From Gompertz to frailty models: Mortality modeling for actuarial applications

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Agenda

1. Introduction
2. The antecedents
3. Gompertz and beyond
4. Perks, Beard, and the logistic models
5. The aging process: Markov models
6. Concluding remarks

1 Introduction

Mortality and longevity modeling: a complex analysis involving several investigation fields:

- statistics
- demography
- medical sciences
- actuarial sciences
- ......

In what follows, nothing original but a roadmap to understand the diversity of “formulae” and “models” and the role of hypotheses in representing the impact of aging

2 The antecedents

See:
S. Haberman. Landmarks in the history of actuarial science (up to 1919). Actuarial Research Paper No. 84, Faculty of Actuarial Science & Insurance, City University, London, 1996. Available at:
`http://openaccess.city.ac.uk/2228/1/84-ARC.pdf`

Data: earliest life tables

- 220 dC: Ulpiano
- ......
- 1662: John Graunt
- 1693: Edmond Halley
- 1740: Nicholas Struyck
- ......

Earliest "formulae"

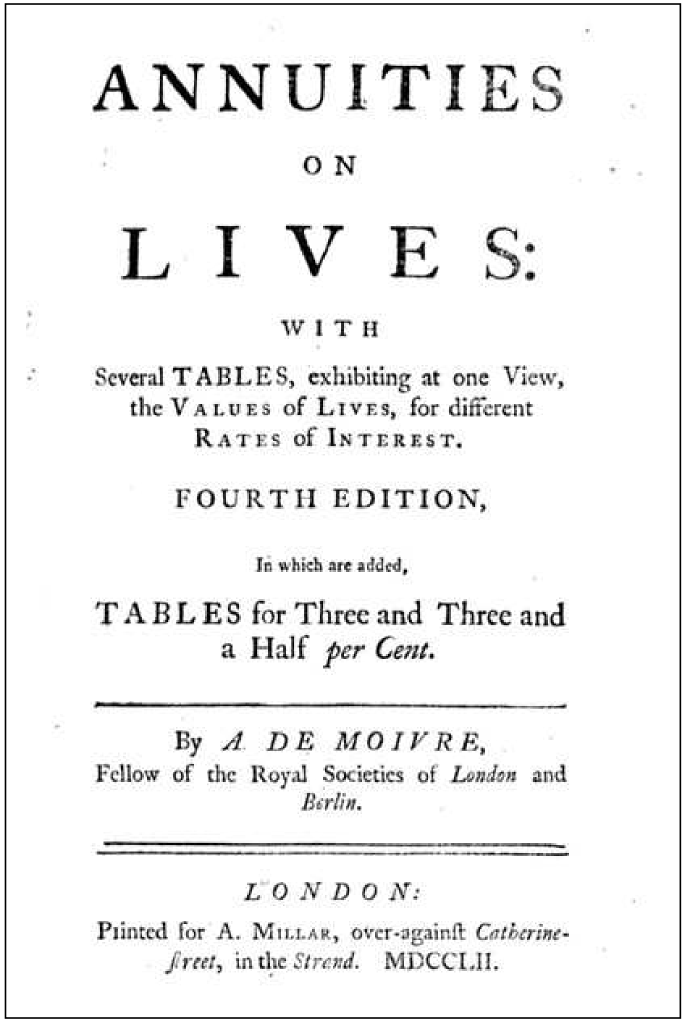
Need for summarizing a hundred of numbers (the life table) via a small set of parameters

Abraham De Moivre

French mathematician, author of pioneering contributions on:

- *probability*
- *trigonometry and complex numbers*
- *approximation formulae*
- *Fibonacci's sequence*
- *...*

Author of
"Annuities on Lives"
(1st edition: 1725)



ANNUITIES

ON

LIVES:

WITH

Several TABLES, exhibiting at one View,
the VALUES of LIVES, for different
RATES of INTEREST.

FOURTH EDITION,

In which are added,

TABLES for Three and Three and
a Half *per Cent.*

By *A. DE MOIVRE*,
Fellow of the Royal Societies of *London* and
Berlin.

LONDON:

Printed for A. MILLAR, over-againſt *Catherine-ſtreet*, in the *Strand*. MDCCLII.

De Moivre's law, 1725:

$$S(x) = 1 - \frac{x}{\omega} \qquad (\text{hypothesis: } \omega = 86)$$

Lambert's law, 1776:

$$S(x) = \left(\frac{a - x}{x}\right)^2 - b\left(\mathrm{e}^{\frac{x}{c}} - \mathrm{e}^{\frac{x}{d}}\right)$$

Babbage's law, 1823:

$$S(x) = 1 - a\,x - b\,x^2$$

3 Gompertz and beyond

Gompertz: the first “model”

Benjamin Gompertz

Self-taught mathematician and actuary

His formula (1825): the first biometric model, based on “ageing effect”

Benjamin Gompertz

The "ageing effect"

In terms of force of mortality:

$$\Delta\mu_x = \beta\,\varphi(x)\,\Delta x + O(\Delta x)$$

with $\beta > 0$ and $\varphi(x)$ increasing with age x

Gompertz observed an exponential behavior (over a broad age range) of the force μ_x

Hence: $\varphi(x) = \mu_x$ and

$$\mu_x = \alpha\,\mathrm{e}^{\beta x}$$

See: Gompertz [1825, 1860], Olshansky and Carnes [1997]

Remark 1

Gompertz stated the hypothesis in terms of survival functions, instead of force of mortality, concept introduced by T. R. Edmonds in 1832

Remark 2

Gompertz however noted that:

- two components contribute to mortality
  - ▷ ageing
  - ▷ "accidental" causes, independent of age

  (idea developed by Makeham)
- it is impossible to represent with one "simple" function the age-pattern of mortality over the whole life span
  (see the contribution by Thiele)

Some generalizations

First Makeham's law, 1867:

$$\mu_x = \gamma + \alpha \, \mathrm{e}^{\beta \, x}$$

Remark

First example of mortality "by causes"; see the following

Second Makeham's law, 1890:

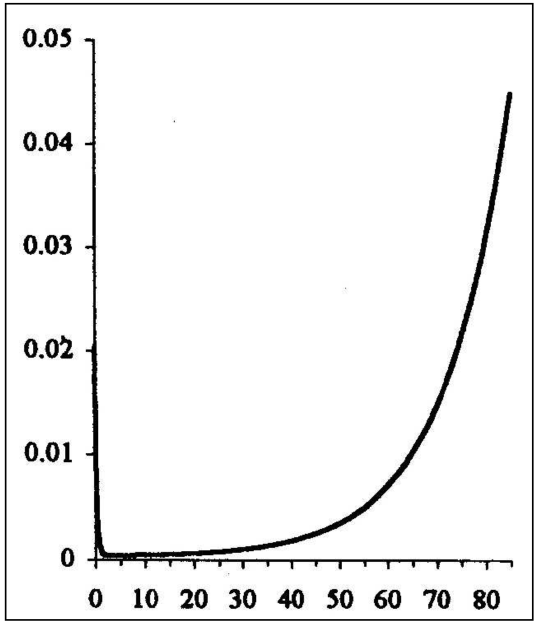
$$\mu_x = \gamma + \delta \, x + \alpha \, \mathrm{e}^{\beta \, x}$$

See: Makeham [1867, 1890]

Lazarus' law, 1867:

$$\mu_x = \alpha_1 \, \mathrm{e}^{-\beta_1 \, x} + \gamma + \alpha_2 \, \mathrm{e}^{\beta_2 \, x}$$

Generalize the first Makeham's law, aiming to represent, via negative exponential term, infant mortality decreasing with age



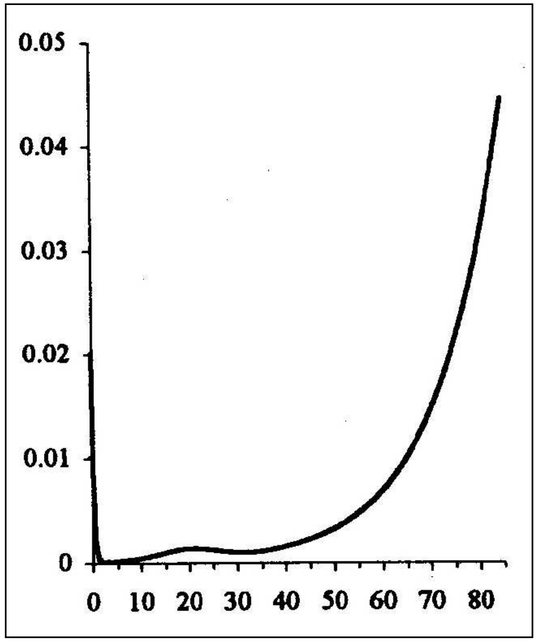
See: Graf [1906]

Remark

Abandoned because of computational intractability, almost unknown at international level, currently known as Siler's law; see Siler [1983]

Thiele's law, 1871:

$$\mu_x = \alpha_1\,\mathrm{e}^{-\beta_1\,x} + \alpha_3\,\mathrm{e}^{-\beta_3(x-\theta)^2} + \alpha_2\,\mathrm{e}^{\beta_2\,x}$$



Generalize first Makeham's law, aiming to represent age-pattern of mortality over the whole life span, infant and young-adult mortality included

See: Thiele [1871]

Remark

Abandoned because of computational intractability; developed by Heligman and Pollard, in terms of odds $\dfrac{q_x}{1-q_x}$. See: Heligman and Pollard [1980]

Among the recent proposals

D. O. Forfar, J. J. McCutcheon, A. D. Wilkie: *General family of "Gompertz-Makeham" models*

$$\mu_x = \mathrm{GM}_x^{(r,s)} = \sum_{i=1}^{r} \alpha_i \, x^{i-1} + \exp\Big(\sum_{i=r+1}^{r+s} \alpha_i \, x^{i-(r+1)} \Big)$$

In particular:

$(r, s) = (0, 2) \;\Rightarrow\;$ Gompertz's law
$(r, s) = (1, 2) \;\Rightarrow\;$ first Makeham's law
$(r, s) = (2, 2) \;\Rightarrow\;$ second Makeham's law

Various $\mathrm{GM}_x^{(r,s)}$ applied to CMI data

See: Forfar et al. [1988]

(A model or a formula . . . ?)

From Gompertz to Makeham: interpretations

Makeham's generalization can be interpreted in terms of a "shock" model (see, for example, Doray [2008])

Alternative interpretation provided by Lindholm [2017]

Assume that:

- all the individuals in a cohort follow a common baseline force of mortality $\mu_y = \mathrm{e}^{\beta\, y}$ (particular case of the Gompertz law), combined with an individual random variable Z, due to unobservable heterogeneity (proportional "frailty")
- Z follows a specific translated gamma distribution

$\Rightarrow$ The force of mortality in the cohort is given by Makeham law, with parameters depending on the parameters of the translated gamma distribution

4 Perks, Beard, and the logistic models

Perks' laws

To fit mortality data, W. Perks proposed:

$$\mu_x = \frac{\alpha\,\mathrm{e}^{\beta x} + \gamma}{\delta\,\mathrm{e}^{\beta x} + 1}$$

$$\mu_x = \frac{\alpha\,\mathrm{e}^{\beta x} + \gamma}{\delta\,\mathrm{e}^{\beta x} + \epsilon\,\mathrm{e}^{-\beta x} + 1}$$

See: Perks [1932]

- $\Rightarrow$ horizontal asymptote for the force of mortality
- $\Rightarrow$ mortality "deceleration"

  - ▷ Should the hypothesis of exponential behavior (Gompertz, Makeham, ecc.) be rejected ?
  - ▷ What is the "object" of deceleration ?

Beard and the "frailty"

See: Pitacco [2019] and references therein

Assume that:

- ▷ a cohort consists of heterogeneous individuals, w.r.t mortality because of unobservable risk factors
- ▷ the heterogeneity effect is quantified, for each individual, by a random positive real number, named *frailty* level
- ▷ the individual frailty level remains unchanged over the whole individual life span

For an individual age y with frailty level $z \Rightarrow$ force of mortality $\mu_y(z)$

Probability distribution of the frailty inside the cohort at age $y \Rightarrow$ pdf $g_y(z)$

Standard force of mortality (that is, for $z = 1$):

$$\mu_y = \mu_y(1)$$

Average force of mortality inside the cohort:

$$\bar{\mu}_y = \int_0^{+\infty} \mu_y(z)\, g_y(z)\, \mathrm{d}z$$

Specific models and results are based on:

1. relation between $\mu_y(z)$ and $\mu_y = \mu_y(1)$
2. distribution of the frailty at a given age x, e,g. $x = 0$: $g_0(z)$
3. model for μ_y

In particular, combining:

1. multiplicative hypothesis: $\mu_y(z) = z\, \mu_y$
2. Gamma distribution of the frailty, with parameters δ, θ
3. Gompertz's law for the standard force of mortality $\mu_y = \alpha\, \mathrm{e}^{\beta\, y}$

we find:

$$\bar{\mu}_y = \frac{\alpha' \, \mathrm{e}^{\beta y}}{\delta' \, \mathrm{e}^{\beta y} + 1}$$

that is, the *Gompertz - Gamma model*: Beard's law $\Rightarrow$ particular case of the first Perks' law, with parameters α', δ' depending on the parameters δ, θ of the frailty distribution

- $\Rightarrow$ logistic force of mortality
- $\Rightarrow$ mortality deceleration in the cohort implied by the frailty model

See: Beard [1959], Vaupel et al. [1979]

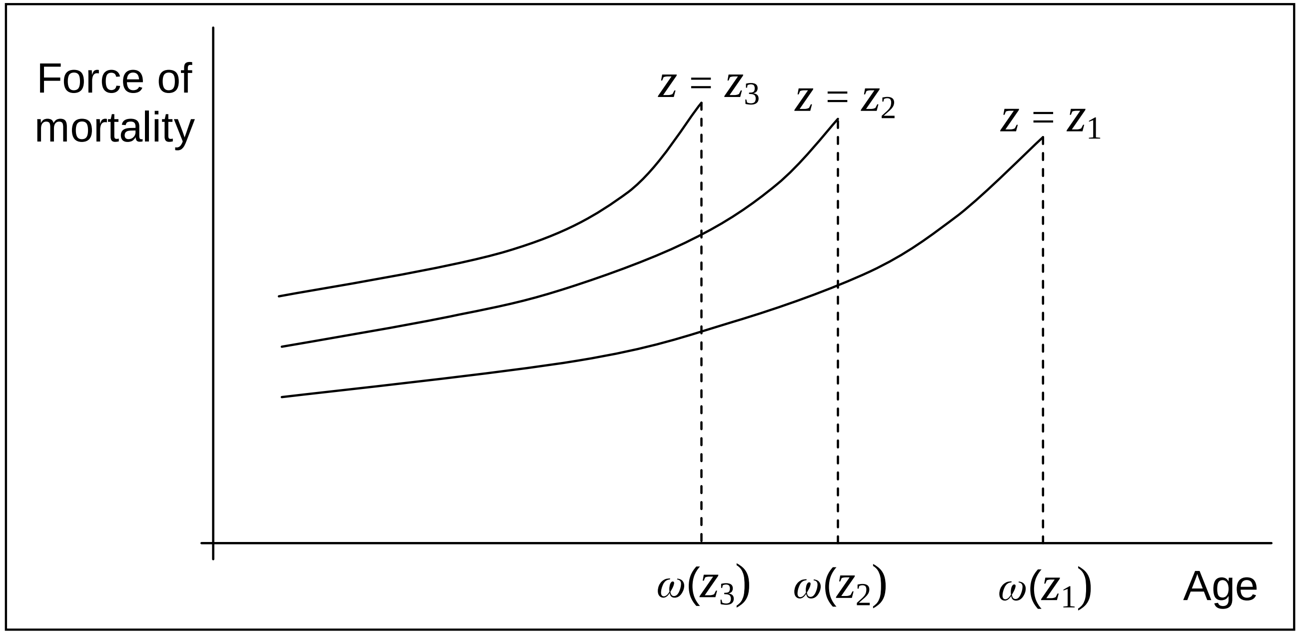
Naïve interpretation of the logistic shape

Assume that a (homogeneous) cohort only consists of individuals with a low frailty level; according to mortality observations:

- ▷ force of mortality $\mu_y(z_1)$
- ▷ maximum attained age $\omega(z_1)$

with z_1 = hypothetical frailty level

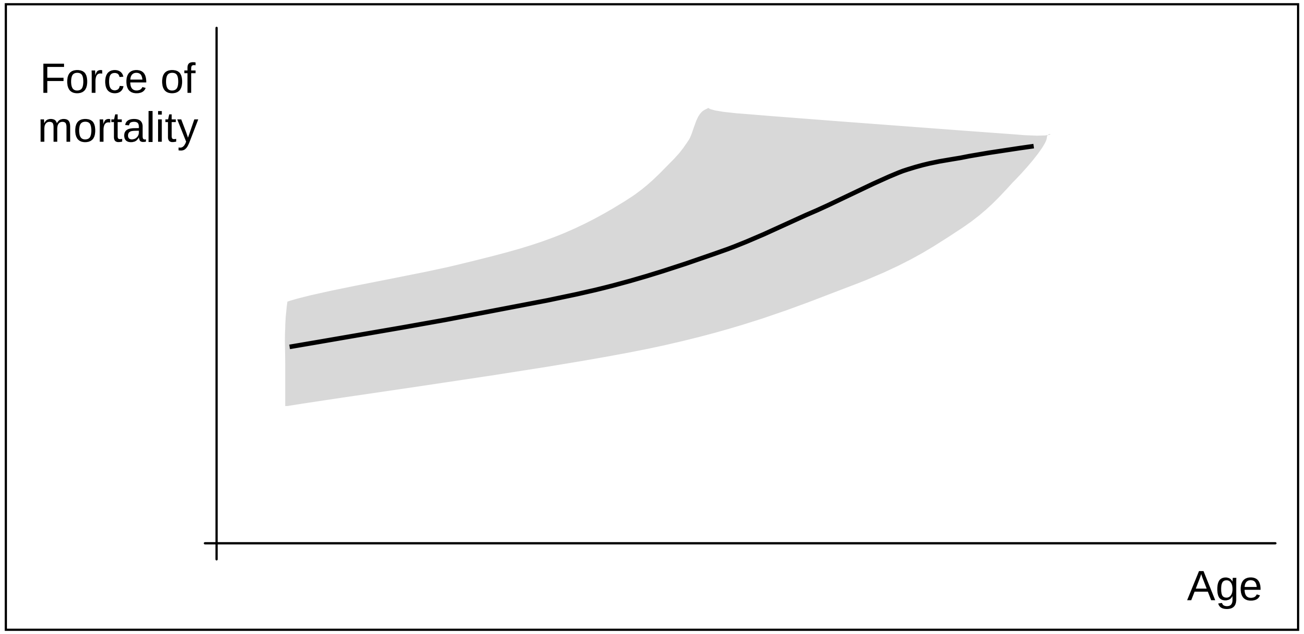
Similar results for homogeneous cohorts only consisting of individuals with mean (z_2) and high (z_3) frailty level respectively



Set of forces of mortality depending on the frailty level z

Referring to a cohort consisting of individuals with diverse frailty levels, the average force of mortality in the cohort increases at a decreasing rate because:

- individual remaining exposed to risk of death have a gradually decreasing frailty level
- the average frailty decreases



Average force of mortality in the cohort

A number of generalizations proposed; for example:

- ▷ force of mortality expressed by
  - ◦ Makeham's law (Beard [1959])
  - ◦ Weibull's law (Manton et al. [1986])
- ▷ frailty distribution given by:
  - ◦ inverse Gaussiam (Hougaard [1984, 1986], Manton et al. [1986], Butt and Haberman [2002, 2004])
  - ◦ shifted Gamma distribution (Martinelle [1987])
  - ◦ generalized Gamma distribution, including, as particular cases, lognormal and Weibull distributions (Balakrishnan and Peng [2006])

For a more general framework, see: Duchateau and Janssen [2008], and Wienke [2003]

Critical aspects of frailty models: Yashin et al. [2001]

A survey on mortality heterogeneity: Pitacco [2019]

Some recent logistic models

Kannisto's law (Kannisto [1994]):

$$\mu_x = \frac{\alpha\, \mathrm{e}^{\beta x}}{\alpha\, \mathrm{e}^{\beta x} + 1}$$

(that is, first Perks' law with $\gamma = 0$ and $\delta = \alpha$)

Thatcher's law (Thatcher [1999]):

$$\mu_x = \frac{\nu\, \alpha\, \mathrm{e}^{\beta x}}{\alpha\, \mathrm{e}^{\beta x} + 1} + \kappa$$

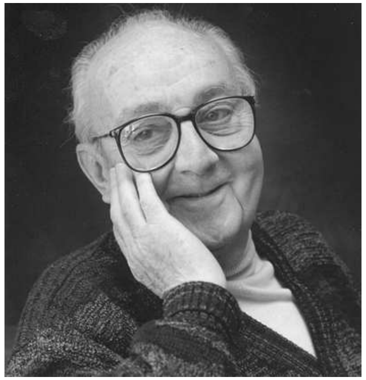
Simplified version, to analyze mortality trends at high ages:

$$\mu_x = \frac{\alpha\, \mathrm{e}^{\beta x}}{\alpha\, \mathrm{e}^{\beta x} + 1} + \kappa$$

Some remarks . . .

All models are wrong,
but some are useful

George E. P. Box (1978)



The practical question is
how wrong do they have to be
to not be useful

George E. P. Box (1987)

Mortality deceleration

And what about the deceleration phenomenon ?

A first achievement:

- the frailty model (under rather general conditions) implies the deceleration of the *average force of mortality* in a cohort

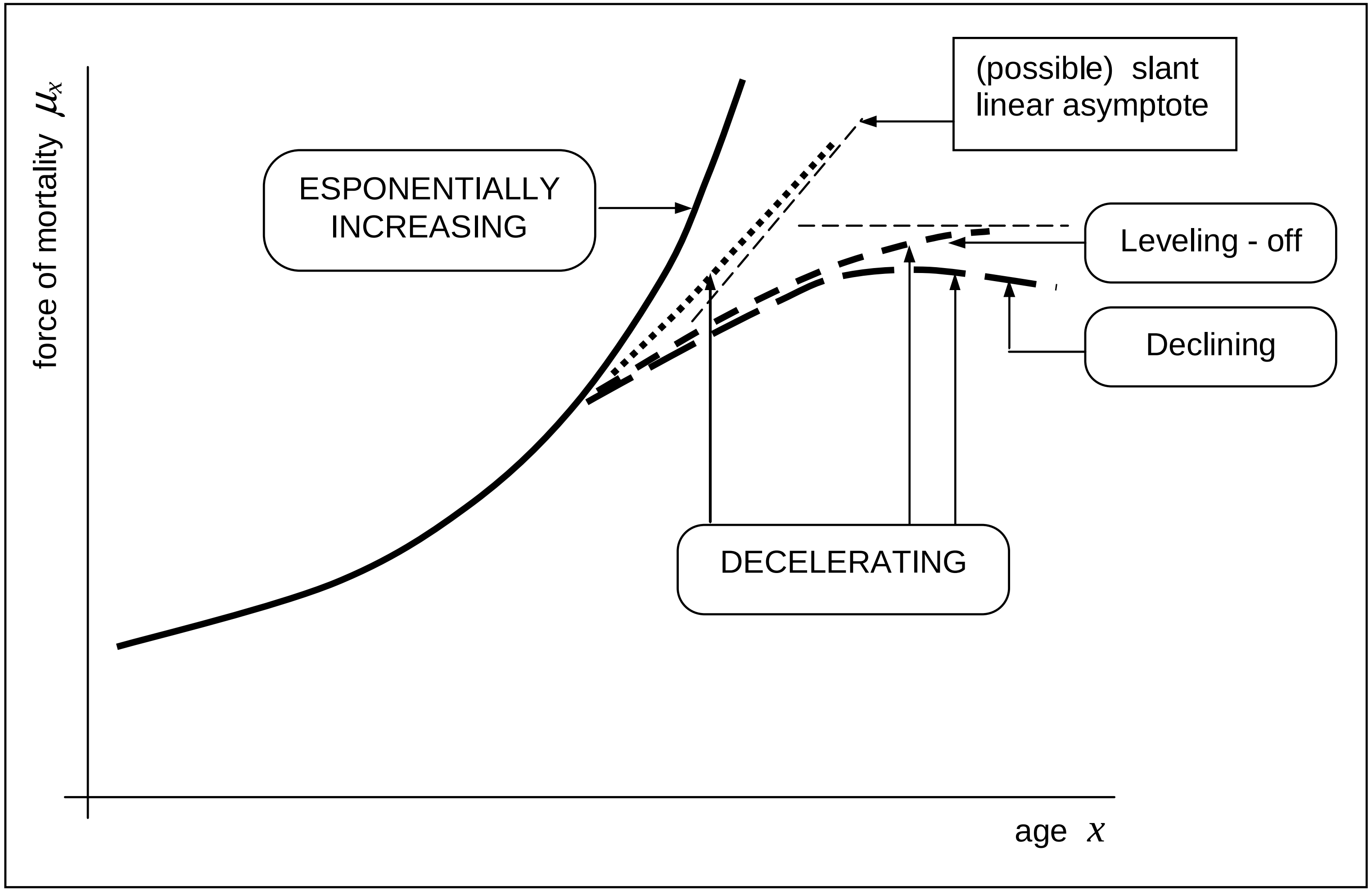
What can be argued by analyzing multi-cohort populations ?

The following question is still open:

- does the deceleration affect the *individual force of mortality* (whatever the frailty level) ?

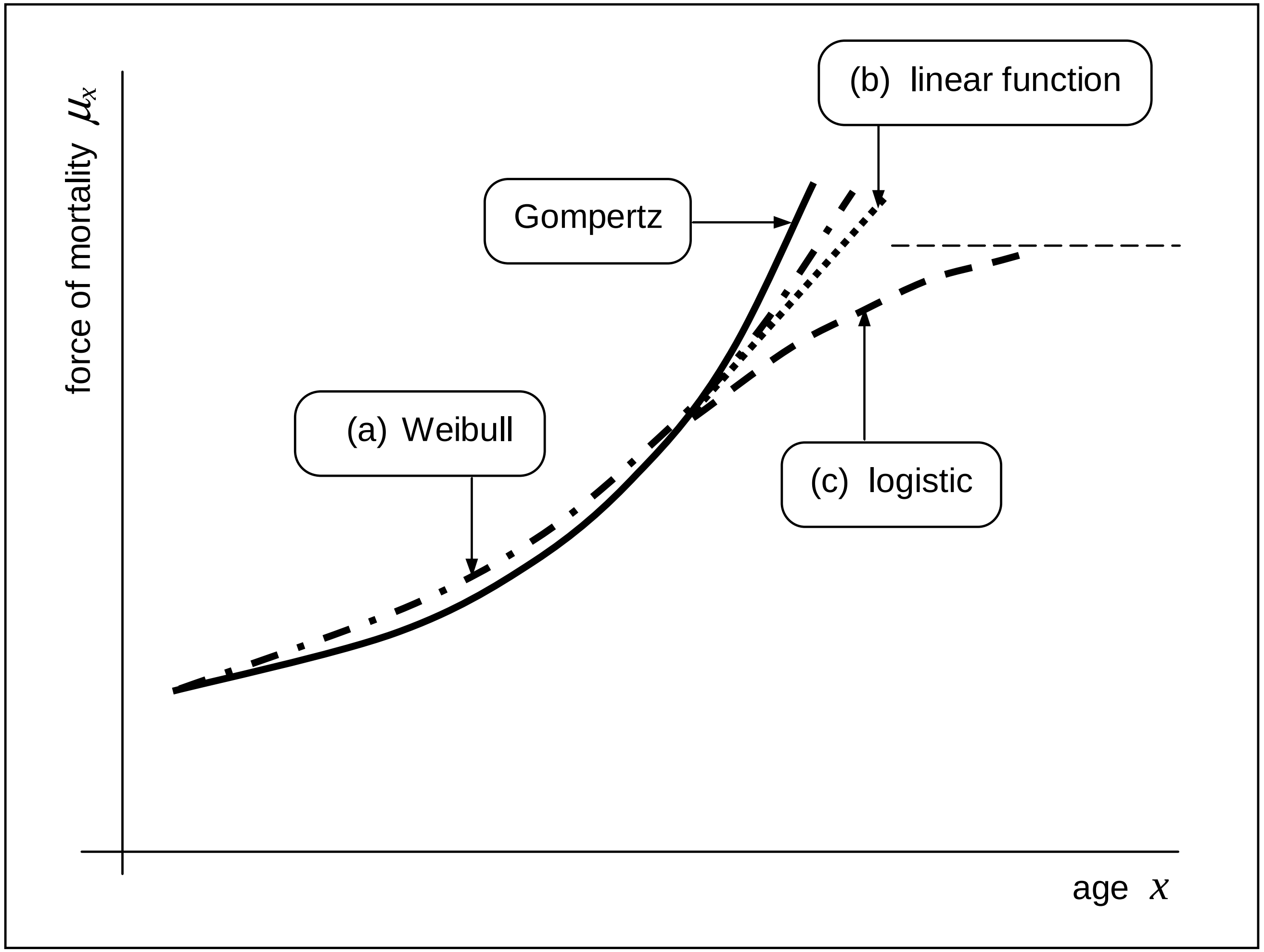
A lively debate, involving statistics, biology, medical sciences, etc.

See: Pitacco [2016] and references therein



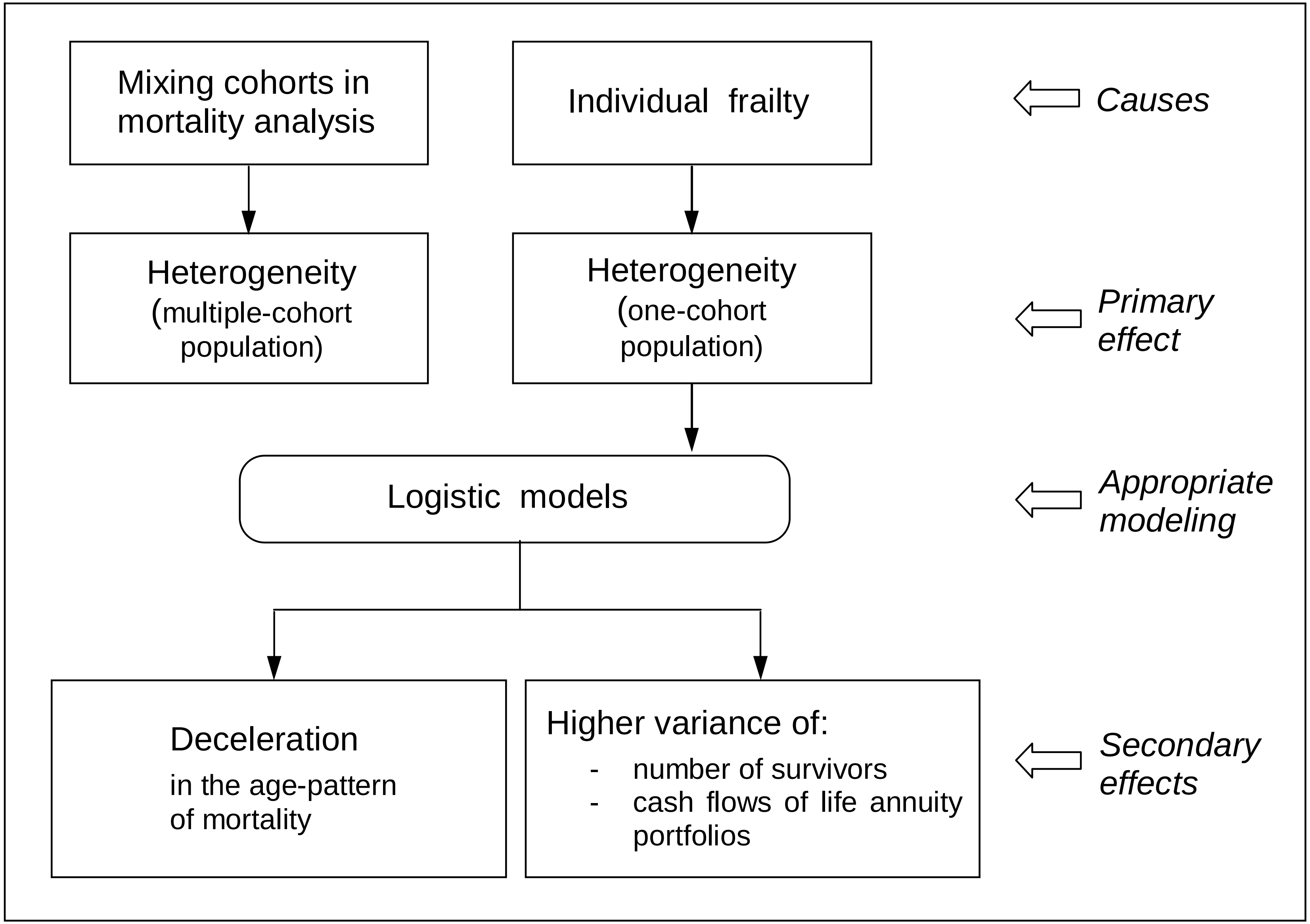
Exponential behavior vs deceleration

Perks, Beard, and the logistic models *(cont'd)*



Mortality deceleration at high ages

Perks, Beard, and the logistic models *(cont'd)*



Among recent contributions, Bebbington et al. [2011] propose a classification of "deceleration" definitions, noting that:

> *Mortality deceleration is the observed but yet to be understood phenomenon that the increase in the late-life death rate slows down after a certain species-related advanced age*

5 The aging process: Markov models

Frailty models proposed by Beard [1959, 1971], Vaupel et al. [1979]
$\Rightarrow$ constant individual frailty over the whole lifespan

- $\triangleright$ frailty due to genetic factors

Alternative approaches $\Rightarrow$ variable individual frailty, i.e. dynamic approaches to capture the individual aging process

A number of interesting contributions, many focusing on actuarial problems, starting from the seminal contribution by Levinson [1959]

Unobservable heterogeneity: dynamic settings

Levinson [1959]:

- ▷ Every population is heterogeneous in respect of mortality; even if split into classes (e.g. the class of insureds accepted as "normal risks"), each class is heterogeneous $\Rightarrow$ homogeneous subclasses
- ▷ Heterogeneous population split into a given number of homogeneous *strata*
- ▷ Definition of *mortality strata* $\Rightarrow$ each stratum consists of individuals with the same probability of death (regardless of age)
- ▷ Individuals move from one stratum to another one, in particular because of ageing, and in general because of *deterioration*
  - $\Rightarrow$ an example of *dynamic setting*
  - $\Rightarrow$ approach which in modern terms may be considered *multistate*
- ▷ Ultimate aim: construction of life tables allowing for strata; application to US life tables

Le Bras [1976]: individual frailty as a Markov process

- Framework: mortality modeling to define the limit age
- Basic idea: a mortality law must be the result of assumptions on the structure of a *process* describing the evolution of individual mortality throughout the whole life
- Assumptions:
  - ▷ each individual has an “initial frailty” (*faute*)
  - ▷ definition of “transition” probabilities: the probability of an increase in frailty (*nouvelle faute*) and the probability of death are proportional to the current frailty level (Markov hypothesis)
  - ▷ the resulting mortality law approx follows the Gompertz pattern up to some age, then tending to a limit ($\Rightarrow$ logistic-like shape)

The Markov framework

Lin and Liu [2007]:

- A finite-state Markov process is adopted to model human mortality
- Individual health status is represented by the *physiological age*, and modeled by the Markov process
  - ▷ each Markov state represents an outcome of the physiological age
- Random time of death then follows a phase-type distribution
- Frailty
  - ▷ measured by the physiological age
  - ▷ distributed according to the distribution of individuals among age classes

Liu and Lin [2012]:

- Generalize the previous model by introducing uncertainty in mortality $\Rightarrow$ subordinated Markov model
  - ▷ the aging process of a life is assumed to follow a finite-state Markov process
  - ▷ stochasticity of mortality is governed by a subordinating gamma process
- The model is applied to the evaluation of mortality-linked securities hedging the longevity risk, i.e. longevity bonds

Su and Sherris [2012]:

- Refer to a portfolio of life annuities
- Mortality in the portfolio alternatively given by:
  - ▷ fixed individual frailty, Gamma distributed or Inverse Gaussian distributed
  - ▷ Markov ageing model
- Both models for heterogeneity have implications for annuity markets
- Extent to which life annuity rates vary with age shows the financial significance of heterogeneity implied by the models

Sherris and Zhou [2014]:

- *Biometric risk components* in a life annuity portfolio
  - ▷ idiosyncratic longevity risk $\Rightarrow$ diversifiable via risk pooling
  - ▷ aggregate longevity risk $\Rightarrow$ systematic risk, non-diversifiable via risk pooling
  - ▷ heterogeneity w.r.t. mortality $\Rightarrow$ weakens the diversification of idiosyncratic longevity risk
- Heterogeneity alternatively represented by fixed-frailty model and dynamic model
- Main result: increasing pool sizes increases tail risk when a mortality model includes systematic risk $\Rightarrow$ higher capital allocation required
  - ▷ effect not captured by standard models of heterogeneity

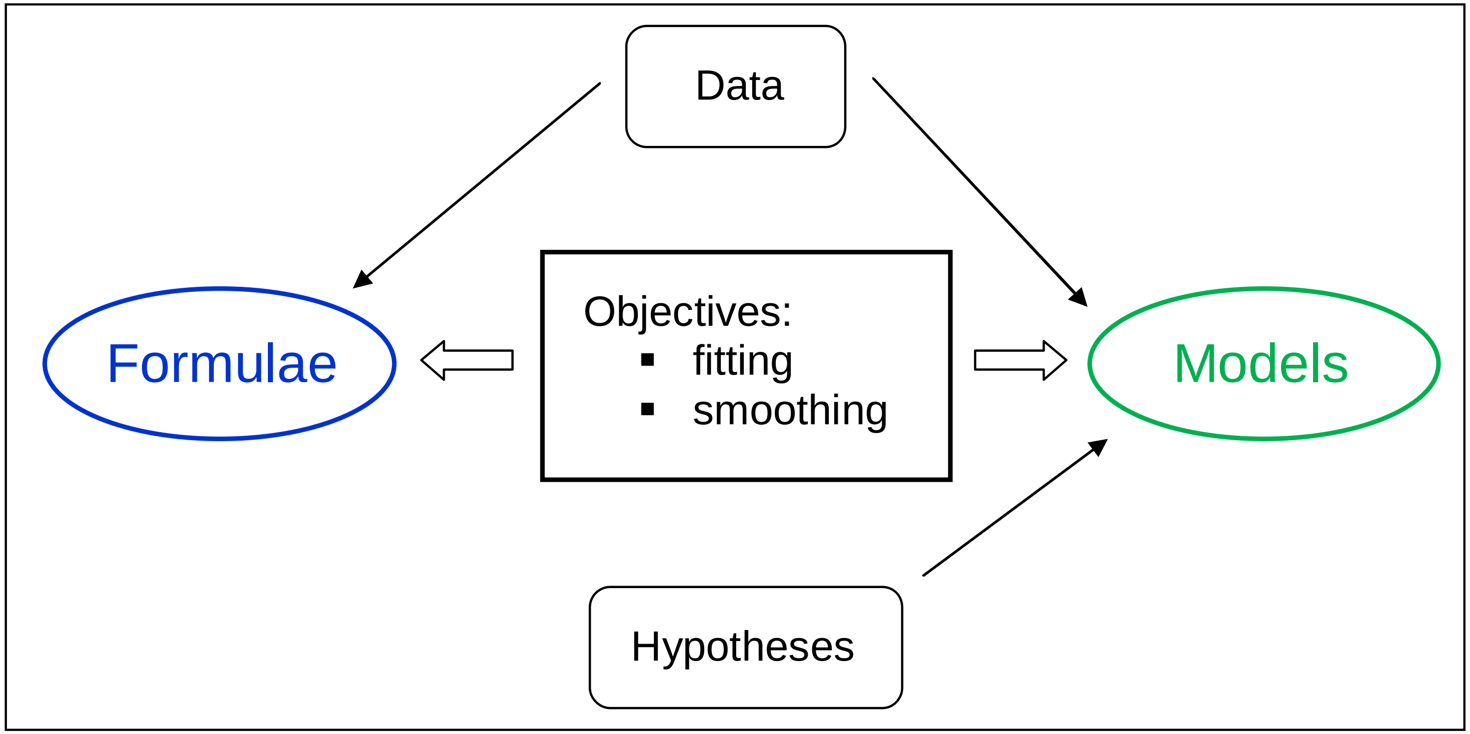
6 Concluding remarks

We have analyzed diverse approaches to representing the age-pattern of mortality

In particular, starting from pioneering contributions, we have singled out:

- the impact of unobservable heterogeneity (viz frailty)
- the deceleration of mortality at high ages
- the biological process of aging

Special attention has been placed on the role of “hypotheses” in defining a mortality model, since Gompertz



Models vs Formulae

References

N. Balakrishnan and Y. Peng. Generalized gamma frailty model. *Statistics in Medicine*, 25(16):2797–2816, 2006

R. E. Beard. Note on some mathematical mortality models. In C. E. W. Wolstenholme and M. O. Connor, editors, *CIBA Foundation Colloquia on Ageing*, volume 5, pages 302–311, Boston, 1959

R. E. Beard. Some aspects of theories of mortality, cause of death analysis, forecasting and stochastic processes. In W. Brass, editor, *Biological aspects of demography*, pages 57–68. Taylor & Francis, London, 1971

M. Bebbington, C.D. Lai, and R. Zitikis. Modelling deceleration in senescent mortality. *Mathematical Population Studies*, 18(1):18–37, 2011

H. Le Bras. Lois de mortalité et age limite. *Population*, 31(3):655–692, 1976

Z. Butt and S. Haberman. Application of frailty-based mortality models to insurance data. Actuarial Research Paper No. 142, Faculty of Actuarial Science & Insurance, City University, London, 2002. Available at: `http://openaccess.city.ac.uk/2279/1/142-ARC.pdf`

Z. Butt and S. Haberman. Application of frailty-based mortality models using generalized linear models. *ASTIN Bulletin*, 34(1):175–197, 2004

L. G. Doray. Inference for logistic-type models for the force of mortality. Presented at the Living to 100 and Beyond Symposium. Orlando, FL, 2008. Available at: `https://www.soa.org/library/monographs/retirement-systems/living-to-100-and-beyond/2008/january/subject-toc.aspx`

L. Duchateau and P. Janssen. *The Frailty Model*. Springer, 2008

D. O. Forfar, J. J. McCutcheon, and A. D. Wilkie. On graduation by mathematical formula. *Journal of the Institute of Actuaries*, 115:1–149, 1988

B. Gompertz. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. *Philosophical Transactions of Royal Society (Series A)*, 115:513–583, 1825

B. Gompertz. On one uniform law of mortality from birth to extreme old age and on the law of sickness. In *International Statistical Congress*, 1860. Reproduced in: 1871, Journal of the Institute of Actuaries, 16:329–344

J. Graf. *Die Rechnungsgrundlagen der k.u.k. priv. Assicurazioni Generali in Triest*. Assicurazioni Generali, Trieste, 1905

S. Haberman. Landmarks in the history of actuarial science (up to 1919). Actuarial Research Paper No. 84, Faculty of Actuarial Science & Insurance, City University, London, 1996. Available at: `http://openaccess.city.ac.uk/2228/1/84-ARC.pdf`

L. Heligman and J. H. Pollard. The age pattern of mortality. *Journal of the Institute of Actuaries*, 107:49–80, 1980

P. Hougaard. Life table methods for heterogeneous populations: distributions describing heterogeneity. *Biometrika*, 71(1):75–83, 1984

P. Hougaard. Survival models for heterogeneous populations derived from stable distributions. *Biometrika*, 73(2):387–396, 1986

V. Kannisto. *Development of Oldest-Old Mortality, 1950-1990: Evidence from 28 Developed Countries*. Odense Monographs on Population Aging, 1. Odense University Press, 1994. Available at: `http://www.demogr.mpg.de/Papers/Books/Monograph1/OldestOld.htm`

L. H. Levinson. A theory of mortality classes. *Transactions of the Society of Actuaries*, 11:46–87, 1959

X. S. Lin and X. Liu. Markov aging process and phase-type law of mortality. *North American Actuarial Journal*, 11:92–109, 2007

M. Lindholm. A note on the connection between some classical mortality laws and proportional frailty. *Stastistics & Probability Letters*, 126:76–82, 2017

X. Liu and X. S. Lin. A subordinated Markov model for stochastic mortality. *European Actuarial Journal*, 2(1):105–127, 2012

W. M. Makeham. On the law of mortality. *Journal of the Institute of Actuaries*, 8: 301–310, 1867

W. M. Makeham. On the further development of Gompertz's law. *Journal of the Institute of Actuaries*, 28:152–159,185–192,316–332, 1890

K. G. Manton, E. Stallard, and J. W. Vaupel. Alternative models for the heterogeneity of mortality risks among the aged. *Journal of the American Statistical Association*, 81: 635–644, 1986

S. Martinelle. *A generalized Perks formula for old-age mortality*. R & D report / Statistics Sweden. Statistiska Centralbyrån, Stockholm, 1987

S. J. Olshansky and B. E. Carnes. Ever since Gompertz. *Demography*, 34(1):1–15, 1997

W. Perks. On some experiments in the graduation of mortality statistics. *Journal of the Institute of Actuaries*, 63:12–57, 1932

E. Pitacco. High age mortality and frailty. Some remarks and hints for actuarial modeling. CEPAR Working Paper 2016/19, 2016. Available at: `http://www.cepar.edu.au/working-papers/working-papers-2016.aspx`

E. Pitacco. Heterogeneity in mortality: a survey with an actuarial focus. *European Actuarial Journal*, 9(1):3–30, 2019

M. Sherris and Q. Zhou. Model risk, mortality heterogeneity, and implications for solvency and tail risk. In O. S. Mitchell, R. Maurer, and P. B. Hammond, editors, *Recreating Sustainable Retirement: Resilience, Solvency, and Tail Risk*. Oxford University Press, 2014

W. Siler. Parameters of mortality in human populations with widely varying life spans. *Statistics in Medicine*, 2:373–380, 1983

S. Su and M. Sherris. Heterogeneity of Australian population mortality and implications for a viable life annuity market. *Insurance: Mathematics & Economics*, 51(2):322–332, 2012

A. R. Thatcher. The long-term pattern of adult mortality and the highest attained age. *Journal of the Royal Statistical Society - Series A*, 162:5–43, 1999

P. N. Thiele. On a mathematical formula to express the rate of mortality throughout the whole of life. *Journal of the Institute of Actuaries*, 16:313–329, 1871

J. W. Vaupel, K. G. Manton, and E. Stallard. The impact of heterogeneity in individual frailty on the dynamics of mortality. *Demography*, 16(3):439–454, 1979

A. Wienke. Frailty models. MPIDR Working Paper WP 2003-032. Max Planck Institute for Demographic Research, Rostock, Germany, 2003. Available at: `http://www.demogr.mpg.de/papers/working/wp-2003-032.pdf`

A. I. Yashin, I. A. Iachine, A. Z. Begun, and J. W. Vaupel. Hidden frailty: Myths and reality. Department of Statistics and Demography, Odense University, Denmark, 2001. Available at: `https://www.researchgate.net/publication/228852668_Hidden_frailty`

Many thanks
for your kind attention !