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Pricing of GMWB Options in Variable Annuities under Stochastic Volatility, Stochastic Interest Rates and Stochastic Mortality via the Componentwise Splitting Method

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Abstract

This paper values Guaranteed Minimum Withdrawal Benefit (GMWB) riders embedded in variable annuities assuming that the underlying fund dynamics evolve under the influence of stochastic interest rates, stochastic volatility, stochastic mortality and equity risk. The valuation problem is formulated as a partial differential equation (PDE) which is solved numerically by employing the operator splitting method. Sensitivity analysis of the fair guarantee fee is performed with respect to various model parameters. We find that (i) the fair insurance fee charged by the product provider is an increasing function of the guarantee rate; (ii) the GMWB price is higher when stochastic interest rates and volatility are incorporated in the model, compared to the case involving static interest rates and volatility; (iii) the GMWB price behaves non-monotonically with changing volatility of variance parameter; (iv) the fair fee increases with increasing volatility of interest rates parameter, and increasing correlation between the underlying fund and the interest rates; (v) the fair fee increases when the speed of mean-reversion of stochastic volatility or the average long-term volatility increase; (vi) the GMWB fee decreases when the speed of mean-reversion of stochastic interest rates or the average long-term interest rates increase. We investigate both, static and dynamic (optimal) policyholder's withdrawal behaviours and present the optimal withdrawal schedule as a function of the withdrawal account and the investment account for varying volatility and interest rates. When incorporating stochastic mortality we find that its impact on the fair guarantee fee is rather small. Our results demonstrate the importance of correct quantification of risks embedded in GMWB riders, and provide guidance to product providers on optimal hedging of various risks associated with the contract.

JEL Classification: C63, G12, G22, G23

Keywords: Variable Annuities; Guaranteed Minimum Withdrawal Benefit (GMWB) riders; stochastic mortality; stochastic volatility; stochastic interest rates; componentwise splitting method

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1 Introduction

Variable annuities (VAs) introduced in 1970s in the US have grown dramatically over the past decades as a response to a growing demand for products that can manage longevity risk attributed to an ageing population.[1] Traditionally, the main form of pension has been the pay as you go (PAYG) systems where governments and public sector funds would foot the retirement bills. Most jurisdictions are currently moving from public sector funded schemes such as the defined benefit (DB) schemes to defined contribution (DC) schemes where individuals are exposed to longevity risk in retirement. As such, there has been various product innovations offered by practitioners and academics, aiming to address challenges related to longevity improvements. Variable annuities (VAs) have emerged as one of the pillars of retirement income streams. The VA are insurance contracts that allow policyholders to invest their retirement savings in mutual funds. The popularity of VA contracts can be attributed to its many attractive characteristics, such as policyholders gain exposure to the equity markets with benefits based on the performance of the underlying funds, along with return guarantees as well as tax advantages. Furthermore, policyholders can elect guarantees to provide minimum benefits with the payment of guarantee fees.

The level of income provided by VA contracts depends on the performance of investments chosen by the policyholder at the inception of the contract. The guarantees embedded in VAs offer protection against the scenario when policyholders outlive their assets. These guarantees exhibit financial option-like features. There are two major classes of guarantees: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs). GMDBs are usually offered during the accumulation phase; they provide guaranteed payments of the accumulated value of premiums to beneficiaries in the event of untimely death of the policyholder. An accumulation phase is a period during which the policyholder makes contributions to a retirement savings account. GMLBs provide principal and/or income guarantees to protect the policyholder's income from declining during the annuitization/retirement phase. The retirement phase is a period over which the policyholder withdrawals a pension from a given account. GMLBs can be further categorized into three subclasses, namely, the GMxB, where "x" stands for maturity (M), income (I) and withdrawal (W). A GMMB guarantees the return of the premium payments made by the policyholder or a higher stepped-up value at the end of the accumulation period.

[1]Improvements in mortality observed across the entire developed world lead to a decrease in the old-age support ratio (i.e. the number of people aged 15 to 64 years old per person aged 65 years old or over, which relates the number of individuals that are capable of providing economic support to the number of older people dependent on the support of others (United Nations, 2013).

A GMIB guarantees a lifetime income stream when a policyholder annuitizes the GMMB regardless of the underlying investment performance. A GMWB guarantees a stream of income payments, regardless of the contract account value and payments can be guaranteed for a specified period or for a lifetime of the policyholder. This paper will focus on the valuation of GMWBs that have been identified by LIMRA (2012) as the products having the highest election rate by investors.

Rentz (1972) and Greene (1973) provide early literature comparing VAs with other insurance contracts. Brennan and Schwartz (1976) value equity-linked life insurance policies (akin to VAs) using finite difference techniques. A formal framework for valuing GMWBs is presented in Milevsky and Salisbury (2006) who provide mathematically rigorous derivations for both, static and dynamic policyholders' withdrawal strategies. Under the static withdrawal case, the authors show that a GMWB rider can be split into a Quanto Asian put option and a term annuity certain. For the dynamic withdrawal case, Milevsky and Salisbury (2006) show that the valuation of the guarantee proceeds by formulating the optimal stopping time problem.

Dai et al. (2008) apply the Hamilton-Jacobi-Bellman (HJB) approach to solve the stochastic control problem in cases of continuous and discrete dynamic withdrawals. Chen et al. (2008) analyse a more generalised framework of sub-optimal behaviour of the policyholder using the approach suggested in Ho et al. (2005). Both Chen et al. (2008) and Huang and Kwok (2014) analyse how optimal withdrawal strategies depend on the relationship between values of the guarantee account and the investment account. Bauer et al. (2008) present a general framework for consistent modelling and simultaneous pricing of variable annuities with various guarantee benefits; the valuation of the contracts is performed assuming deterministic, probabilistic and stochastic policyholders' behaviour. Liu (2010) derive semi-static strategies for hedging GMWB with periodic static withdrawals and show that such strategies outperform delta-hedging if the value of the underlying asset jumps randomly. The generalised framework with stochastic interest rates, stochastic volatility and jumps has been considered by various authors (Chen et al., 2008; Donnelly et al., 2014; Liu, 2010; Luo and Shevchenko, 2016; Peng et al., 2012). Yang and Dai (2013) and Dai et al. (2015) consider stochastic mortality in the context of GMWB contracts and conclude that ignoring mortality risk leads to overpricing of the rider. In the literature on GMWBs it is predominantly assumed that the policyholder is charged a constant fee by the insurer, which is proportional to the value of an individual's investment account. Delong (2014) relaxes this assumption and considers pricing and hedging of variable annuities with state-dependent fees. The non-constant fees are argued to be beneficial to both, the insurers and the policyholders as there would be less incentive for the policyholder to surrender the policy early under the strong market conditions, and the insurer will be capable of hedging the risks more accurately.

The objective of this paper is to incorporate several risk factors into the valuation of GMWBs embedded in VAs, namely, stochastic volatility, stochastic interest rates and stochastic mortality to highlight the impact of such factors on the value of the VA contract. We devise a numerical technique for pricing GMWBs assuming that the underlying investment fund evolves under the influence of stochastic volatility and stochastic interest rates. Our objective is to find the fair insurance fee and to analyse price sensitivity with respect to various input parameters associated with different types of risks that are incorporated in our modelling framework simultaneously. Two withdrawal strategies related to the policyholder's behaviour, namely; the static and the dynamic (optimal) withdrawals are incorporated in the valuation framework. We adopt an operator splitting approach, which has proved to be computationally efficient in the physical sciences and mathematical finance when applied to pricing financial options (Duffy, 2006; Ikonen and Toivanen, 2007, 2009; Jeong and Kim, 2013; Yanenko, 1971). More precisely, and operator splitting approach is used for solving multi-dimensional pricing partial differential equations (PDEs) in order to find an expected value of the investment portfolio and estimate the fair value of the insurance fee.

Our results show that the fair insurance fee charged by insurance providers is an increasing function of the guarantee rate, which is the consequence of higher risks associated with selling the guarantees with higher guarantee rates. The increase in the fair insurance fee is more pronounced for the static withdrawal strategy compared to the dynamic withdrawal strategy. The GMWB price is higher when stochastic interest rates and stochastic volatility are introduced to the model, compared to when deterministic parameters are used. We document that the fair fee increases with increasing volatility of interest rates, and increasing correlation between the underlying fund and the interest rate. The GMWB price is an increasing function of the speed of mean-reversion of stochastic volatility and a decreasing function of the speed of mean-reversion of stochastic interest rates. The fair guarantee fee also increases with increasing average long-term volatility; and decreases with increasing average long-term interest rates. Generally, the GMWB pricing function behaves non-linearly with changing volatility of variance parameter. The impact of mortality on the fair insurance fee is rather small overall; but is higher when the withdrawal rates are higher. When incorporating stochastic mortality we find that the fair insurance fees for GMWBs are lower compared to the case of no mortality; but generally, the impact of mortality is rather small. When assessing the impact of model parameters we present the optimal withdrawal schedule as a function of the guarantee account and the investment account for varying volatility and interest rates. Our results provide a comprehensive analysis of benefits and risks embedded in GMWBs, and could be of potential interest to insurance companies offering GMWBs.

The remainder of the paper is structured as follows: Section 2 formulates the valuation framework for a VA contract embedded with a GMWB rider. More specifically, Section 2.2 summarises the procedure for determining the fair guarantee fee in the case where policyholder makes dynamic withdrawals. Section 3 outlines the operator splitting method used to solve the pricing PDE, which includes sections on space and time discretisation and the actual componentwise splitting procedure. Section 4 shows how to incorporate mortality in the valuation framework. Section 5 presents empirical results for the static and dynamic policyholder withdrawal behaviour, along with a sensitivity analysis of the fair guarantee fee to model parameters. Section 6 concludes the papers and provides final remarks.

2 Modelling framework

In this section we formulate the valuation framework for a variable annuity (VA) contract embedded with a guaranteed minimum withdrawal benefit (GMWB) rider. The VA consists of a mutual fund and a GMWB rider which promises the policyholder to recoup at least the original investment amount, W_0, through the periodic withdrawals regardless of the performance of the underlying mutual fund where the funds are invested. The withdrawal amounts γ_t occur at discrete times $t = 1, 2, \cdots, T$. The VA provider charges a policyholder a fee, α, to fund both, the guarantee and the mutual fund management fees. The valuation problem involves determining an optimal guarantee fee α which sets the initial account value equal to the discounted expected value of all future cashflows. We assume that the risk neutral dynamics of the underlying investment fund S_t evolves according to the following system of stochastic differential equations (SDEs):[2]

$$dS_t = r_t S_t dt + \sqrt{v_t} S_t d\hat{Z}_t^1, \tag{2.1}$$

$$dv_t = \zeta_v(v_t, t)dt + \sigma_v(v_t, t)d\hat{Z}_t^2, \tag{2.2}$$

$$dr_t = \zeta_r(r_t, t)dt + \sigma_r(r_t, t)d\hat{Z}_t^3, \tag{2.3}$$

[2]Note that we will specify model parameters from the volatility and interest rate dynamics when assessing different modelling assumptions in the numerical results sections.

where v_t is the instantaneous variance and r_t is the instantaneous interest rate at time t. Here, $\zeta_v(v_t,t)$ and $\sigma_v(v_t,t)$ are the drift and the diffusion terms in the stochastic variance process, respectively; $\zeta_r(r_t,t)$ and $\sigma_r(r_t,t)$ are the drift and the diffusion terms of the interest rate process, respectively, while $\hat{Z}_t^1$, $\hat{Z}_t^2$ and $\hat{Z}_t^3$ are correlated standard Wiener processes with correlations ρ_{sv}, ρ_{sr} and ρ_{vr}.

2.1 Pricing under the static withdrawal strategy

In line with the existing literature (Chen et al., 2008; Dai et al., 2008; Huang and Kwok, 2014; Luo and Shevchenko, 2014) there are two accounts associated with the GMWB contract; the investment account, W_t, which is sometimes referred to as a "personal" account; and the guarantee account, A_t. If A_t is the account balance of the guarantee at time t, the value A_0 will correspond to the initial investment, W_0. At any time t prior to maturity the account balance is determined as

$$A_t = A_0 - \int_0^t \gamma_s ds, \quad 0 \le \gamma_s \le A_s, \tag{2.4}$$

where γ_t is the withdrawal rate and A_s is the maximum allowable withdrawal at any given time. For valuation purposes, it is convenient to transform the Wiener processes in the SDE system (2.1)-(2.3) to a corresponding system which is expressed in terms of independent Wiener processes whose increments we denote as dZ_t^j for $j = 1, 2, 3$. This transformation is accomplished by performing the Cholesky decomposition such that

$$\begin{pmatrix} d\hat{Z}_t^1 \\ d\hat{Z}_t^2 \\ d\hat{Z}_t^3 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ 0 & \rho_{22} & \rho_{23} \\ 0 & 0 & \rho_{33} \end{pmatrix} \begin{pmatrix} dZ_t^1 \\ dZ_t^2 \\ dZ_t^3 \end{pmatrix},$$

where $\rho_{33} = 1$, $\rho_{23} = \rho_{vr}$, $\rho_{22} = \sqrt{1-\rho_{vr}^2}$, $\rho_{13} = \rho_{sr}$, $\rho_{12} = \frac{\rho_{sv}-\rho_{vr}\rho_{sr}}{\sqrt{1-\rho_{vr}^2}}$ and $\rho_{11} = \sqrt{1-\rho_{sr}^2 - \left(\frac{\rho_{sv}-\rho_{vr}\rho_{sr}}{\sqrt{1-\rho_{vr}^2}}\right)^2}$.

This results in the following representation:

$$dS_t = r_t S_t dt + \rho_{11}\sqrt{v_t} S_t dZ_t^1 + \rho_{12}\sqrt{v_t} S_t dZ_t^2 + \rho_{13}\sqrt{v_t} S_t dZ_t^3, \tag{2.5}$$

$$dv_t = \zeta_v(v_t,t)dt + \rho_{22}\sigma_v(v_t,t)dZ_t^2 + \rho_{23}\sigma_v(v_t,t)dZ_t^3, \tag{2.6}$$

$$dr_t = \zeta_r(r_t,t)dt + \sigma_r(r_t,t)dZ_t^3. \tag{2.7}$$

An investment account, W_t, derives its value from the value of the underlying fund, S_t, and evolves according to

$$\begin{cases} dW_t = ((r_t - \alpha)W_t - G)dt + \rho_{11}\sqrt{v_t}W_t dZ_t^1 + \rho_{12}\sqrt{v_t}W_t dZ_t^2 + \rho_{13}\sqrt{v_t}W_t dZ_t^3, \text{ if } \quad t < \tau_0 \\ dv_t = \zeta_v(v_t, t)dt + \rho_{22}\sigma_v(v_t, t)dZ_t^2 + \rho_{23}\sigma_v(v_t, t)dZ_t^3, \\ dr_t = \zeta_r(r_t, t)dt + \sigma_r(r_t, t)dZ_t^3, \\ W_t = 0, \text{ if } t \geq \tau_0, \end{cases} \tag{2.8}$$

where $0 \leq v_t, r_t < \infty$, $\tau_0 = \inf_{t \in (0,T)}[W_t = 0]$ is the ruin time of the investment account, and $\gamma_t = G$ is the contractually agreed withdrawal amount, which is connected to the initial value of the investment account through the withdrawal rate $g = \frac{G}{W_0}$. The investment account is secured against the downside risk by the GMWB rider, which implies that W_t can never drop below zero. When the process reaches zero at time τ_0, it remains zero.

Proposition 1. *The solution of the SDE system* (2.8) *can be represented as*

$$W_t = \Phi_t^{-1} e^{\int_0^t (r_u - \alpha)du} \max\left[\left(W_0 - G\int_0^t e^{-\int_0^u (r_s - \alpha)ds}\Phi_u du\right), 0\right]. \tag{2.9}$$

It can also be shown that

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_0^T r_u du} W_T\right] = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_0^T r_u du}\max(\tilde{W}_T, 0)], \tag{2.10}$$

where

$$\tilde{W}_t = \Phi_t^{-1} e^{\int_0^t (r_u - \alpha)du}\left(W_0 - G\int_0^t e^{-\int_0^u (r_s - \alpha)ds}\Phi_u du\right), \tag{2.11}$$

with

$$\Phi_t = \exp\left(-\rho_{11}\int_0^t \sqrt{v_u}dZ_u^1 - \rho_{12}\int_0^t \sqrt{v_u}dZ_u^2 - \rho_{13}\int_0^t \sqrt{v_u}dZ_u^3 + \frac{1}{2}\int_0^t v_u du\right),$$

being a stochastic integrating factor associated with the SDE system in (2.8).

Proof. Refer to Appendix A.1 □

Using no-arbitrage arguments, the initial investment of the guarantee contract must be equal to the discounted sum of all expected future cashflows. Thus, as presented in Milevsky and Salisbury (2006), the GMWB contract can be decomposed as follows

$$W_0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^s r_u du} G ds\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du}\max(\tilde{W}_T, 0)\right], \tag{2.12}$$

where W_0 is the initial investment of the policyholder. The first term on the right hand side (RHS) of Eq. (2.12) represents an annuity certain whose valuation is a trivial exercise, and the second term is the discounted terminal value of the investment account which is a typical Asian-type option. In what follows, we will concentrate on the valuation of the second expectation on the RHS of (2.12) and we denote this term by $U(t, \tilde{W}_t, v_t, r_t)$ such that

$$U(t, \tilde{W}_t, v_t, r_t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u du} \max(\tilde{W}_T, 0)\right]. \tag{2.13}$$

In the following, we adopt Heston (1993) stochastic volatility model[3] by letting

$$\phi(t, s_t, r_t) = \Big((r_t - \alpha)\tilde{W}_t - G\Big), \quad \psi(t, s_t) = \sqrt{v_t}\tilde{W}_t,$$
$$\zeta_v(v_t, t) = \kappa_v(\eta_v - v_t), \quad \beta_v(t, v_t) = \sigma_v\sqrt{v_t}.$$

For the interest rate process, we adopt the Cox et al. (1985) interest rate model by letting

$$\zeta_r(r_t, t) = \kappa_r(\eta_r - r_t) \quad \text{and} \quad \beta_r(t, r_t) = \sigma_r(r_t, t)\sqrt{r_t}.$$

These affine square root models are assumed for stochastic volatility and stochastic interest rates processes, because they can address the skewness and excess kurtosis, and better reflect the typical dynamics of interest rates and volatility observed in the financial market (Cox et al., 1985; Heston, 1993; Scott, 1987). Moreover, the square root models prohibit negative interest rates and volatility. These models are widely used in financial asset pricing (Donnelly et al., 2014; Grzelak and Oosterlee, 2011; Kim, 2001). The framework can be generalised to more complex stochastic volatility models with various affine/non-affine diffusion terms and linear/non-linear drift terms, resulting, however, in more elaborating derivations.

Using techniques developed in Shreve (2008), evaluating (2.13) is equivalent to solving the following PDE

$$\begin{aligned}\frac{\partial U}{\partial \tau} &= \phi(\tau, s, r)\frac{\partial U}{\partial s} + \frac{1}{2}\psi(\tau, s, v)^2\frac{\partial^2 U}{\partial s^2} + \rho_{sv}\psi(\tau, s, v)\beta_v(\tau, v)\frac{\partial^2 U}{\partial s \partial v}\\ &+ \rho_{sr}\psi(\tau, s, v)\beta_r(\tau, r)\frac{\partial^2 U}{\partial s \partial r} + \xi_v(\tau, v)\frac{\partial U}{\partial v} + \frac{1}{2}\beta_v(\tau, v)^2\frac{\partial^2 U}{\partial v^2}\\ &+ \xi_r(\tau, r)\frac{\partial U}{\partial r} + \frac{1}{2}\beta_r(\tau, r)^2\frac{\partial^2 U}{\partial r^2} + \rho_{vr}\beta_v(\tau, v)\beta_r(\tau, r)\frac{\partial^2 U}{\partial v \partial r} - rU,\end{aligned} \tag{2.14}$$

where $\tau = T - t$ is the time to maturity of the contract. The PDE (2.14) is solved subject to the initial condition

$$U(0, s, v, r) = \max(s, 0). \tag{2.15}$$

[3]For notational convenience we let $s_t \equiv \tilde{W}_t$.

To avoid arbitrage opportunities, all second order partial derivatives and cross derivative terms of $U(\tau, s, v, r)$ with respect to state variables along the boundaries are equal to zero. In solving the PDE in (2.14), we use the componentwise splitting method, a technique that has proven to be fast and efficient in option pricing as presented in Duffy (2006). We outline this numerical technique in the next two sections.

2.2 Pricing under the dynamic withdrawal strategy

In this section we outline the procedure for determining the fair guarantee fee for a GWMB rider in the case where the policyholder can make dynamic withdrawals that maximize the value of the variable annuity contract at any given time during the life of the contract. The guarantee provides cumulative withdrawal of at least the initial investment, W_0, during the life of the contract. We assume that at each withdrawal instant, t_j, for $j = 1, ..., N$, the policyholder can optimally withdraw an amount, γ_{t_j}, where $0 < t_j < T$ with T being the maturity of the contract.

Recall that the investment account is denoted by W_t, and is sometimes referred to as a "personal" account; and the guarantee account is denoted by A_t. At initial time, both accounts are equal to the initial premium paid by the policyholder. Within each time interval (t_j, t_{j+1}), W_t satisfies the PDE in Eq. (2.14), while the guarantee account value, A_t, does not change implying that[4] $A_{t_j^+} = A_{t_{j+1}^-}$. At each withdrawal instant, t_j, the value of investment account drops from $W_{t_j^-}$ to $W_{t_j^+} = \max(W_{t_j^-} - \gamma_{t_j}, 0)$.

The value of the guarantee account immediately before and after the withdrawal are related as follows

$$A_{t_j^+} = h(A_{t_j^-}, W_{t_j^-}, \gamma_{t_j}, G) = \begin{cases} A_{t_j^-} - \gamma_{t_j} & \text{if } 0 \leq \gamma_{t_j} \leq G \\ \min\{A_{t_j^-} - \gamma_{t_j}, \max\{W_{t_j^-} - \gamma_{t_j}, 0\}\} & \text{if } \gamma_{t_j} > G, \end{cases} \tag{2.16}$$

where the second condition in Eq. (2.16) is referred to as a reset provision (Chen et al., 2008; Milevsky and Salisbury, 2006). The reset provision is imposed in order to disincentive the policyholder from making extensive withdrawals: if the policyholder chooses to make withdrawal greater that the guaranteed value G, the value of the guarantee account can be reduced by the value greater than the withdrawal value, depending on the relative size of the investment account, W_t, and the guarantee account, A_t, at the moment of withdrawal (see the second equation in the system above).

It is a common practise for insurance providers to charge a penalty fee, $\tilde{\kappa}_{t_j}$, to the excess withdrawals above the contractually agreed amount, G. Denoting the withdrawal at time t_j by γ_{t_j}, the policyholder

[4] In what follows t_j^- (t_j^+) is a moment of time immediately before (after) the withdrawal time t_j.

receives an amount

$$f(\gamma_{t_j}) = \begin{cases} \gamma_{t_j}, & \text{if } 0 \le \gamma_{t_j} \le G, \\ G + (1-\tilde{\kappa}_{t_j})(\gamma_{t_j} - G), & \text{if } G \le \gamma_{t_j} \le A_{t_j^-}. \end{cases} \quad (2.17)$$

Upon maturity of the contract, the policyholder receives the maximum of the terminal value of the investment account and the balance of the guarantee account subject to the surrender fee, that is, $\max(W_T, (1-\tilde{\kappa}_T)A_T)$.

The policyholder will dynamically select an optimal withdraw strategy, γ_{t_j}, for $j = 1, \cdots, N$, which maximizes the present value of cash-flows such that

$$U(0, W_0, v_0, r_0, A_0) = W_0 = \max_{\bar{\gamma}} \mathbb{E}^{\mathbb{Q}} \left[\sum_{j=1}^{N} e^{-\int_0^{t_j} r_u du} f(\gamma_{t_j}) + e^{-\int_0^T r_u du} \max(W_T, (1-\tilde{\kappa}_T)A_T) \right], \quad (2.18)$$

where[5] $\bar{\gamma} = (\gamma_{t_1}, \gamma_{t_2}, ..., \gamma_{t_N})$. In order to maximise this value, at each t_j, the withdrawal amount, γ_{t_j}, has to be chosen as a solution to the problem

$$U(t_j, W_{t_j^-}, r_{t_j^-}, v_{t_j^-}, A_{t_j^-}) = \max_{\gamma_{t_j}} [f(\gamma_{t_j}) + U(t_j^+, W_{t_j^+}, r_{t_j^+}, v_{t_j^+}, A_{t_j^+})], \quad (2.19)$$

where

$$U(t_j, W_{t_j^+}, r_{t_j^+}, v_{t_j^+}, A_{t_j^+}) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_{t_j^+}^{t_{j+1}^-} r_s ds} U(t_{j+1}^-, W_{t_{j+1}^-}, r_{t_{j+1}^-}, v_{t_{j+1}^-}, A_{t_{j+1}^-}) | W_{t_j^+}, r_{t_j^+}, v_{t_j^+}, A_{t_j^+} \right], \quad (2.20)$$

with $W_{t_j^+} = \max(W_{t_j^-} - \gamma_{t_j}, 0)$, $A_{t_j^+}$ as given in Eq. (2.16) and $A_{t_{j+1}^-} = A_{t_j^+}$. For convenience, we assume that the contract terminates when the guarantee account reaches zero.

At the beginning of the contract's life the insurer usually imposes high penalties on excess withdrawals above the guaranteed amount, G, meant to disincentive the policyholder from making excessive withdrawals. Insurance companies impose high penalty charges to cover transaction costs associated with termination of the contract and liquidation of the fund. The most frequently adopted penalty fee structure is a decreasing fee with decreasing time to maturity (Milevsky and Salisbury (2001) and Chen et al. (2008)). In our numerical experiments we will adopt a penalty fee structure presented in Table 1 (adopted in Chen et al. (2008)).

[5] In the case of dynamic withdrawals the value of the contract $U(\cdot)$ depends on t, W_t, v_t, r_t and the value of the guarantee account, A_t as well.

$\tilde{\kappa}_\tau$	$\tau = T - t$
0.00	$0 \leq \tau < \frac{1}{9}T$
0.01	$\frac{1}{9}T \leq \tau < \frac{2}{9}T$
0.02	$\frac{2}{9}T \leq \tau < \frac{3}{9}T$
0.03	$\frac{3}{9}T \leq \tau < \frac{4}{9}T$
0.04	$\frac{4}{9}T \leq \tau < \frac{5}{9}T$
0.05	$\frac{5}{9}T \leq \tau < \frac{6}{9}T$
0.06	$\frac{6}{9}T \leq \tau < \frac{7}{9}T$
0.07	$\frac{7}{9}T \leq \tau < \frac{8}{9}T$
0.08	$\frac{8}{9}T \leq \tau < T$

Table 1: Surrender fee, $\tilde{\kappa}_\tau$, as a function of time to maturity, $\tau = T - t$.

3 Operator splitting method

In this section we present the operator splitting method which is used to solve the pricing PDE. This includes Section 3.1 which discusses space discretisation and Section 3.2 which outlines time discretisation and the actual componentwise splitting procedure.

3.1 Space discretisation

The function $U(\tau, s, v, r)$ introduced in Section 2 is defined on the unbounded domain $[0,T]\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^+$, however, in order to implement our numerical procedure, we need to truncate the region into a finite computational domain $[0,T]\times[0,s_{\max}]\times[0,v_{\max}]\times[0,r_{\max}]$ where $s_{\max}$, $v_{\max}$ and $r_{\max}$ are set to be large enough such that the truncation errors are negligible. In this domain we define a grid $\{\tau_n, s_i, v_j, r_k\}$ with uniform steps in each direction: $\tau_n = n\Delta\tau$ for $n = \{0, 1, ..., N_\tau = \frac{T}{\Delta\tau}\}$, $s_i = i\Delta s$ for $i = \{0, 1, ..., N_s = \frac{s_{\max}}{\Delta s}\}$, $v_j = j\Delta v$ for $j = \{0, 1, ..., N_v = \frac{v_{\max}}{\Delta v}\}$, and $r_k = k\Delta r$ for $k = \{0, 1, ..., N_r = \frac{r_{\max}}{\Delta r}\}$. Thus, each grid point is denoted by

$$U^n_{i,j,k} \approx U(\tau_n, s_i, v_j, r_k). \tag{3.1}$$

We now present finite difference approximations for the derivatives of $U(\tau_n, s_i, v_j, r_k)$ with respect to s, v and r. For the first derivative with respect to s we use upwinding second order approximations[6]

$$\frac{\partial U(\tau_n, s_i, v_j, r_k)}{\partial s} \approx \begin{cases} \frac{U^n_{i-2,j,k} - 4U^n_{i-1,j,k} + 3U^n_{i,j,k}}{2\Delta s}, & \text{for } i = \{1, 2, ..., i^* - 1\}, \\ \frac{U^n_{i+1,j,k} - U^n_{i-1,j,k}}{2\Delta s}, & \text{for } i = \{i^*, ..., N_s\}, \end{cases} \tag{3.2}$$

where $i^* = \min\{i : \phi(\tau_n, s_i, r_j) \geq 0\}$. For the initial point, s_0, we use a third order forward approximation, which involves the points s_{-1}, s_0, s_1, s_2 such that[7]

$$\frac{\partial U(\tau_n, s_0, v_j, r_k)}{\partial s} \approx \frac{-8U^n_{-1,j,k} - 3U^n_{0,j,k} + 12U^n_{1,j,k} - U^n_{2,j,k}}{18\Delta s}. \tag{3.3}$$

We use central differencing for the second derivative of $U(\tau, s, v, r)$ with respect to s such that

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial s^2} \approx \frac{U^n_{i+1,j,k} - 2U^n_{i,j,k} + U^n_{i-1,j,k}}{(\Delta s)^2}, \text{ for } i = \{1, ..., N_s - 1\}. \tag{3.4}$$

At the boundaries s_0 and s_{N_s} we have

$$\frac{\partial^2 U(\tau_n, s_0, v_j, r_k)}{\partial s^2} \approx \frac{U^n_{1,j,k} - 2U^n_{0,j,k} + U^n_{-1,j,k}}{(\Delta s)^2} = 0, \tag{3.5}$$

and

$$\frac{\partial^2 U(\tau_n, s_{N_s}, v_j, r_k)}{\partial s^2} = \frac{U^n_{N_s+1,j,k} - 2U^n_{N_s,j,k} + U^n_{N_s-1,j,k}}{(\Delta s)^2} = 0, \tag{3.6}$$

such that $U^n_{-1,j,k} = 2U^n_{0,j,k} - U^n_{1,j,k}$ and $U^n_{N_s+1,j,k} = 2U^n_{N_s,j,k} - U^n_{N_s-1,j,k}$ for $n = \{0, ..., N_\tau\}$, $j = \{0, ..., N_v\}$ and $k = \{0, ..., N_r\}$.

Using similar arguments as applied to the discretisations with respect to s, the first derivative of $U(\tau, s, v, r)$ with respect to v can be represented as

$$\frac{\partial U(\tau_n, s_i, v_j, r_k)}{\partial v} \approx \begin{cases} \frac{U^n_{i,j+1,k} - U^n_{i,j-1,k}}{2\Delta v}, & \text{for } j = \{0, ..., j^* - 1\}, \\ \frac{U^n_{i,j-2,k} - 4U^n_{i,j-1,k} + 3U^n_{i,j,k}}{2\Delta v}, & \text{for } j = \{j^*, ..., N_v\}, \end{cases} \tag{3.7}$$

[6]An upwind finite-difference scheme attempts to discretise the PDE by using differencing biased in the direction determined by the sign of the associated coefficients of the partial derivatives. We refer to Appendix A.2 for the derivation of the result in (3.2).

[7]In order to apply a second order backward approximation at the boundary point s_0 using $\frac{\partial U(\tau_n, s_0, v_j, r_k)}{\partial s} \approx \frac{U^n_{-2,j,k} - 4U^n_{-1,j,k} + 3U^n_{0,j,k}}{2\Delta s}$ we should use the "fictitious" point $U_{-2,j,k}$ which falls outside the domain; unlike that of $U_{-1,j,k}$, the value of $U_{-2,j,k}$ cannot be found directly from the boundary conditions. To avoid using the point $U_{-2,j,k}$ we increase accuracy by using the third order approximation, which involves points s_{-1}, s_0, s_1, s_2 only as presented in Eq. (3.3). Derivations of the third order finite difference approximation is presented in Appendix A.2.

where $j^* = \min\{j : \xi_v(\tau_n, v_j) \leq 0\}$. Central differencing is applied to the second derivative with respect to v resulting in

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial v^2} \approx \frac{U^n_{i,j+1,k} - 2U^n_{i,j,k} + U^n_{i,j-1,k}}{(\Delta v)^2}, \text{ for } j = \{1, ..., N_v - 1\}. \tag{3.8}$$

Using similar arguments as utilised for Eq. (3.5) and (3.6), at the the boundaries v_0 and v_{N_v} we obtain

$$U^n_{i,-1,k} = 2U^n_{i,0,k} - U^n_{i,1,k} \quad \text{and} \quad U^n_{i,N_v+1,k} = 2U^n_{i,N_v,k} - U^n_{i,N_v-1,k}, \tag{3.9}$$

for $n = \{0, ..., N_\tau\}$, $i = \{0, ..., N_s\}$ and $k = \{0, ..., N_r\}$.

In the r-direction we obtain the first derivative approximation as

$$\frac{\partial U(\tau_n, s_i, v_j, r_k)}{\partial r} \approx \begin{cases} \frac{U^n_{i,j,k+1} - U^n_{i,j,k-1}}{2\Delta r}, & \text{for } k = \{0, ..., k^* - 1\}, \\ \frac{U^n_{i,j,k-2} - 4U^n_{i,j,k-1} + 3U^n_{i,j,k}}{2\Delta r}, & \text{for } k = \{k^*, ..., N_r\}, \end{cases} \tag{3.10}$$

where $k^* = \min\{j : \xi_r(\tau_n, r_k) \leq 0\}$. The second derivative with respect to r can also be represented as

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial r^2} \approx \frac{U^n_{i,j,k+1} - 2U^n_{i,j,k} + U^n_{i,j,k-1}}{(\Delta r)^2}, \text{ for } k = \{1, ..., N_r - 1\}. \tag{3.11}$$

At the boundaries r_0 and r_{N_r} we use

$$U^n_{i,j,-1} = 2U^n_{i,j,0} - U^n_{i,j,1} \quad \text{and} \quad U^n_{i,j,N_r+1} = 2U^n_{i,j,N_r} - U^n_{i,j,N_r-1}, \tag{3.12}$$

for $n = \{0, ..., N_\tau\}$, $i = \{0, ..., N_s\}$ and $j = \{0, ..., N_v\}$.

Central differencing is applied to the cross-derivative terms yielding

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial s \partial v} \approx \frac{U^n_{i+1,j+1,k} - U^n_{i-1,j+1,k} - U^n_{i+1,j-1,k} + U^n_{i-1,j-1,k}}{4\Delta s \Delta v}, \tag{3.13}$$

for $i = \{1, ..., N_s - 1\}$, $j = \{1, ..., N_v - 1\}$ and $k = \{0, ..., N_r\}$;

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial s \partial r} \approx \frac{U^n_{i+1,j,k+1} - U^n_{i-1,j,k+1} - U^n_{i+1,j,k-1} + U^n_{i-1,j,k-1}}{4\Delta s \Delta r}, \tag{3.14}$$

for $i = \{1, ..., N_s - 1\}$, $j = \{0, ..., N_v\}$ and $k = \{1, ..., N_r - 1\}$;

$$\frac{\partial^2 U(\tau_n, s_i, v_j, r_k)}{\partial v \partial r} \approx \frac{U^n_{i,j+1,k+1} - U^n_{i,j-1,k+1} - U^n_{i,j+1,k-1} + U^n_{i,j-1,k-1}}{4\Delta v \Delta r}, \tag{3.15}$$

for $i = \{0, ..., N_s\}$, $j = \{2, ..., N_v - 1\}$ and $k = \{1, ..., N_r - 1\}$. Discretisations at the boundary points are handled in a similar fashion as presented in Eq. (3.5) and (3.6).

The discretised version of Eq. (2.14) can be represented as

$$\frac{U^{n+1}_{i,j,k} - U^n_{i,j,k}}{\Delta \tau} = \Big[\mathcal{L}^n_s + \mathcal{L}^n_v + \mathcal{L}^n_r\Big] U^{n+1}_{i,j,k} + \Big[\mathcal{L}^n_{sv} + \mathcal{L}^n_{sr} + \mathcal{L}^n_{vr}\Big] U^n_{i,j,k}, \tag{3.16}$$

where the functional forms of the differential operators ($\mathcal{L}^{(\cdot)}_{(\cdot)}$) on the right hand side of Eq. (3.16) and the associated boundary conditions are presented in Appendix A.3. In particular, to avoid arbitrage opportunities, all second order partial derivatives and cross derivatives with respect to state variables along the boundaries are equal to zero. Eq. (3.16) is solved subject to the initial condition

$$U^0_{i,j,k} = \max(s_i, 0). \tag{3.17}$$

Having outlined the discretisation steps for the PDE in Eq. (2.14), we are now faced with solving the discrete PDE in (3.16) subject to the initial condition in Eq. (3.17) and associated boundary conditions derived from second order and cross derivatives terms as highlighted above. We will adopt the componentwise splitting method for solving Eq. (3.16), as outlined in the next section.

3.2 Time discretisation and componentwise splitting method

The discrete problem outlined in Eq. (3.16) can be rewritten using more compact notation in matrix form as

$$(\boldsymbol{I} - \triangle\tau \boldsymbol{D_1})\boldsymbol{u}^{n+1} = (\boldsymbol{I} + \triangle\tau \boldsymbol{D_2})\boldsymbol{u}^n, \tag{3.18}$$

where $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are sparse matrices of size $(N_s+1)(N_v+1)(N_r+1) \times (N_s+1)(N_v+1)(N_r+1)$ whose elements are components of the differential operators appearing in Eq. (3.16) and $\boldsymbol{I}$ is an identity matrix. Solving problems like the one in Eq. (3.18) is computationally intensive especially when the grid points are made finer in order to reduce discretisation errors. It is therefore imperative to devise alternative computationally efficient schemes for solving this problem. One such scheme is the componentwise splitting method, which is based on the decomposition of matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ into simpler matrices such that

$$\boldsymbol{D_1} = \boldsymbol{D_{11}} + \boldsymbol{D_{12}} + \boldsymbol{D_{13}} \quad \text{and} \quad \boldsymbol{D_2} = \boldsymbol{D_{21}} + \boldsymbol{D_{22}} + \boldsymbol{D_{23}} \tag{3.19}$$

where matrix $\boldsymbol{D_{11}}$ contains coupling of finite difference stencil in the s-direction, $\boldsymbol{D_{21}}$ contains half of the stencil in the sv-direction and half in the sr-direction. The matrix $\boldsymbol{D_{12}}$ contains the stencil in the v-direction, $\boldsymbol{D_{22}}$ contains the other half in the sv-direction and half in the vr-direction. In addition, matrix $\boldsymbol{D_{13}}$ contains the stencil in the r-direction, with $\boldsymbol{D_{23}}$ containing the other halves in the sr- and vr-directions[8]. Martices $\boldsymbol{D_{11}}$, $\boldsymbol{D_{12}}$, $\boldsymbol{D_{13}}$, $\boldsymbol{D_{21}}$, $\boldsymbol{D_{22}}$ and $\boldsymbol{D_{23}}$ can be transformed into tridiagonal

[8]The coupled system is presented in Appendix A.4 for completeness. Here, we have chosen to express the system in terms of the differential operators in scalar form. The elements of the matrix $\boldsymbol{D}$ are also presented in Appendix A.5.

matrices by reordering their elements, see Ikonen and Toivanen (2007) for a detailed discussion.

By applying the componentwise splitting method, we reduce the three-dimensional problem in (3.18), into a sequence of one-dimensional problems which can be solved efficiently. Applying this method to Eq. (3.18) yields

$$\boldsymbol{A_1}\boldsymbol{u}^{n+\frac{1}{3}} = \boldsymbol{B_1}\boldsymbol{u}^{n} \tag{3.20}$$

$$\boldsymbol{A_2}\boldsymbol{u}^{n+\frac{2}{3}} = \boldsymbol{B_2}\boldsymbol{u}^{n+\frac{1}{3}} \tag{3.21}$$

$$\boldsymbol{A_3}\boldsymbol{u}^{n+1} = \boldsymbol{B_3}\boldsymbol{u}^{n+\frac{2}{3}}, \tag{3.22}$$

where

$$\boldsymbol{A_1} = (\boldsymbol{I} + \triangle\tau \boldsymbol{D_{11}}), \quad \boldsymbol{A_2} = (\boldsymbol{I} + \triangle\tau \boldsymbol{D_{12}}), \quad \boldsymbol{A_3} = (\boldsymbol{I} + \triangle\tau \boldsymbol{D_{13}}), \tag{3.23}$$

and

$$\boldsymbol{B_1} = (\boldsymbol{I} - \triangle\tau \boldsymbol{D_{21}}), \quad \boldsymbol{B_2} = (\boldsymbol{I} - \triangle\tau \boldsymbol{D_{22}}), \quad \boldsymbol{B_3} = (\boldsymbol{I} - \triangle\tau \boldsymbol{D_{23}}). \tag{3.24}$$

In Eq. (3.23), all the six matrices, $\boldsymbol{A_1}$, $\boldsymbol{A_2}$, $\boldsymbol{A_3}$, $\boldsymbol{B_1}$, $\boldsymbol{B_2}$ and $\boldsymbol{B_3}$ are tridiagonal. Instead of solving the sparse block tridiagonal matrices $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ we are now faced with solving much simpler matrices, which results in significant reduction of the computational time. In our numerical experiments, we use a combination of the implicit and explicit Euler schemes in such a way that optimises the computational speed, that is, for each time substep, we apply the implicit scheme in one direction and half of the explicit computation in the mixed directions, and then use the result from the previous substep in the next substep.

4 Incorporating stochastic mortality

Considering the long-term nature of the GMWB contract it is natural to incorporate mortality risk in the valuation framework. We adopt the stochastic mortality model proposed in Dahl and Moller (2006) due to its many attractive features which include; tractability as it allows for closed-form expressions for survivor probabilities and ability of capturing mortality evolutions across all ages. Under the framework of Dahl and Moller (2006) the risk neutral dynamics of the stochastic mortality for a person aged x at time $t = 0$ is given by

$$d\mu(x,t) = (\gamma^{\mu}(x,t) - \delta^{\mu}(x,t)\mu(x,t))dt + \sigma^{\mu}(x,t)\sqrt{\mu(x,t)}dZ_t^4, \tag{4.1}$$

which is a time-inhomogeneous Cox-Ingersoll-Ross (CIR) process (Cox et al., 1985). In Eq. (4.1), dZ_t^4 represents the increments of a Wiener process independent from Z_t^1, Z_t^2 and Z_t^3 associated with the financial risk factors and

$$\gamma^\mu(x,t) = \tilde{\delta} e^{-\tilde{\gamma} t} \mu^\circ(x+t), \quad \delta^\mu(x,t) = \tilde{\delta} - \frac{\frac{d}{dt}\mu^\circ(x+t)}{\mu^\circ(x+t)}, \quad \sigma^\mu(x,t) = \tilde{\sigma}\sqrt{\mu^\circ(x+t)}.$$

Dahl and Moller (2006) assume that $\mu^\circ(x+t)$ corresponds to the Gompertz (1825) mortality law implying that

$$\mu^\circ(x+t) = \alpha^\mu + \beta^\mu c^{x+t}. \tag{4.2}$$

In all our numerical experiments, we adopt the parameters presented in Dahl and Moller (2006) and reported in Table 2 for $x = 30$ as one of our main objectives is to highlight the impact of stochastic mortality on the valuation of GMWB riders.

α^μ	β^μ	c	$\tilde{\delta}$	$\tilde{\gamma}$	$\tilde{\sigma}$
0.000233	0.0000658	1.0959	0.2	0.008	0.02

Table 2: Parameters for the mortality model obtained in Dahl and Moller (2006), based on the Danish mortality data for 30 years old males for 1980.

As highlighted above, the most superiour feature of the Dahl and Moller (2006) model is that the survival probabilities can be expressed in closed-form which, using current parametrization can be represented as

$$S(x,t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \mu(x,s)ds}\right] = e^{A^\mu(x,t,T) - B^\mu(x,t,T)\mu(x,t)}, \tag{4.3}$$

where

$$\frac{\partial}{\partial t_1} B^\mu(x,t,T) = \delta^\mu(x,t) B^\mu(x,t,T) + \frac{1}{2}(\sigma^\mu(x,t))^2 (B^\mu(x,t,T))^2 - 1, \tag{4.4}$$

$$\frac{\partial}{\partial t_1} A^\mu(x,t,T) = \gamma^\mu(x,t) B^\mu(x,t,T), \tag{4.5}$$

are the Riccati ordinary differential equations that can be solved subject to the terminal conditions $B^\mu(x,T,T) = 0$ and $A^\mu(x,T,T) = 0$, respectively. Also of particular importance in modelling survivor curves under the Dahl and Moller (2006)'s framework: the forward force of mortality which is the density function for the remaining future lifetime can be expressed in closed-form as

$$f^\mu(x,t,T) = -\frac{\partial}{\partial T}\log S(x,t,T) = \mu(x,t)\frac{\partial}{\partial T}\log B^\mu(x,t,T) - \frac{\partial}{\partial T}\log A(x,t,T). \tag{4.6}$$

Incorporating the above stochastic mortality model into the static valuation framework yields

$$W_0^{mort} = \int_0^T f(x,0,u)\left(\int_0^u D(0,s)Gds + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^u r_s ds}\max(\tilde{W}_u,0)\right]\right)du$$
$$+ S(x,0,T)\left(\int_0^T D(0,u)Gdu + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\max(\tilde{W}_T,0)\right]\right), \tag{4.7}$$

where the first term on the RHS is the weighted sum of the payments to beneficiaries in the event of untimely death of the policyholder, and the second term is the discounted sum of all cash flows including the final payment if the policyholder survives to maturity of the contract. Due to independence between mortality and financial risk parameters, the expectation in Eq. (4.7) can be solved using the operator splitting method presented in Section 3.

For the dynamic withdrawal case, the value of a variable annuity contract can be expressed as[9]

$$U^{mort}(0,W_0,v,r,A_0) = \int_0^T f(x,0,u)U(u,W_u,v,r,A_u)du + S(x,0,T)U(0,W_0,v,r,A_0), \tag{4.8}$$

where $U(0,W_0,v,r,A_0)$ is the solution of Eq. (2.19) for the case where no mortality is incorporate in the model.

5 Numerical results

For all numerical experiments that follow, we use the financial parameter set presented in Table 3. We have also assumed that the standard guarantee rate g corresponds to 7% p.a.

5.1 Static withdrawal case

In this section we present results for the static policyholder withdrawal behaviour. We first assess the accuracy of our approach by comparing it with the existing literature in Section 5.1.1. We are going to analyse the impact of various model parameters in Section 5.1.2.

5.1.1 Numerical Comparisons

We start by assessing the performance of our approach relative to two existing frameworks developed in Luo and Shevchenko (2014) and Liu (2010). Both papers consider the valuation of variable annuity

[9] Note that we have expressed the value in terms of running time, $t \in [0,T]$ for convenience.

v_t−Parameter	Value	r_t−Parameter	Value
κ_v	0.005	κ_r	0.3
η_v	0.04	η_r	0.05
σ_v	0.1	σ_r	0.1
ρ_{sv}	−0.6	ρ_{sr}	0.3
ρ_{vr}	0.15		

Table 3: Parameters used for assessing policyholder's behavior on the GMWB rider. The first two columns contain parameters and the corresponding values of the stochastic variance process whilst the last two columns contain parameters and corresponding values of the stochastic interest rate process. These parameters are adopted in order to be consistent with parameters obtained in Ignatieva et al. (2015), Donnelly et al. (2014), Luo and Shevchenko (2014) and Luo and Shevchenko (2016).

contracts embedded with GMWB riders under the static withdrawal case when the underlying fund evolves according to the geometric Brownian motion (GBM) process. To match the GBM parameters, we first analyse the behaviour of the density functions for either the interest rate process (Cox et al., 1985) or the stochastic variance process (Heston, 1993); both are square root processes. If x_t represents a square root process, the corresponding dynamics can be written as

$$dx_t = \kappa(\eta - x_t)dt + \sigma\sqrt{x_t}dZ_t, \tag{5.1}$$

where Z_t is a Wiener process with κ, η and σ being the speed of mean reversion, the long-run mean and the diffusion parameter, respectively.[10] The corresponding density function has been derived in Maghsoodi (1996) as

$$f(x_t|x_0) = c\exp\{-c(x_0e^{-\kappa_x t} + x_t)\}\left(\frac{x_t}{(x_0e^{-\kappa t})}\right)^{q/2} I_q(2\sqrt{c^2(x_0e^{-\kappa t}x_t)}), \tag{5.2}$$

where $c = \frac{2\kappa}{(1-e^{-\kappa t})\sigma^2}$, $q = \frac{2\kappa\eta}{\sigma^2} - 1$ and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. From Figure 1 which shows the density function for the CIR stochastic process for various values of σ (and fixed $\kappa = 1$ and $\eta = 0.04$), we note that for $\sigma = 0.05$, the density function has light right tail and concentrates around the long-term average value, $\eta = 0.04$. When σ increases from 0.05 to 0.25 the probability mass shifts to the left, and the right tail of the distribution becomes thicker. This implies that for some "intermediate" values of the volatility of variance parameter the square-root process takes

[10] In the case of a stochastic variance process, σ will be used to denote the volatility of variance parameter.

extreme values with high probability. As the volatility of variance parameter σ increases further from 0.25 to 0.5, the probability mass function concentrates around zero and the right tail gets lighter, which means that the process takes low values with high probability. It should be emphasised that for $\sigma = 0.5$ the Feller's condition is violated.

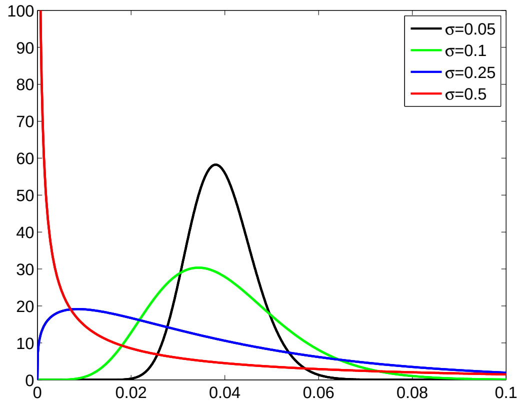

Figure 1: Density function for the CIR stochastic process for various values of σ ($\kappa = 1$ and $\eta = 0.04$).

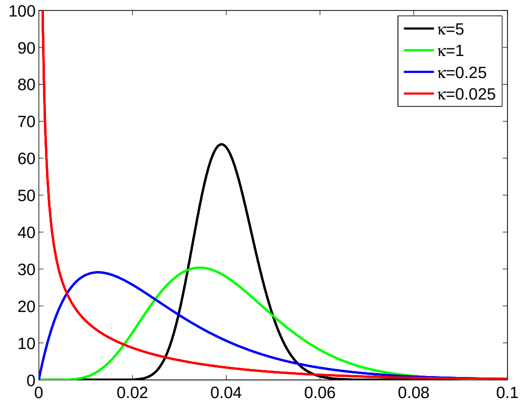

Figure 2: Density function for the CIR stochastic process for various values of κ ($\sigma = 0.1$ and $\eta = 0.04$).

Feller (1951) shows that when $\frac{2\kappa\eta}{\sigma^2} \geq 1$, the square root is strictly positive and when $\frac{2\kappa\eta}{\sigma^2} < 1$, the process can reach zero but will always remain non-negative. Moreover, for the square-root process it can be shown that

$$\mathbb{E}^{\mathbb{Q}}[x_t|x_0] = x_0 e^{-\kappa t} + \eta(1 - e^{-\kappa t}), \tag{5.3}$$

refer to Maghsoodi (1996). This suggests that, in order to compare the results between the stochastic variance/interest rate case and the deterministic case, we must set $x_0 = \eta$ and $\sigma \to 0$.

In Figure 2 we also analyse the behaviour of the transition density function for varying speed of mean-reversion, κ (and fixed $\sigma = 0.1$ and $\eta = 0.04$). From this figure we note that for low values of κ, the density function is shifted towards the left. When κ is increasing, the density function becomes more centred around the long-run mean of x_t.

Having analysed the properties of the density functions, we now present numerical comparisons for the GMWB fees for various approaches, when the guarantee rate, g, is varying. Table 4 summarises the results from our approach (assuming stochastic volatility, stochastic interest and static withdrawals, abbreviated as SVSI SW), comparing them with the results presented in Luo and Shevchenko (2014) and Liu (2010), who consider discrete deterministic withdrawal at frequencies (fr) of 4 and 12 times per year. The fair guarantee fees are reported in basis points.[11]

g	SVSI SW (fr$=\infty$)	Liu (2010) (fr$=12$)	L & S (2014) (fr$=4$)
5%	29.89	28.51	28.33
6%	41.88	40.61	40.33
7%	55.66	53.78	53.31
10%	100.25	96.65	95.81

Table 4: Comparison of the fair guarantee fees generated by the componentwise splitting method with the frameworks presented in Luo and Shevchenko (2014) and Liu (2010). The following parameters have been chosen for the numerical comparisons: $\sigma_v = 10^{-7}$, $\sigma_r = 10^{-7}$, $\kappa_v = 1$, $\kappa_r = 0.3$, $v_0 = 0.04$, $r_0 = 0.05$, $\eta_v = 0.04$, $\eta_r = 0.05$, $\rho_{sv} = -0.6$, $\rho_{sr} = 0.3$, $\rho_{vr} = 0.15$.

From Table 4 we note that the fair insurance fee is an increasing function of the guarantee rate, g. This behaviour is consistent across all three frameworks presented in the table. By selling riders with higher guarantee rates, the annuity provider is taking more risk. This risk can be offset/compensated by charging higher premiums, which explains the results in the table. When the volatility of variance parameters of stochastic interest rate process and stochastic volatility process are close to zero ($\sigma_r = 10^{-7}$ and $\sigma_v = 10^{-7}$), the fair insurance fees computed by the componentwise splitting method are close to those resulted from the Geometric Brownian Motion (GBM) cases analysed in Luo and Shevchenko (2014) and Liu (2010) for different frequencies (fr) of discrete withdrawals [12].

5.1.2 Analysis of the static withdrawal case

Having performed numerical comparisons highlighting the competency of our approach as presented in Section 5.1.1, we now analyse the impact of various model parameters on the fair guarantee fee, assuming static withdrawal strategy of the policyholder. For all numerical experiments that follow in this section

[11] Note that 1 basis point (b.p.) is equivalent to 0.01%. Also note that for comparison purposes, we do not incorporate mortality so as to explicitly highlight the impact of stochastic volatility and stochastic interest rates.

[12] Liu (2010) find (see Table 2.3) that as the frequency of withdrawals increases the fair insurance fee increases and approaches the fair fee for the continuous withdrawals case.

we assume that the mortality risk is incorporated in the model as discussed in Section 4, and, unless specified otherwise, we have used the parameter set of the mortality component as presented in Table 2 and the parameter set of the financial component as presented in Table 3.

Figure 3 shows the impact of varying σ_v and σ_r on the fair guarantee fee. From this figure we note that the fair fee increases with increasing σ_r. A highly volatile interest rate environment implies that interest rates become less predictable, which in the context of the long-term insurance contracts leads to higher interest rate risk. This in turn results in higher insurance fees required to cover the annuity provider from the associated risk. From Figure 4 we notice that the fair guarantee fee is an increasing function of the correlation between the underlying fund and the interest rate process, ρ_{sr}. The correlation effects are more pronounced for higher σ_r as reflected by the figure. This result is consistent with findings reported in Dai et al. (2015) and Luo and Shevchenko (2016). Kang and Ziveyi (2016) also note that an increase in σ_r results in higher zero-coupon bond prices which in turn leads to higher management fees.

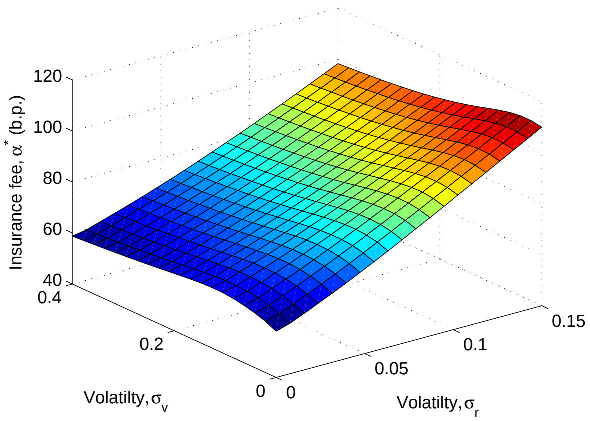

Figure 3: Fair insurance fee, α^*, as a function of the volatility of variance parameter, σ_v, and the volatility of interest rate, σ_r. All other parameters are as presented in Table 2 and Table 3.

Figure 5 shows how the fair guarantee fees is affected by varying volatility of variance parameter, σ_v and the correlation between the underlying fund and the volatility processes, ρ_{sv}. The results from this figure suggest that there is a non-monotonic dependence between the fair insurance fees and σ_v. As it has been discussed above (see Figure 1), when the volatility of variance parameter increases, the probability of “extremely” high values of the stochastic volatility process increases, which is reflected by fatter right tail. As the volatility of variance parameter, σ_v, continues to increase, the probability mass tends to

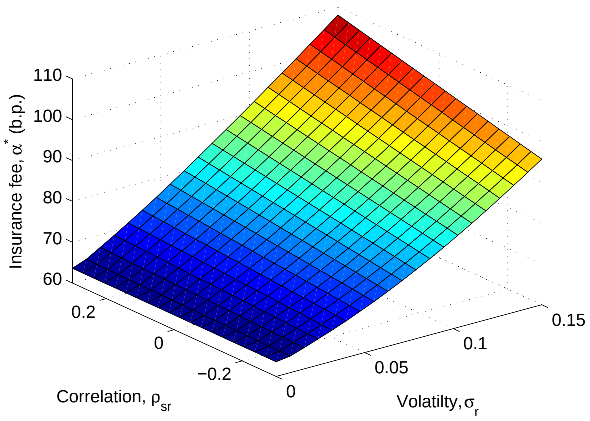

Figure 4: Fair insurance fee, α^*, as a function of the correlation, ρ_{sr}, and the volatility of interest rate, σ_r. All other parameters are as presented in Table 2 and Table 3.

shift to the left and concentrate around zero. Thus, for "moderate" values of σ_v there is high volatility risk, whereas "extreme" values of σ_v imply higher probability of observing lower value of the stochastic volatility process. The findings in Figure 5 are consistent with the results presented in Donnelly et al. (2014) who also note the non-monotonic dependence between volatility of variance and the fair insurance fees[13] when valuing GMWBs under the Heston (1993) modelling framework.

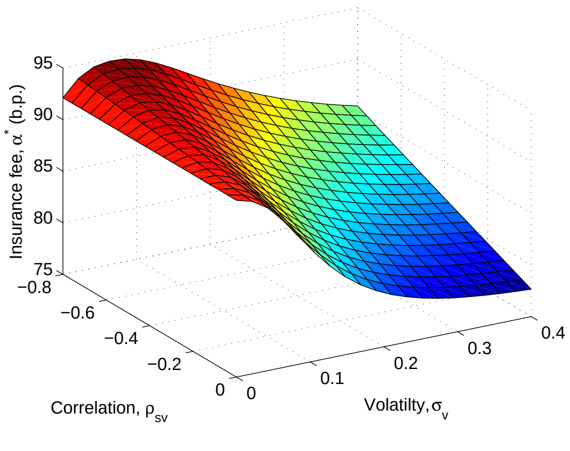

Figure 5: Fair insurance fee, α^*, as a function of the correlation, ρ_{sv}, and the volatility of variance parameter, σ_v. All other parameters are as presented in Table 2 and in Table 3.

[13]See Figure 4 in Donnelly et al. (2014).

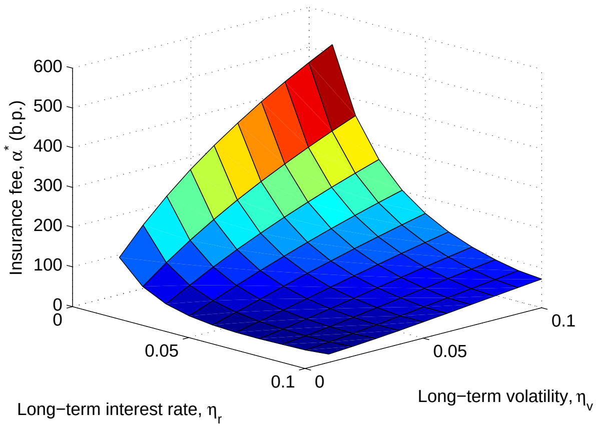

Figure 6: Fair insurance fee, α^*, as a function of the long-term volatility, η_v, and long-term interest rate, η_r. All other parameters are as presented in Table 2 and in Table 3.

From Figure 6 we note that the fair insurance fee, α^*, is an increasing function of the average long-term volatility, η_v, and a decreasing function of the average long-term interest rate, η_r. When η_v is high, the volatility risk is higher implying higher ruin probability. In contrast, when η_r is high, the average growth of the investment account is high, which leads to lower probability of ruin and lower risks for the insurance provider.

We now assess the impact of the speed of mean-reversion parameters, κ_v and κ_r, on insurance fees as presented in Tables 5 and 6. From Table 5 we note that when σ_v is very close to zero, the fees are insensitive to changes in κ_v. When the volatility of variance is very small, the uncertainty associated with the diffusion component of Eq. (2.1) is negligible, leaving the fund dynamics to be mainly dictated by the interest rate movements. For non-zero σ_v, we note that the fair management fee is an increasing function of κ_v. As noted from Figure 3 and 5, for any given level of κ_v, the fair management fee is a non monotonic function of σ_v.

The results in Table 6 also reveal that when σ_r is very close to zero, the fair insurance fees are insensitive to changes in κ_r. However, for non-zero values of σ_r, the fair fee becomes a decreasing function of κ_r; this is consistent with the analysis of the density function presented in Figure 2.

The impact of mortality risk can be assessed by comparing the fair insurance fee to the case when no mortality is incorporate in the modelling framework. From Table 7 we observe that the GWMB fee in the case of no mortality is higher compared to the case when mortality is incorporated in the model. In

	$\sigma_v = 10^{-7}$	$\sigma_v = 0.1$	$\sigma_v = 0.2$	$\sigma_v = 0.3$	$\sigma_v = 0.4$
$\kappa_v = 0.005$	91.28	91.60	87.42	84.62	82.91
$\kappa_v = 0.05$	91.28	94.14	90.31	86.94	84.84
$\kappa_v = 0.1$	91.28	95.97	93.47	89.56	86.95

Table 5: Fair insurance fee as a function of the volatility of variance parameter, σ_v, and speed of mean-reversion, κ_v, of stochastic variance. All other parameters are as presented in Table 2 and in Table 3.

	$\sigma_r = 10^{-5}$	$\sigma_r = 0.0375$	$\sigma_r = 0.075$	$\sigma_r = 0.1125$	$\sigma_r = 0.15$
$\kappa_r = 0.1$	63.43	74.23	89.58	108.19	125.89
$\kappa_r = 0.5$	63.43	71.28	80.28	89.59	98.88
$\kappa_r = 1$	63.43	69.35	75.94	82.67	89.25
$\kappa_r = 1.5$	63.43	68.33	73.80	79.35	84.84

Table 6: Fair insurance fee as a function of the interest rate volatility parameter, σ_r, and speed of mean-reversion parameter, κ_r, of stochastic interest rate. All other parameters are as presented in Table 2 and in Table 3.

g	SVSI SW with mortality	SVSI SW without mortality
5%	61.74	62.29
6%	76.22	76.89
7%	91.59	92.35
10%	141.65	142.60

Table 7: Comparison of the fair guarantee fees in the cases with mortality and without mortality for various values of the guarantee rate, g. All other parameters are as presented in Table 2 and in Table 3.

line with the discussion presented in Milevsky and Salisbury (2006) lower fees in the case of incorporated mortality risk can be explained by the fact that there is non-zero probability for termination of the contract before maturity, when the policyholder dies. This implies that there is a lower ruin probability of the investment account, so the insurer faces lower risk and can charge lower insurances fees. Generally, we notice that the difference between the fair insurance fees in the case of with and without mortality is relatively small, and this difference is higher when withdrawal rates are higher.

5.2 Dynamic withdrawal case

This section analyses the results for the dynamic withdrawal strategy of the policyholder. Here we assume that there is no mortality risk in the model. The reason is that as it has been highlighted in Milevsky and Salisbury (2006) and also demonstrated in the static withdrawal case analysis, the importance of stochastic mortality risk is relatively small.

We begin by assessing the impact of stochastic volatility and stochastic interest rates on the fair insurance fees and perform numerical comparisons with the static withdrawal strategy (Section 5.2.1). We then perform some experiments to highlight how the optimal policyholder withdrawal behaviour is influenced by changes in various model parameters (Section 5.2.2). In all numerical experiments that follow, we adopt the surrender fee schedule presented in Table 1, with all other parameters as specified in Table 3.

5.2.1 Numerical comparisons

We now assess how changes in the guarantee rate g affect the fair insurance fee α^* under various model settings. Table 8 shows six different models.[14] Column 2 contains insurance fees for the dynamic withdrawal (DW) strategy when the underlying fund evolves under the influence of stochastic volatility and stochastic interest rates (SVSI) as presented in Section 2.2. Columns 3-5 contain three special cases: Column 3 reports the results for the stochastic volatility and stochastic interest rates (SVSI) model under the static withdrawals (SW) case presented in Eq. (2.1)-(2.3); column 4 corresponds to the special case of the dynamic withdraw strategy (DW) of the policyholder where the underlying fund is driven by stochastic volatility (SV) while the interest rates are deterministic; and column 5 refers to the model with stochastic volatility (SV), deterministic interest rates and static withdrawals (SW). In the last two columns we

[14]SVSI stands for stochastic volatility and stochastic interest rates; SV - stochastic volatility; SI - stochastic interest; DW - dynamic withdrawals; SW - static withdrawals.

make comparisons with the results presented in Luo and Shevchenko (2014) and Liu (2010) that were introduced in Section 5.1.1. From columns 2-5 we observe that the fair insurance fee α^* increases as the guarantee rate g increases. This increase is more pronounced for the static withdrawal cases (columns 3 and 5), compared to the dynamic withdrawal case (columns 2 and 4) where we note that by varying the guarantee rate from 5%-10% the corresponding fees increase by the factor of approximately 2.5 and 3 for SVSI SW case and SV SW case, respectively. By comparing the fair insurance fees for a fixed guarantee rate we can conclude that the increase is the most sound, when we move from the static withdrawal schedule (columns 3 and 5) to the dynamic one (columns 2 and 4) for a given modelling setup (SV or SVSI). This allows us to conclude that the behavioural risk associated with the policyholder's withdrawal schedule is the most important comparing to the interest rate risk (when we move form columns 4 and 5 to columns 2 and 3) and volatility risk (when we move from columns 6 and 7, which correspond to geometric Brownian motion case, to columns 4 and 5). Our results are consistent with those reported in Luo and Shevchenko (2014) and Liu (2010) that value VA contracts embedded with GMWB riders under the static withdrawals case and assuming GBM dynamics for the the underlying fund.

g	SVSI DW	SVSI SW	SV DW	SV SW	L&S (2014)	Liu (2010)
5%	181.93	51.95	178.78	32.89	28.33	28.51
6%	205.46	65.94	202.35	44.40	40.33	40.61
7%	214.77	81.02	211.75	56.80	53.31	53.78
10%	244.41	131.08	241.97	97.75	95.81	96.65

Table 8: Fair insurance fee α^* as a function of changing guarantee rate g for the GMWB contract under the dynamic withdrawals, compared to static withdrawals with financial parameters as given in Table 3. SVSI stands for stochastic volatility, stochastic interest rates; SV - stochastic volatility; SI - stochastic interest; DW - dynamic withdrawals; SW - static withdrawals.

5.2.2 Assessing the impact of model parameters

Having analysed the effectiveness of our approach in computing fair insurance fees, we now provide a detailed analysis of the optimal withdrawal behaviour for varying volatility and interest rate levels (Figure 7). We also provide insights on how the optimal withdrawal regions change over time (Figure 8).

Figure 7 shows the optimal withdrawal schedule as a function of the withdrawal account, A_t, and the investments account, W_t, for varying values of the instantaneous variance, v (increasing from top to bottom), and varying values of instantaneous interest rates, r (increasing from left to right). The contour lines represent the withdrawal amounts for any given combinations of A_t and W_t with the triangular region (dark blue area) on each panel denoting the region where it is optimal to withdrawal the contractually agreed amount, G. The changing contour colours (from dark blue to red) represent the magnitude of excessive withdrawals as either A_t or W_t dominate each other. The region on the left of each triangle represents the case where the guarantee account, A_t, dominates the investment account, W_t while the right segment corresponds to the case where W_t dominates A_t. Given the surrender schedule in Table 1 we observe that when W_t is sufficiently small relative to A_t, it is optimal for the policyholder to excessively withdraw the variable annuity account (this is consistent with findings in Chen et al. (2008) and Huang and Kwok (2014) who consider the valuation framework under the GBM setting). From the figure we also notice that for any given level of interest rates, the triangular region widens as the volatility level increases. With increasing volatility the investment account becomes more dominant compared to the guarantee account, as it is evident from the shrinking region on the left. The widening triangular region also implies that there are more combinations of (W_t, A_t) where it is optimal to withdraw the contractually agreed amount as the volatility increases. As highlighted in Chen et al. (2008), when W_t dominates A_t, the guarantee is out-of-the-money rendering the policyholder to withdraw an amount which minimises both, the guarantee fee and the early surrender charges. However, for a fixed volatility level, we notice that the withdrawal strategy is not significantly affected by changes in the interest rate level.

Figure 8 infers the optimal withdrawal regions through time when the interest rate and the volatility are fixed. We note that as the contract approaches maturity, the region where it is optimal to withdrawal the contractually agreed amount shrinks to a point where there is only a line ($t = 9.9$ in the right bottom panel) separating the regions where A_t and W_t dominate each other. This can be explained by the surrender charge which is decreasing with decreasing time-to-maturity. From Table 1, the fee during the final phase ($0 \leq \tau < \frac{1}{9}T$) of withdrawals is set to zero implying that during this phase the policyholder can immediately withdraw everything at the next withdrawal opportunity at no cost, when either A_t dominates W_t or vice versa.

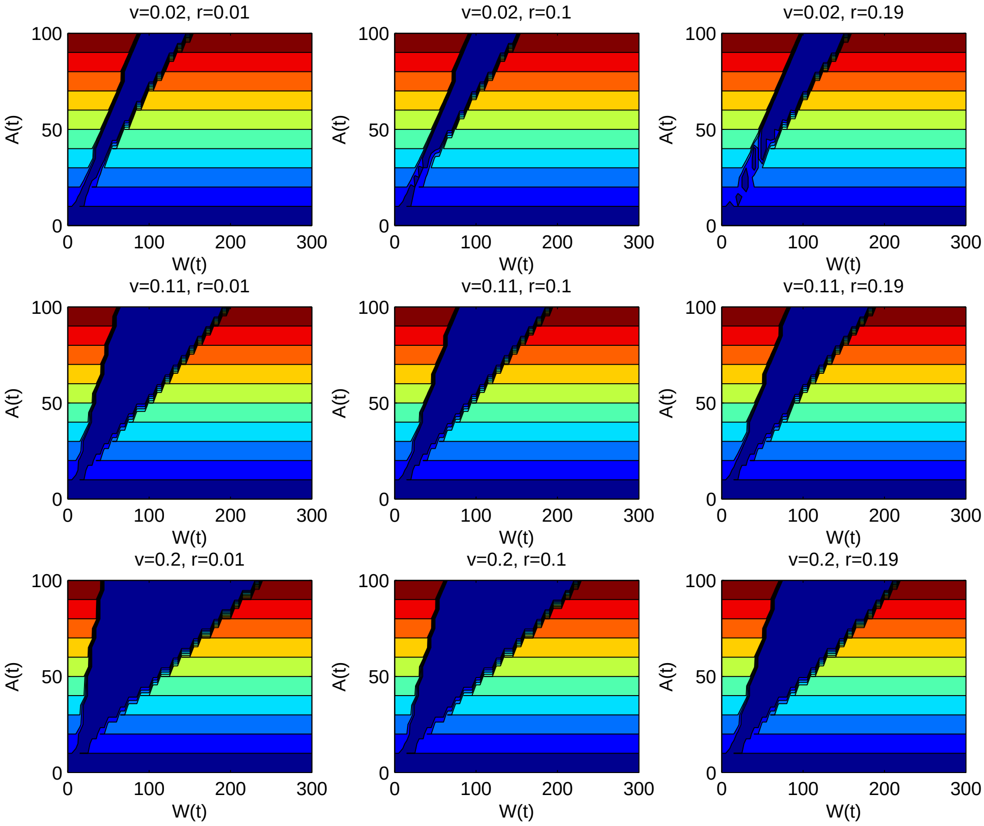

Figure 7: Optimal withdrawal schedule as a function of the withdrawal account, A, and the investments account, W, for various values of the stochastic volatility process, $v(t)$, and the stochastic interest rate process, $r(t)$, at $t = 8.72$.

6 Conclusion

This paper deals with the valuation of GMWBs embedded in variable annuities under the influence of several risk factors, namely, stochastic interest rates, stochastic volatility, stochastic mortality and equity risk. Pricing of the guarantee is performed numerically via the componentwise operator splitting method, which is computationally efficient. We compute the fair insurance fee and analyse its sensitivity with respect to different model parameters associated with various risk factors. Both, static and dynamic withdrawal strategies are analysed.

We find that the fair insurance fee charged by the guarantee provider increases with increasing guarantee

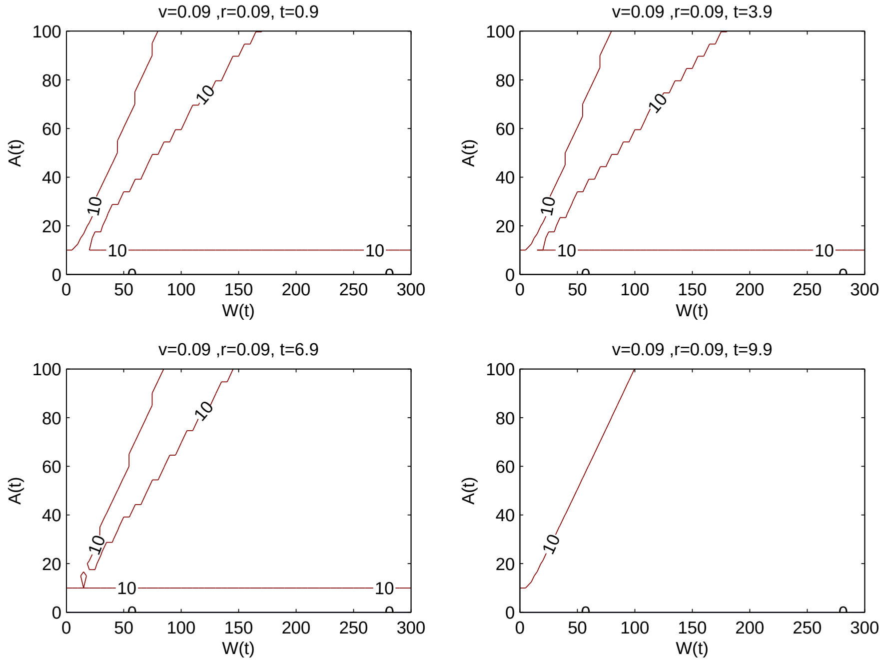

Figure 8: Optimal withdrawal schedule as a function of withdrawal account, A, and investments account, W, for various values of the stochastic volatility process, $v(t)$, and the stochastic interest rate process, $r(t)$, over time.

rate, while a larger increase is observed for the case of static withdrawals, compared to the case of dynamic withdrawals. Stochastic risk factors (stochastic interest rates and stochastic volatility) lead to a higher GMWB price, compared to the case when deterministic parameters are used. The fair guarantee fee is also an increasing function of the volatility of interest rates and a non-monotonic function of the volatility of variance parameter. Furthermore, it increases when the correlation between the underlying fund and the interest rate increases, and when the speed of mean-reversion of stochastic volatility increases. The GMWB price is a decreasing function of the speed of mean-reversion of stochastic interest rates. The fair guarantee fee also increases with increasing average long-term volatility; and decreases with increasing average long-term interest rates. We document a rather small (but negative) impact of incorporated mortality on the fair insurance fee, that is, the fair insurance fee decreases when mortality is incorporated in the model. Our analysis also provides recommendation for the optimal withdrawal schedule depending on the values of the withdrawal and the investments account, and varying volatility and interest rates.

Our paper provides a comprehensive analysis of GMBWs, highlighting the risk management benefits and costs for insurance providers offering these guarantees. Several extensions and further analysis could be of interest. In particular, the dynamic withdrawal behaviour could be analysed within the modelling framework which incorporates stochastic mortality. In addition, all model parameters could be calibrated to real data. We leave these problems for future work.

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A Appendices

A.1 Proof of Proposition 1

Consider the following stochastic processes where we introduce $\tilde{W}_t$ on an unrestricted domain such that

$$\begin{aligned} d\tilde{W}_t &= ((r_t-\alpha)\tilde{W}_t - G)dt + \rho_{11}\sqrt{v_t}\tilde{W}_t dZ_t^1 + \rho_{12}\sqrt{v_t}\tilde{W}_t dZ_t^2 + \rho_{13}\sqrt{v_t}\tilde{W}_t dZ_t^3, \\ dv_t &= \zeta_v(v_t,t)dt + \rho_{22}\sigma_v(v_t,t)dZ_t^2 + \rho_{23}\sigma_v(v_t,t)dZ_t^3, \\ dr_t &= \zeta_r(r_t,t)dt + \sigma_r(r_t,t)dZ_t^3, \end{aligned} \tag{A1}$$

with $\tilde{W}_t \in (-\infty,\infty)$, $v_t \in (0,\infty)$ and $r_t \in (0,\infty)$.

In finding the solution to Eq. (A1) we consider the stochastic integrating factor associate with this system which can be represented as

$$\Phi_t = \exp\left(-\rho_{11}\int_0^t \sqrt{v_u}dZ_u^1 - \rho_{12}\int_0^t \sqrt{v_u}dZ_u^2 - \rho_{13}\int_0^t \sqrt{v_u}dZ_u^3 + \frac{1}{2}\int_0^t v_u du\right). \tag{A2}$$

The above equation can be expressed in differential form as

$$d\Phi_t \quad = \quad \Phi_t\left(-\rho_{11}\sqrt{v_t}dZ_t^1 - \rho_{12}\sqrt{v_t}dZ_t^2 - \rho_{13}\sqrt{v_t}dZ_t^3 + v_t dt\right).$$

Consider a stochastic process given by the product, $\Phi_t\tilde{W}_t$. Applying the Itô's lemma to this product yields

$$\begin{aligned} d(\Phi_t\tilde{W}_t) &= d\Phi_t\tilde{W}_t + \Phi_t d\tilde{W}_t + d\Phi_t d\tilde{W}_t \\ &= \Phi_t\left(-\rho_{11}\sqrt{v_t}dZ_t^1 - \rho_{12}\sqrt{v_t}dZ_t^2 - \rho_{13}\sqrt{v_t}dZ_t^3 + v_t dt\right)\tilde{W}_t \\ &+ \Phi_t\left(((r_t-\alpha)\tilde{W}_t - G)dt + \rho_{11}\sqrt{v_t}dZ_t^1 + \rho_{12}\sqrt{v_t}dZ_t^2 + \rho_{13}\sqrt{v_t}dZ_t^3\right) + \Phi_t(-v_t dt)\tilde{W}_t \\ &= \Phi_t((r_t-\alpha)\tilde{W}_t - G)dt. \end{aligned}$$

We now consider the discounted process $e^{-\int_0^t (r_u-\alpha)du}\Phi_t\tilde{W}_t$. Applying Itô's lemma to the discounted process yields

$$\begin{aligned} d\left(e^{-\int_0^t (r_u-\alpha)du}\Phi_t\tilde{W}_t\right) &= -(r_t-\alpha)e^{-\int_0^t (r_u-\alpha)du}dt\Phi_t\tilde{W}_t + e^{-\int_0^t (r_u-\alpha)du}d(\Phi_t\tilde{W}_t) \\ &= -(r_t-\alpha)e^{-\int_0^t (r_u-\alpha)du}dt\Phi_t\tilde{W}_t + e^{-\int_0^t (r_u-\alpha)du}\Phi_t((r-\alpha)\tilde{W}_t - G)dt \\ &= -e^{-\int_0^t (r_u-\alpha)du}\Phi_t G dt, \end{aligned} \tag{A3}$$

implying that

$$\tilde{W}_t = \Phi_t^{-1}e^{\int_0^t (r_u-\alpha)du}\left(W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du\right), \tag{A4}$$

which is the solution to the system of equations in (A1).

Now, comparing W_t and $\tilde{W}_t$ as presented in the systems (2.8) and (A1), respectively, we notice that both processes are the same in the positive domain. However, W_t, can never go below zero due the guarantee feature implying that zero is an absorbing state for W_t. Letting $\tilde{W}_0 = W_0$, for $0 \leq t < \tau_0 = \inf_{t\in(0,T)}[W_t = 0]$ we note that $d\tilde{W}_t \equiv dW_t$. It has been shown in (A4) above that the solution to Eq.(A1) is (A4) implying that $\tilde{W}_t \leq 0$ if and only if $(W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du) \leq 0$, since $\Phi_t^{-1}e^{-\int_0^t (r_u-\alpha)du} > 0$ for all t. By definition of τ_0 we have that $W_0 - G\int_0^{\tau_0} e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du = 0$. Moreover,

$$\frac{\partial\left(W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du\right)}{\partial t} = -(r-\alpha)e^{-\int_0^u (r_s-\alpha)ds}\Phi_t < 0,$$

for all t, which implies that

$$W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du$$

is a monotonically decreasing function in t. Thus $W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du \le 0$ for $\forall t \ge \tau_0$, which implies that $\tilde{W}_t \le 0$ for $\forall t \ge \tau_0$. Therefore we obtain the result that

$$Prob\left(\tilde{W_T} > 0 | \tilde{W_t} \le 0, t < T\right) = 0.$$

Since $d\tilde{W}_t \equiv dW_t$ for $0 \le t < \tau_0$ and $\tilde{W}_0 = W_0$, we note that probability density functions of $\tilde{W}_t$ and W_t are equivalent such that $p_{\tilde{W_T}}(x) \equiv p_{W_T}(x)$ for $x > 0$ and $Prob(W_T = 0) = \int_{-\infty}^{0-} p_{\tilde{W_T}}(x)dx + Prob(\tilde{W_T} = 0)$. Thus, by definition of expectation we have:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[W_T] &= 0 \times Prob(W_T = 0) + \int_{0^+}^{+\infty} x p_{W_T}(x)dx \\ &= \int_{0^+}^{+\infty} x p_{W_T}(x)dx = \int_{0^+}^{+\infty} x p_{\tilde{W_T}}(x)dx = \mathbb{E}^{\mathbb{Q}}[\max(\tilde{W_T}, 0)].\end{aligned}$$

Moreover, we have that the solution to Eq.(2.8) for the restricted process W_t is given by

$$W_t = \Phi_t^{-1} e^{\int_0^t (r_u-\alpha)du} \max\left[W_0 - G\int_0^t e^{-\int_0^u (r_s-\alpha)ds}\Phi_u du, 0\right]. \tag{A5}$$

A.2 Backward approximation of the second and third order

Let $U(x,y,z)$ be a function $\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ having a finite nth derivative $\frac{\partial^n U(x,y,z)}{\partial x^n}$ everywhere in the open interval (a,b) and assume the $(n-1)$th derivative $\frac{\partial^{n-1} U(x,y,z)}{\partial x^{n-1}}$ is continuous on the closed interval $[a,b]$. Apostol (1965) states that for every $x \in [a,b]$, $x \ne x_0$, then

$$U(x,y,z) = U(x_0,y,z) + \sum_{i=1}^{n} \frac{(x-x_0)^i}{i!}\frac{\partial^i U(x_0,y,z)}{\partial x^i} + o((x-x_0)^n), \tag{A6}$$

where the function $o(x-x_0)^n$ is defined as $\lim_{n\to\infty}\frac{o((x-x_0)^n)}{(x-x_0)^n} = 0$. This implies that

$$\begin{aligned}U(t_n, s_i \pm k\Delta s, v_j) = U(t_n,s_i,v_j) &\pm k\Delta s\frac{\partial U(t_n,s_i,v_j)}{\partial s} + \frac{(k\Delta s)^2}{2}\frac{\partial^2 U(t_n,s_i,v_j)}{\partial s^2} \\ &\pm \frac{(k\Delta s)^3}{3!}\frac{\partial^3 U(t_n,s_i,v_j)}{\partial s^3} + \frac{(k\Delta s)^4}{4!}\frac{\partial^4 U(t_n,s_i,v_j)}{\partial s^4} + o((k\Delta s)^4),\end{aligned} \tag{A7}$$

For points s_i, s_{i-1} and s_{i-2} we have

$$\begin{aligned}U(t_n,s_{i-2},v_j) - 4U(t_n,s_{i-1},v_j) + U(t_n,s_i,v_j) &= U(t_n, s_i - 2\Delta s, v_j) \\ - 4U(t_n, s_i - \Delta s, v_j) + U(t_n,s_i,v_j) &= 2\Delta s\frac{\partial U(t_n,s_i,v_j)}{\partial s} + o((k\Delta s)^2).\end{aligned} \tag{A8}$$

This yields

$$\frac{\partial U(\tau_n,s_i,v_j)}{\partial s} \approx \frac{U_{i-2,j}^n - 4U_{i-1,j}^n + 3U_{i,j}^n}{2\Delta s} + o((k\Delta s)^2), \tag{A9}$$

which is Eq.(3.2).

Similarly, for points s_{-1}, s_0, s_1 and s_2 we have

$$\begin{aligned}-8U(t_n,s_{-1},v_j) - 3U(t_n,s_0,v_j) + 12U(t_n,s_1,v_j) + U(t_n,s_2,v_j) &= -8U(t_n, s_0-\Delta s, v_j) - 3U(t_n,s_0,v_j) \\ + 12U(t_n, s_0+\Delta s, v_j) + U(t_n, s_0+2\Delta s, v_j) &= 18\Delta s\frac{\partial U(t_n,s_0,v_j)}{\partial s} + o((k\Delta s)^3),\end{aligned} \tag{A10}$$

which yields

$$\frac{\partial U(\tau_n,s_0,v_j)}{\partial s} \approx \frac{-8U_{-1,j}^n - 3U_{0,j}^n + 12U_{1,j}^n - U_{2,j}^n}{18\Delta s} + o((k\Delta s)^3), \tag{A11}$$

as presented in Eq. (3.3).

A.3 Discrete differential operators

We substitute approximations presented in Duffy (2006) to obtain discrete differential operators $\mathcal{L}_s^n U_{i,j,k}^n$, $\mathcal{L}_v^n U_{i,j,k}^n$, $\mathcal{L}_r^n U_{i,j,k}^n$, $\mathcal{L}_{sv}^n U_{i,j,k}^n$, $\mathcal{L}_{sr}^n U_{i,j,k}^n$ and $\mathcal{L}_{vr}^n U_{i,j,k}^n$.

In the s-direction we have[15]

$$\mathcal{L}_s^n U_{0,j,k}^n = \phi(\tau_n, s_0, r_k) \frac{-19U_{0,j,k}^n + 20U_{1,j,k}^n - U_{2,j,k}^n}{18\Delta s} \tag{A12}$$

$$\mathcal{L}_s^n U_{1,j,k}^n = \phi(\tau_n, s_1, r_k) \frac{-U_{0,j,k}^n + U_{1,j,k}^n}{\Delta s} + \frac{1}{2}\psi(\tau_n, s_1, v_j)^2 \frac{U_{2,j,k}^n - 2U_{1,j,k}^n + U_{0,j,k}^n}{(\Delta s)^2}, \tag{A13}$$

$$\begin{aligned} \mathcal{L}_s^n U_{i,j,k}^n &= \phi(\tau_n, s_i, r_k) \frac{U_{i-2,j,k}^n - 4U_{i-1,j,k}^n + 3U_{i,j,k}^n}{2\Delta s} \\ &+ \frac{1}{2}\psi(\tau_n, s_i, v_j)^2 \frac{U_{i+1,j,k}^n - 2U_{i,j,k}^n + U_{i-1,j,k}^n}{(\Delta s)^2}, \text{ for } i = \{2, ..., i^* - 1\}, \end{aligned} \tag{A14}$$

$$\begin{aligned} \mathcal{L}_s^n U_{i,j,k}^n &= \phi(\tau_n, s_i, r_k) \frac{U_{i+1,j,k}^n - U_{i-1,j,k}^n}{2\Delta s} \\ &+ \frac{1}{2}\psi(\tau_n, s_i, v_j)^2 \frac{U_{i+1,j,k}^n - 2U_{i,j,k}^n + U_{i-1,j,k}^n}{(\Delta s)^2}, \text{ for } i = \{i^*, ..., N_s - 1\}, \end{aligned} \tag{A15}$$

$$\mathcal{L}_s^n U_{N_s,j,k}^n = \phi(\tau_n, s_{N_s}, r_k) \frac{U_{N_s,j,k}^n - U_{N_s-1,j,k}^n}{\Delta s}, \tag{A16}$$

In the v-direction we have

$$\mathcal{L}_v^n U_{i,0,k}^n = \xi_v(\tau_n, v_0) \frac{U_{i,1,k}^n - U_{i,0,k}^n}{\Delta v}, \tag{A17}$$

$$\begin{aligned} \mathcal{L}_v^n U_{i,j,k}^n &= \xi_v(\tau_n, v_j) \frac{U_{i,j+1,k}^n - U_{i,j-1,k}^n}{2\Delta v} \\ &+ \frac{1}{2}\beta_v(\tau_n, v_j)^2 \frac{U_{i,j+1,k}^n - 2U_{i,j,k}^n + U_{i,j-1,k}^n}{\Delta v^2}, \text{ for } j = \{1, ..., j^* - 1\}, \end{aligned} \tag{A18}$$

$$\begin{aligned} \mathcal{L}_v^n U_{i,j,k}^n &= \xi_v(\tau_n, v_j) \frac{U_{i,j-2,k}^n - 4U_{i,j-1,k}^n + 3U_{i,j,k}^n}{2\Delta v} \\ &+ \frac{1}{2}\beta_v(\tau_n, v_j)^2 \frac{U_{i,j+1,k}^n - 2U_{i,j,k}^n + U_{i,j-1,k}^n}{\Delta v^2}, \text{ for } j = \{j^*, ..., N_v - 1\}, \end{aligned} \tag{A19}$$

$$\mathcal{L}_v^n U_{i,N_v,k}^n = \xi_v(\tau_n, v_{N_v}) \frac{U_{i,N_v-2,k}^n - 4U_{i,N_v-1,k}^n + 3U_{i,N_v,k}^n}{2\Delta v}, \tag{A20}$$

In the r-direction we obtain

$$\mathcal{L}_r^n U_{i,j,0}^n = \xi_r(\tau_n, r_0) \frac{U_{i,j,1}^n - U_{i,j,0}^n}{\Delta r}, \tag{A21}$$

$$\begin{aligned} \mathcal{L}_r^n U_{i,j,k}^n &= \xi_r(\tau_n, r_k) \frac{U_{i,j,k+1}^n - U_{i,j,k-1}^n}{2\Delta r} \\ &+ \frac{1}{2}\beta_r(\tau_n, r_k)^2 \frac{U_{i,j,k+1}^n - 2U_{i,j,k}^n + U_{i,j,k-1}^n}{\Delta r^2}, \text{ for } k = \{1, ..., k^* - 1\}, \end{aligned} \tag{A22}$$

$$\begin{aligned} \mathcal{L}_r^n U_{i,j,k}^n &= \xi_r(\tau_n, r_k) \frac{U_{i,j,k-2}^n - 4U_{i,j,k-1}^n + 3U_{i,j,k}^n}{2\Delta r} \\ &+ \frac{1}{2}\beta_r(\tau_n, r_k)^2 \frac{U_{i,j,k+1}^n - 2U_{i,j,k}^n + U_{i,j,k-1}^n}{\Delta r^2}, \text{ for } k = \{k^*, ..., N_r - 1\}, \end{aligned} \tag{A23}$$

$$\mathcal{L}_r^n U_{i,j,N_r}^n = \xi_r(\tau_n, r_{N_r}) \frac{U_{i,j,N_r-2}^n - 4U_{i,j,N_r-1}^n + 3U_{i,j,N_r}^n}{2\Delta r}, \tag{A24}$$

In the mixed $sv-$ direction we have

$$\mathcal{L}_{sv}^n U_{i,j,k}^n = \rho_{12}\psi(\tau_n, s_i, v_j)\beta_v(\tau_n, v_j) \left(\frac{U_{i+1,j+1,k}^n - U_{i-1,j+1,k}^n - U_{i+1,j-1,k}^n + U_{i-1,j-1,k}^n}{4\Delta s \Delta v} \right), \tag{A25}$$

[15] In what follows, we will make use of the fact that the second derivative terms with respect to all state variables vanish at the boundary points.

for $i = \{1, ..., N_s - 1\}$, $j = \{1, ..., N_v - 1\}$ and $k = \{0, ..., N_r\}$.

$$\mathcal{L}^n_{sv} U^n_{i,j,k} = 0, \text{ when } i = \{0, N_s\} \text{ or } j = \{0, N_v\}. \tag{A26}$$

In the mixed $sr-$ direction we have

$$\mathcal{L}^n_{sr} U^n_{i,j,k} = \rho_{13}\psi(\tau_n, s_i, v_j)\beta_r(\tau_n, r_k)\left(\frac{U^n_{i+1,j,k+1} - U^n_{i-1,j,k+1} - U^n_{i+1,j,k-1} + U^n_{i-1,j,k-1}}{4\Delta s \Delta r}\right), \tag{A27}$$

for $i = \{1, ..., N_s - 1\}$, $k = \{1, ..., N_r - 1\}$ and $j = \{0, ..., N_v\}$.

$$\mathcal{L}^n_{sr} U^n_{i,j,k} = 0, \text{ when } i = \{0, N_s\} \text{ or } k = \{0, N_r\}. \tag{A28}$$

In the mixed $vr-$ direction we have

$$\mathcal{L}^n_{vr} U^n_{i,j,k} = \rho_{23}\beta_v(\tau_n, v_j)\beta_r(\tau_n, r_k)\left(\frac{U^n_{i,j+1,k+1} - U^n_{i,j+1,k-1} - U^n_{i,j-1,k+1} + U^n_{i,j-1,k-1}}{4\Delta v \Delta r}\right), \tag{A29}$$

for $i = \{0, ..., N_s\}$, $j = \{1, ..., N_v - 1\}$ and $k = \{1, ..., N_r - 1\}$.

$$\mathcal{L}^n_{vr} U^n_{i,j,k} = 0, \text{ when } k = \{0, N_r\} \text{ or } j = \{0, N_v\}. \tag{A30}$$

A.4 The coupled differenced PDE system

From the discrete problem formulated in Eq. (3.16), we can develop an operator splitting algorithm for approximating the solution of the PDE in Eq. (2.14). We divide each time step from time τ_n to τ_{n+1} into three substeps such that

$$\frac{U^{n+\frac{1}{3}}_{i,j,k} - U^n_{i,j,k}}{\Delta\tau} = \mathcal{L}^n_s U^{n+\frac{1}{3}}_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{sv} U^n_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{sr} U^n_{i,j,k}, \tag{A31}$$

$$\frac{U^{n+\frac{2}{3}}_{i,j,k} - U^{n+\frac{1}{3}}_{i,j,k}}{\Delta\tau} = \mathcal{L}^n_v U^{n+\frac{2}{3}}_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{sv} U^{n+\frac{1}{3}}_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{vr} U^{n+\frac{1}{3}}_{i,j,k}, \tag{A32}$$

$$\frac{U^{n+1}_{i,j,k} - U^{n+\frac{2}{3}}_{i,j,k}}{\Delta\tau} = \mathcal{L}^n_r U^{n+1}_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{sr} U^{n+\frac{2}{3}}_{i,j,k} + \frac{1}{2}\mathcal{L}^n_{vr} U^{n+\frac{2}{3}}_{i,j,k}. \tag{A33}$$

Thus, moving backwards from $n = 0$, for which we have terminal condition presented in Eq. (2.15) enables us to estimate the values $U^n_{i,j,k}$ for $n = \{1, ..., N_\tau\}$.

A.5 Elements of the matrices

Here $\boldsymbol{D_1}$ is a sparse $(N_s + 1) \times (N_s + 1)$ matrix given by

$$\boldsymbol{D_1} = \begin{pmatrix}
\Gamma^{n+\frac{1}{3}}_{0jk} & \Sigma^{n+\frac{1}{3}}_{0jk} & \Lambda^{n+\frac{1}{3}}_{0jk} & 0 & .... & .... & .... & .... & 0 \\
\Theta^{n+\frac{1}{3}}_{1jk} & \Gamma^{n+\frac{1}{3}}_{1jk} & \Sigma^{n+\frac{1}{3}}_{1jk} & 0 & .... & .... & .... & .... & 0 \\
\Upsilon^{n+\frac{1}{3}}_{2jk} & \Theta^{n+\frac{1}{3}}_{2jk} & \Gamma^{n+\frac{1}{3}}_{2jk} & \Sigma^{n+\frac{1}{3}}_{2jk} & .... & .... & .... & .... & 0 \\
.... & .... & .... & .... & .... & .... & .... & .... & .... \\
0 & .... & \Upsilon^{n+\frac{1}{3}}_{i^*-1jk} & \Theta^{n+\frac{1}{3}}_{i^*-1jk} & \Gamma^{n+\frac{1}{3}}_{i^*-1jk} & \Sigma^{n+\frac{1}{3}}_{i^*-1jk} & .... & .... & 0 \\
0 & .... & .... & 0 & \Theta^{n+\frac{1}{3}}_{i^*jk} & \Gamma^{n+\frac{1}{3}}_{i^*jk} & \Sigma^{n+\frac{1}{3}}_{i^*jk} & .... & 0 \\
.... & .... & .... & .... & .... & .... & .... & .... & .... \\
0 & .... & .... & .... & .... & .... & \Theta^{n+\frac{1}{3}}_{N_s-1jk} & \Gamma^{n+\frac{1}{3}}_{N_s-1jk} & \Sigma^{n+\frac{1}{3}}_{N_s-1jk} \\
0 & .... & .... & .... & .... & .... & .... & \Theta^{n+\frac{1}{3}}_{N_sjk} & \Gamma^{n+\frac{1}{3}}_{N_sjk}
\end{pmatrix},$$

where

$$\Gamma_{0jk}^{n+\frac{1}{3}} = \frac{19\phi(\tau_{n+\frac{1}{3}}, s_0, r_k)}{18\Delta s} + \frac{1}{\Delta\tau},\ \Sigma_{0jk}^{n+\frac{1}{3}} = -\frac{20\phi(\tau_{n+\frac{1}{3}}, s_0, r_k)}{18\Delta s},\ \Lambda_{0jk}^{n+\frac{1}{3}} = \frac{\phi(\tau_{n+\frac{1}{3}}, s_0, r_k)}{18\Delta s},$$

$$\Theta_{1jk}^{n+\frac{1}{3}} = \frac{\phi(\tau_{n+\frac{1}{3}}, s_1, r_k)}{\Delta s} - \frac{\psi(\tau_{n+\frac{1}{3}}, s_1, r_k)^2}{2(\Delta s)^2},\ \Gamma_{1jk}^{n+\frac{1}{3}} = -\frac{\phi(\tau_{n+\frac{1}{3}}, s_1, r_k)}{\Delta s} + \frac{\psi(\tau_{n+\frac{1}{3}}, s_1, v_j)^2}{(\Delta s)^2} + \frac{1}{\Delta\tau},$$

$$\Sigma_{1jk}^{n+\frac{1}{3}} = -\frac{\psi(\tau_{n+\frac{1}{3}}, s_1, v_j)^2}{2(\Delta s)^2},\quad \Upsilon_{ijk}^{n+\frac{1}{3}} = -\frac{\phi(\tau_{n+\frac{1}{3}}, s_i, r_k)}{2\Delta s},\ \Theta_{ijk}^{n+\frac{1}{3}} = \frac{2\phi(\tau_{n+\frac{1}{3}}, s_i, r_k)}{\Delta s} - \frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{2(\Delta s)^2},$$

$$\Gamma_{ijk}^{n+\frac{1}{3}} = -\frac{3\phi(\tau_{n+\frac{1}{3}}, s_i, r_k)}{2\Delta s} + \frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{(\Delta s)^2} + \frac{1}{\Delta\tau},\ \Sigma_{ijk}^{n+\frac{1}{3}} = -\frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{2(\Delta s)^2},$$

for $i = \{2, ..., i^* - 1\}$,

$$\Theta_{ijk}^{n+\frac{1}{3}} = \frac{\phi(\tau_{n+\frac{1}{3}}, s_i, r_k)}{2\Delta s} - \frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{2(\Delta s)^2},\ \Gamma_{ijk}^{n+\frac{1}{3}} = \frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{(\Delta s)^2} + \frac{1}{\Delta\tau},$$

$$\Sigma_{ijk}^{n+\frac{1}{3}} = -\frac{\phi(\tau_{n+\frac{1}{3}}, s_i, r_k)}{2\Delta s} - \frac{\psi(\tau_{n+\frac{1}{3}}, s_i, v_j)^2}{2(\Delta s)^2},$$

for $i = \{i^*, ..., N_s - 1\}$, and

$$\Theta_{N_sjk}^{n+\frac{1}{3}} = \frac{\phi(\tau_{n+\frac{1}{3}}, s_{N_s}, r_k)}{\Delta s},\ \Gamma_{N_sjk}^{n+\frac{1}{3}} = -\frac{\phi(\tau_{n+\frac{1}{3}}, s_{N_s}, r_k)}{\Delta s} + \frac{1}{\Delta\tau}.$$

The matrix $\boldsymbol{D_2}$ is a sparse $(N_v + 1) \times (N_v + 1)$ matrix given as

$$\boldsymbol{D_2} = \begin{pmatrix} \Gamma_{i0k}^{n+\frac{2}{3}} & \Sigma_{i0k}^{n+\frac{2}{3}} & 0 & .... & .... & .... & .... & .... & 0 \\ \Theta_{i1k}^{n+\frac{2}{3}} & \Gamma_{i1k}^{n+\frac{2}{3}} & \Sigma_{i1k}^{n+\frac{2}{3}} & 0 & .... & .... & .... & .... & 0 \\ .... & .... & .... & .... & .... & .... & .... & .... & .... \\ 0 & .... & 0 & \Theta_{ij^*-1k}^{n+\frac{2}{3}} & \Gamma_{ij^*-1}^{n+\frac{2}{3}} & \Sigma_{ij^*-1k}^{n+\frac{2}{3}} & .... & .... & 0 \\ 0 & .... & .... & \Upsilon_{ij^*k}^{n+\frac{2}{3}} & \Theta_{ij^*k}^{n+\frac{2}{3}} & \Gamma_{ij^*k}^{n+\frac{2}{3}} & \Sigma_{ij^*k}^{n+\frac{2}{3}} & .... & 0 \\ .... & .... & .... & .... & .... & .... & .... & .... & .... \\ 0 & .... & .... & .... & .... & \Upsilon_{iN_v-1k}^{n+\frac{2}{3}} & \Theta_{iN_v-1k}^{n+\frac{2}{3}} & \Gamma_{iN_v-1k}^{n+\frac{2}{3}} & \Sigma_{iN_v-1k}^{n+\frac{2}{3}} \\ 0 & .... & .... & .... & .... & .... & \Upsilon_{iN_vk}^{n+\frac{2}{3}} & \Theta_{iN_vk}^{n+\frac{2}{3}} & \Gamma_{iN_vk}^{n+\frac{2}{3}} \end{pmatrix},$$

where

$$\Gamma_{i0k}^{n+\frac{2}{3}} = \frac{\xi_v(\tau_{n+\frac{2}{3}}, v_0)}{\Delta v} + \frac{1}{\Delta\tau},\ \Sigma_{i0k}^{n+\frac{2}{3}} = -\frac{\xi_v(\tau_{n+\frac{2}{3}}, v_0)}{\Delta v},$$

$$\Theta_{ijk}^{n+\frac{2}{3}} = \frac{\xi_v(t_{n+\frac{2}{3}}, v_j)}{\Delta v} - \frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{2\Delta v^2},\ \Gamma_{ijk}^{n+\frac{2}{3}} = \frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{\Delta v^2} + \frac{1}{\Delta\tau},$$

$$\Sigma_{ijk}^{n+\frac{2}{3}} = -\frac{\xi_v(t_{n+\frac{2}{3}}, v_j)}{\Delta v} - \frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{2\Delta v^2},$$

for $j = \{1, ..., j^* - 1\}$,

$$\Upsilon_{ijk}^{n+\frac{2}{3}} = -\frac{\xi_v(t_{n+\frac{2}{3}}, v_j)}{2\Delta v},\ \Theta_{ijk}^{n+\frac{2}{3}} = \frac{2\xi_v(t_{n+\frac{2}{3}}, v_j)}{\Delta v} - \frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{2\Delta v^2},$$

$$\Gamma_{ijk}^{n+\frac{2}{3}} = -\frac{3\xi_v(t_{n+\frac{2}{3}}, v_j)}{2\Delta v} + \frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{\Delta v^2} + \frac{1}{\Delta\tau},\ \Sigma_{ijk}^{n+\frac{2}{3}} = -\frac{\beta_v(t_{n+\frac{2}{3}}, v_j)^2}{2\Delta v^2},$$

for $j = \{j^*, ..., N_v - 1\}$, and

$$\Upsilon_{iN_vk}^{n+\frac{2}{3}} = -\frac{\xi_v(t_{n+\frac{2}{3}}, v_{N_v})}{2\Delta v},\ \Theta_{iN_vk}^{n+\frac{2}{3}} = \frac{2\xi_v(t_{n+\frac{2}{3}}, v_{N_v})}{\Delta v},\quad \Gamma_{iN_v}^{n+\frac{2}{3}} = -\frac{3\xi_v(t_{n+\frac{2}{3}}, v_{N_v})}{2\Delta v} + \frac{1}{\Delta\tau}.$$

The third matrix, $\boldsymbol{D_3}$, is a sparse $(N_r+1)\times(N_r+1)$ matrix given as

$$\boldsymbol{D_3} = \begin{pmatrix} \Gamma_{ij0}^{n+1} & \Sigma_{ij0}^{n+1} & 0 & .... & .... & .... & .... & .... & 0 \\ \Theta_{ij1}^{n+1} & \Gamma_{ij1}^{n+1} & \Sigma_{ik1}^{n+1} & 0 & .... & .... & .... & .... & 0 \\ .... & .... & .... & .... & .... & .... & .... & .... & .... \\ 0 & .... & 0 & \Theta_{ijk^*-1}^{n+1} & \Gamma_{ijk^*-1}^{n+1} & \Sigma_{ijk^*-1}^{n+1} & .... & .... & 0 \\ 0 & .... & .... & \Upsilon_{ijk^*}^{n+1} & \Theta_{ijk^*}^{n+1} & \Gamma_{ijk^*}^{n+1} & \Sigma_{ijk^*}^{n+1} & .... & 0 \\ .... & .... & .... & .... & .... & .... & .... & .... & .... \\ 0 & .... & .... & .... & .... & \Upsilon_{ijN_r-1}^{n+1} & \Theta_{ijN_r-1}^{n+1} & \Gamma_{ijN_r-1}^{n+1} & \Sigma_{ijN_r-1}^{n+1} \\ 0 & .... & .... & .... & .... & .... & \Upsilon_{ijN_r}^{n+1} & \Theta_{ijN_r}^{n+1} & \Gamma_{ijN_r}^{n+1} \end{pmatrix},$$

where

$$\begin{aligned} \Gamma_{ij0}^{n+1} &= \frac{\xi_r(\tau_{n+1}, r_0)}{\Delta r} + \frac{1}{\Delta\tau},\ \Sigma_{ij0}^{n+1} = -\frac{\xi_r(\tau_{n+1}, r_0)}{\Delta r}, \\ \Theta_{ijk}^{n+1} &= \frac{\xi_r(\tau_{n+1}, r_k)}{\Delta r} - \frac{\beta_r(\tau_{n+1}, r_k)^2}{2\Delta r^2},\ \Gamma_{ijk}^{n+1} = \frac{\beta_r(\tau_{n+1}, r_k)^2}{\Delta r^2} + \frac{1}{\Delta\tau}, \\ \Sigma_{ijk}^{n+1} &= -\frac{\xi_r(t_{n+1}, r_k)}{\Delta r} - \frac{\beta_r(\tau_{n+1}, r_k)^2}{2\Delta r^2}, \end{aligned}$$

for $k = \{1, ..., k^*-1\}$,

$$\begin{aligned} \Upsilon_{ijk}^{n+1} &= -\frac{\xi_r(\tau_{n+1}, r_k)}{2\Delta r},\ \Theta_{ijk}^{n+1} = \frac{2\xi_r(\tau_{n+1}, r_k)}{\Delta r} - \frac{\beta_r(\tau_{n+1}, v_j)^2}{2\Delta r^2}, \\ \Gamma_{ijk}^{n+1} &= -\frac{3\xi_r(\tau_{n+1}, r_k)}{2\Delta r} + \frac{\beta_r(\tau_{n+1}, r_k)^2}{\Delta r^2} + \frac{1}{\Delta\tau},\ \Sigma_{ijk}^{n+1} = -\frac{\beta_r(\tau_{n+1}, r_k)^2}{2\Delta r^2}, \end{aligned}$$

for $k = \{k^*, ..., N_r-1\}$, and

$$\Upsilon_{ijN_r}^{n+1} = -\frac{\xi_r(\tau_{n+1}, r_{N_r})}{2\Delta r},\ \Theta_{ijN_r}^{n+1} = \frac{2\xi_r(\tau_{n+1}, r_{N_r})}{\Delta r},\quad \Gamma_{ijN_r}^{n+1} = -\frac{3\xi_r(\tau_{n+1}, r_{N_r})}{2\Delta r} + \frac{1}{\Delta\tau}.$$