Index 2005	15.26	15.96	0.70
1942	73	CoRI Index 2006	15.94	16.41	0.47
1943	72	CoRI Index 2007	16.61	16.89	0.28
1944	71	CoRI Index 2008	17.28	17.40	0.12
1945	70	CoRI Index 2009	17.95	17.93	-0.02
1946	69	CoRI Index 2010	18.60	18.48	-0.12
1947	68	CoRI Index 2011	19.26	19.05	-0.21
1948	67	CoRI Index 2012	19.93	19.64	-0.29
1949	66	CoRI Index 2013	20.59	20.24	-0.35
1950	65	CoRI Index 2014	21.25	20.85	-0.40
1951	64	CoRI Index 2015	22.19	21.03	-1.16
1952	63	CoRI Index 2016	21.50	20.66	-0.84
1953	62	CoRI Index 2017	20.93	20.29	-0.64
1954	61	CoRI Index 2018	20.35	19.93	-0.42
1955	60	CoRI Index 2019	19.73	19.57	-0.16
1956	59	CoRI Index 2020	19.11	19.21	0.10
1957	58	CoRI Index 2021	18.52	18.85	0.33
1958	57	CoRI Index 2022	17.98	18.50	0.52
1959	56	CoRI Index 2023	17.50	18.13	0.63
1960	55	CoRI Index 2024	16.93	17.77	0.84

*The CoRI Index data is obtained from BlackRock on 31 March 2015.

Table 6: CoRI index level and the risk-neutral index level at the market prices of longevity risk given in Table 5.

Wrap Up

- Introduced continuous-time mortality models including an AFNS cohort mortality model with interpretable latent stochastic factors for level, slope and curvature of the survival curve.
  - The model is based on factor loadings multiplied by (latent) factors, where the factors are equivalent to stochastic parameters and the factor loadings determine how the factors impact different ages.
- Outlined the dynamics of the mortality rates and the affine survival curves.
- Outlined the estimation of the models using the Gaussian Kalman filter.
- Outlined how the models can capture Poisson variation in the estimation.
- Introduced a multi-cohort continuous time mortality model with cohort factors with calibrated prices of risk to Blackrock CORI data.

Some comments on the Models

- Empirical results show that the independent-factor AFNS cohort mortality model is:
  - Parsimonious, captures the variation in cohort mortality rates in US data, producing a better fit at older ages than the independent-factor Blackburn-Sherris model, and has good predictive performance.
  - As a Gaussian model it is easy to implement with closed-form expressions for survival probabilities, ieasy to estimate using the Kalman filter, and can be readly implemented using simulation.
  - Negative mortality rates have very low probability.
  - Factors that better fit historical data dynamics and have intuitive factor interpretation (Level, Slope, Curvature).
  - Multi-factor age-cohort models, and particularly the AFNS model, is well suited for financial and insurance applications.
- Work to be done: incorporating imcomplete cohorts inrto estimation, better capturing Poisson variation, age-dependence in trend and covariance, CIR model estimation and forecasting.

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