Longevity and Mortality Models

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Longevity and Long-Term Care Risks and Products

Agenda

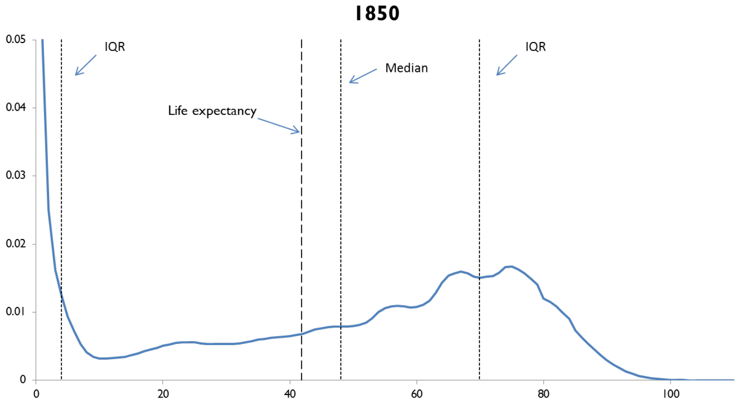
- Motivation and Preliminaries
- Single population (discrete) stochastic mortality models
- Mortality Improvement rate models
- Multipopulation mortality models
- Continuous time mortality models
- Recent developments and outlook

Motivation and Preliminaries

Main demographic trends

- Expansion over time
- Rectangularisation over time
- Increasing trend over time in life expectancy
- Downward trend over time in death rates

Male life table distribution of deaths (d_x), England and Wales 1850-2009



Source: Human Mortality Database

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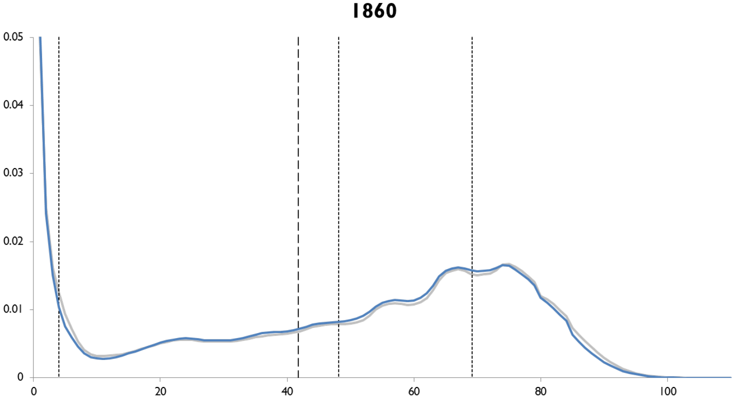

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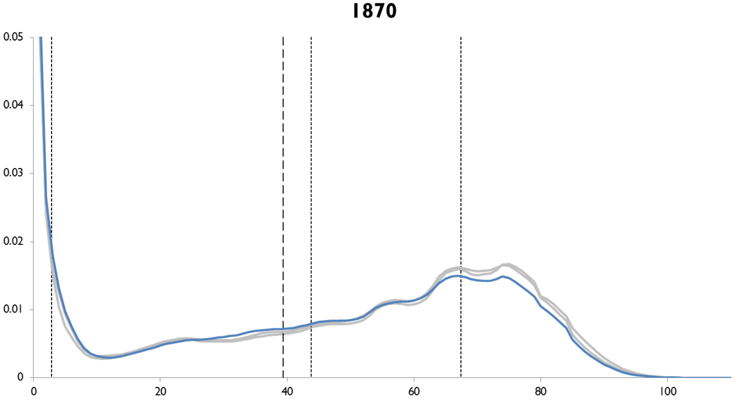

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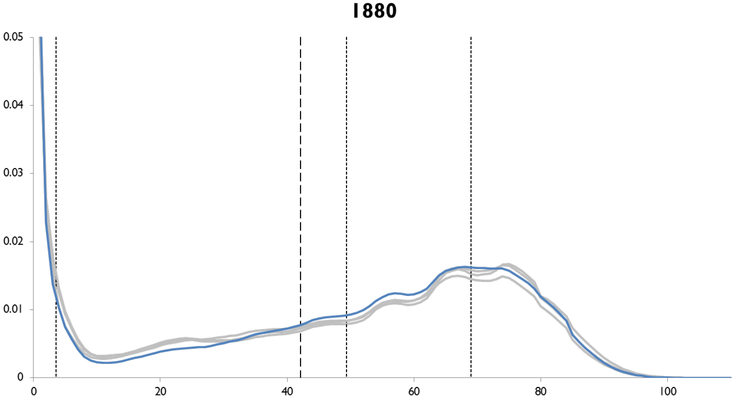

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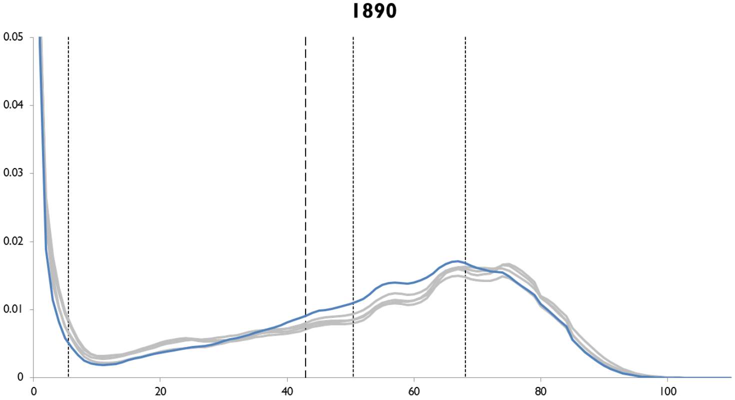

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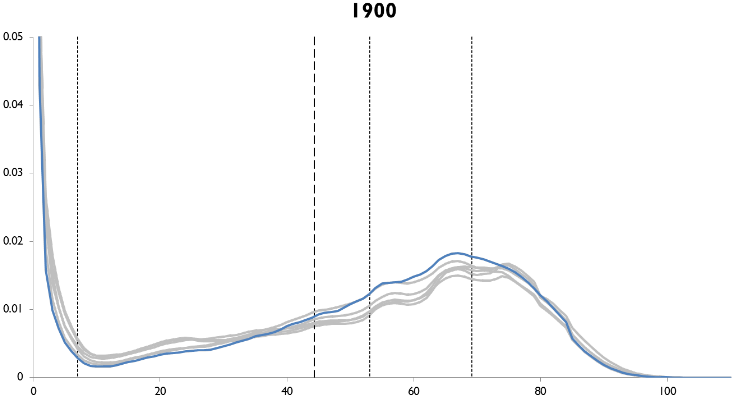

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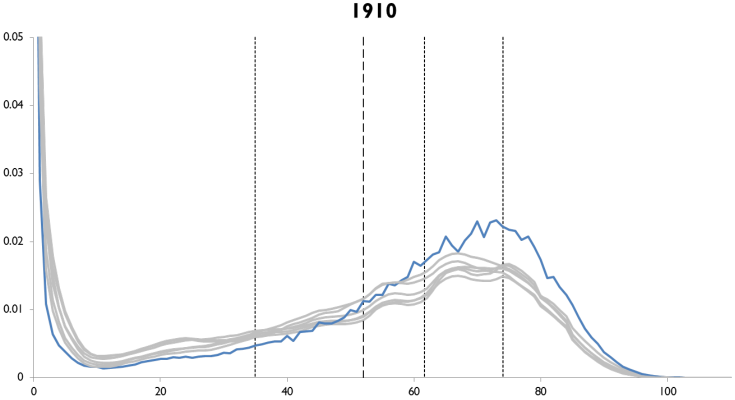

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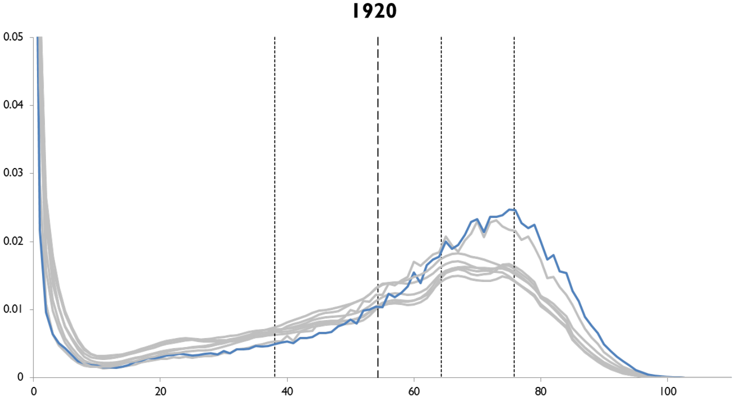

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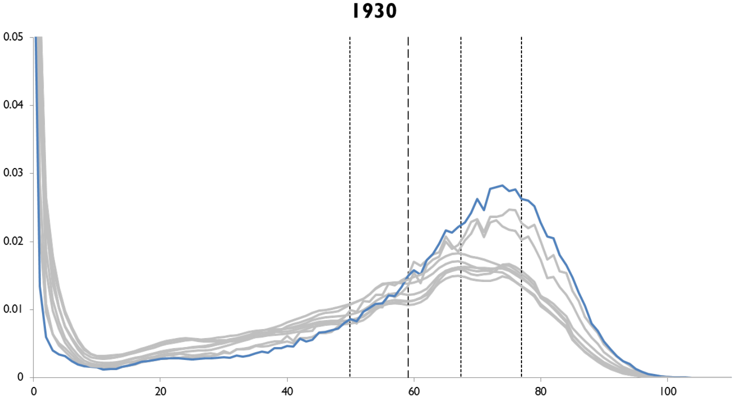

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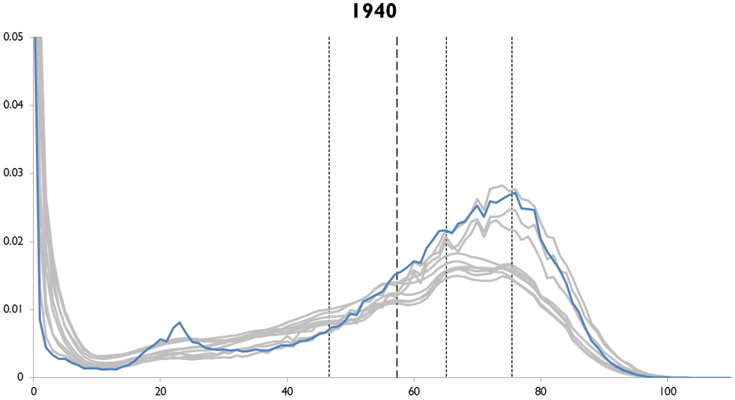

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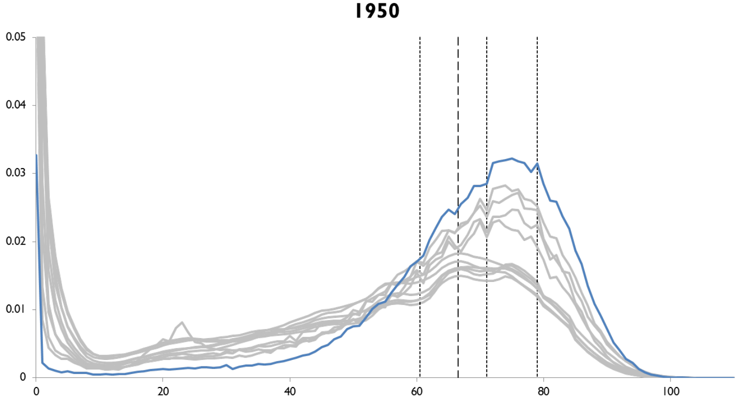

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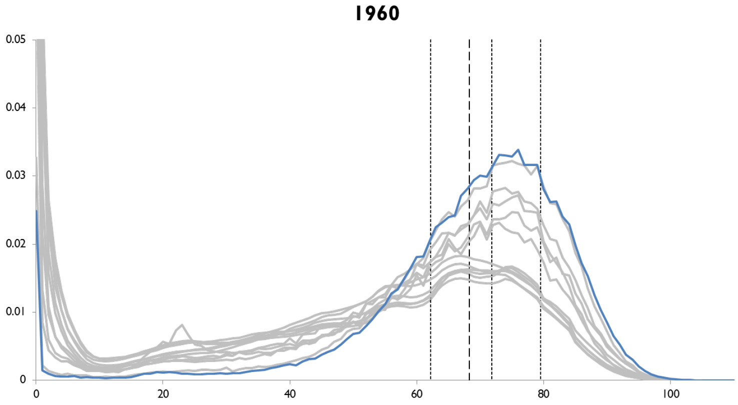

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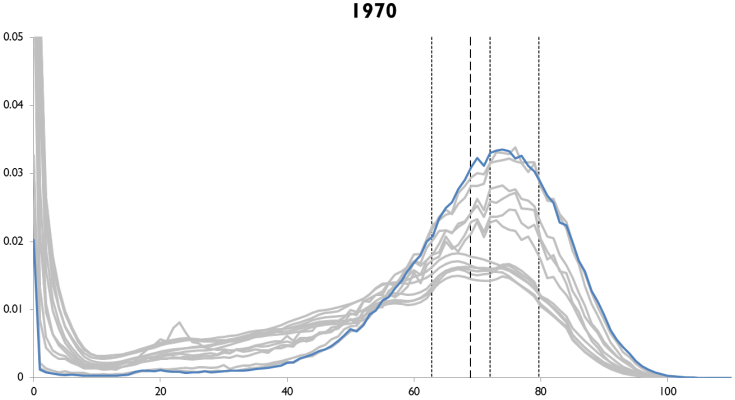

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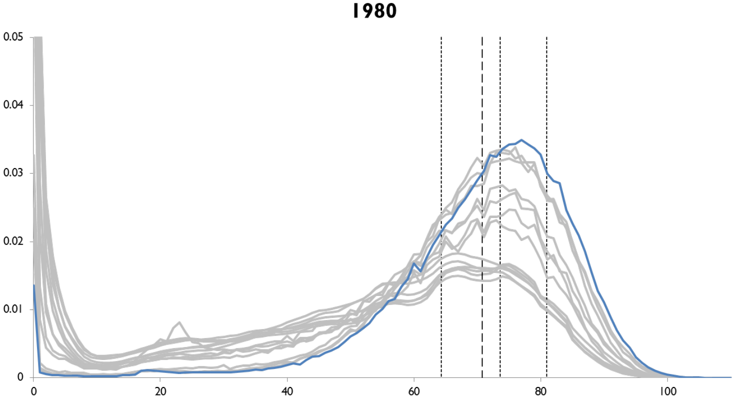

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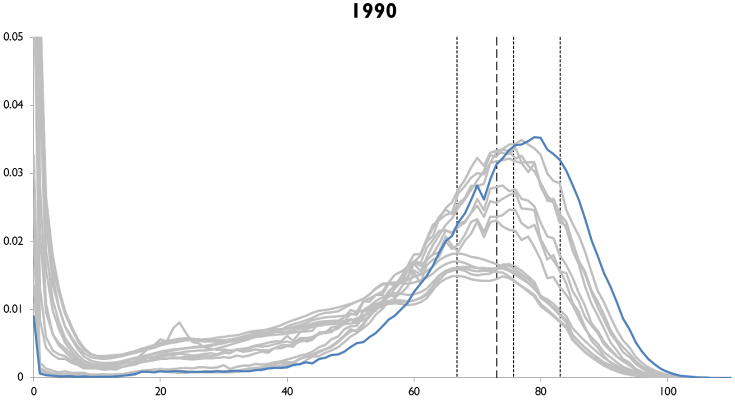

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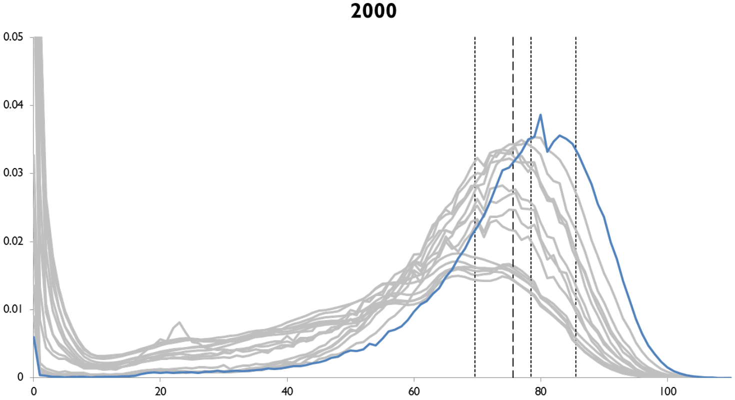

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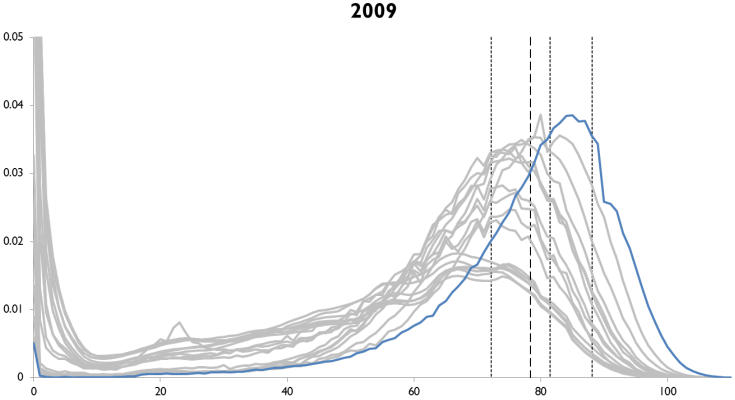

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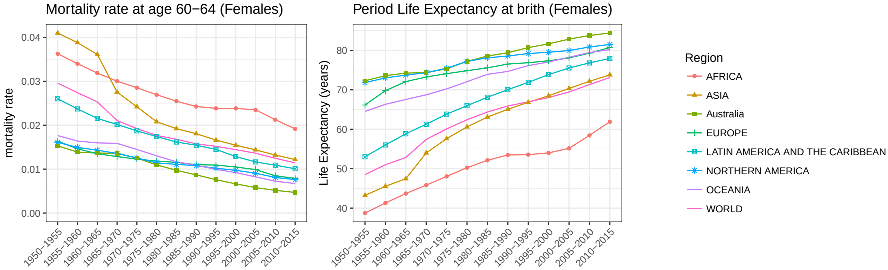
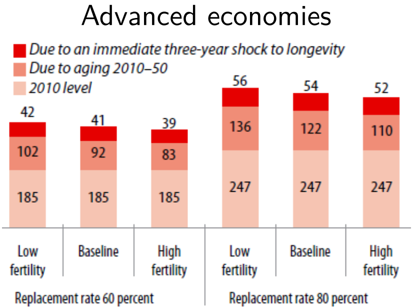
Recent trends in mortality and life expectancy

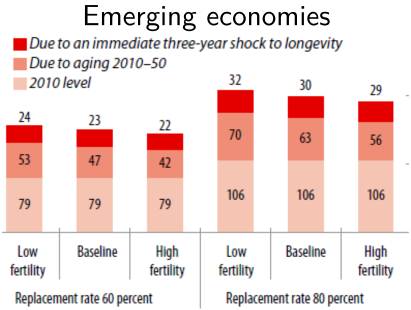


United Nations World Population Prospects 2017

- Good-news!
- Important social and financial implications for governments, insurers, individuals.
- Need to model and project these trends

IMF assessment of global of three year longevity shock longevity





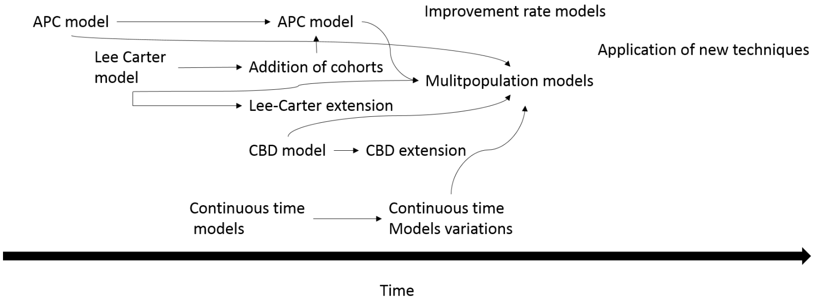
Source: IMF (2012)

Mortality forecasting methodologies

A good overview of methodologies is given in the review papers Booth and Tickle (2008), Wong-Fupuy and Haberman (2004), Pitacco (2004) and in book Pitacco et al. (2009)

- Expert based
- Explanatory
  - Structural Modelling (Explanatory or Econometric).
  - Cause of death decomposition
- **Extrapolation**
  - Trend modelling

A timeline of "recent" mortality modelling methodologies



Single population (discrete) stochastic mortality models

Discussion based on:

Villegas, A. M., Millossovich, P., & Kaishev, V. K. (2018). StMoMo: An R Package for Stochastic Mortality Modelling. JSS Journal of Statistical Software, 84(3). https://doi.org/10.18637/jss.v084.i03

Advances in single population mortality modelling

- **Lee-Carter model** (Lee and Carter, 1992)
  - Add more bilinear age-period components (Renshaw and Haberman, 2003)
  - Add a cohort effect (Renshaw and Haberman, 2006)
- Two factor **CBD model** (Cairns et al., 2006)
  - Add cohort effect, quadratic age term (Cairns et al., 2009)
  - Combine with features of the Lee-Carter (Plat, 2009b)
- **Many more models** proposed in the literature (e.g. Aro and Pennanen (2011), O'Hare and Li (2012), Börger et al. (2013), Alai and Sherris (2014))

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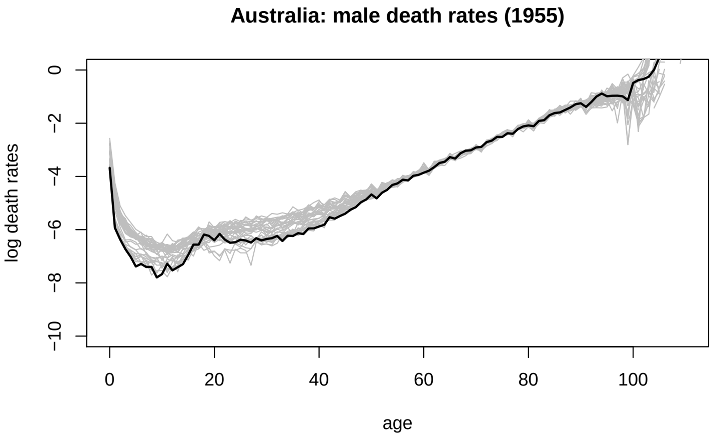

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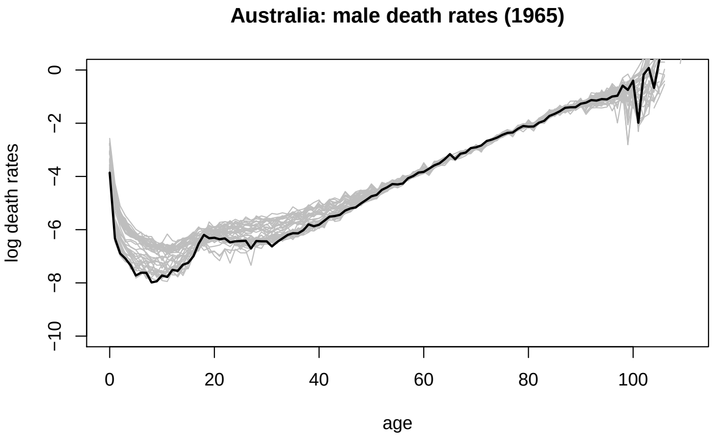

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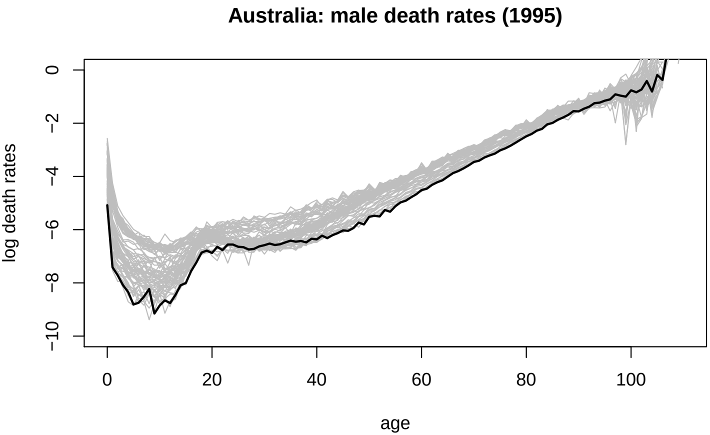

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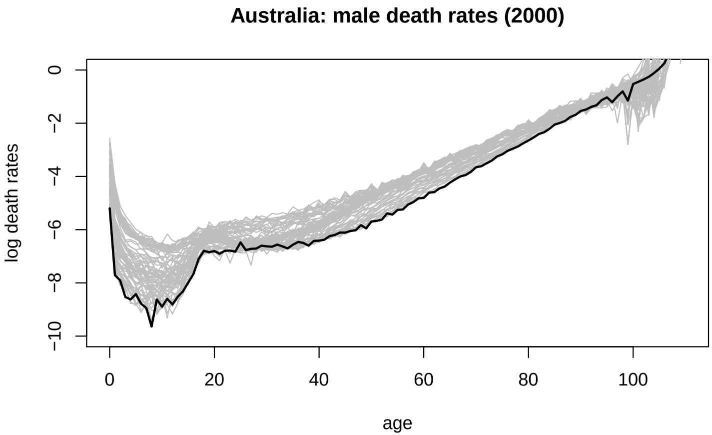

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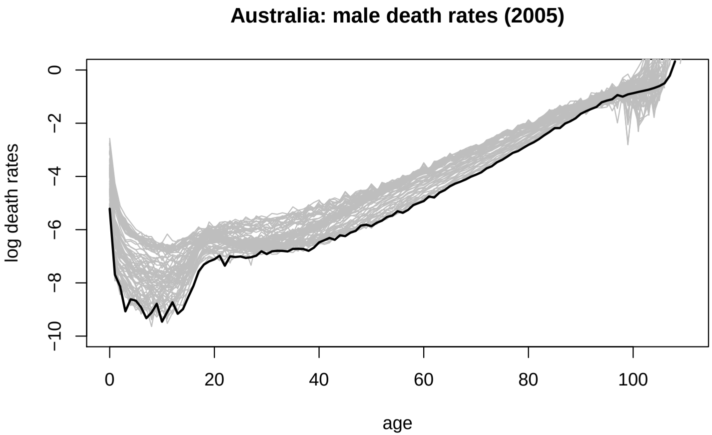

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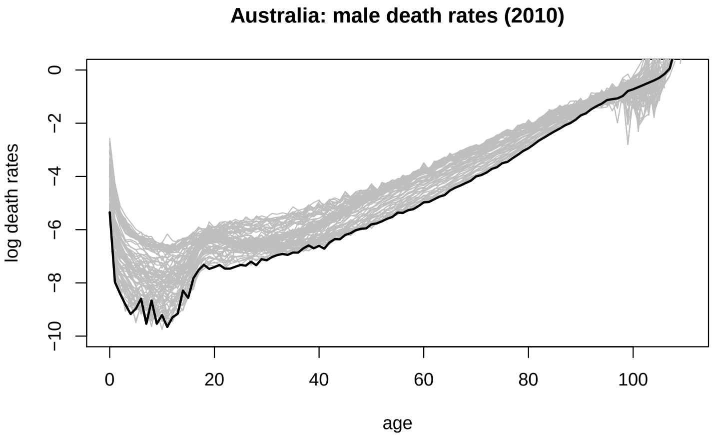

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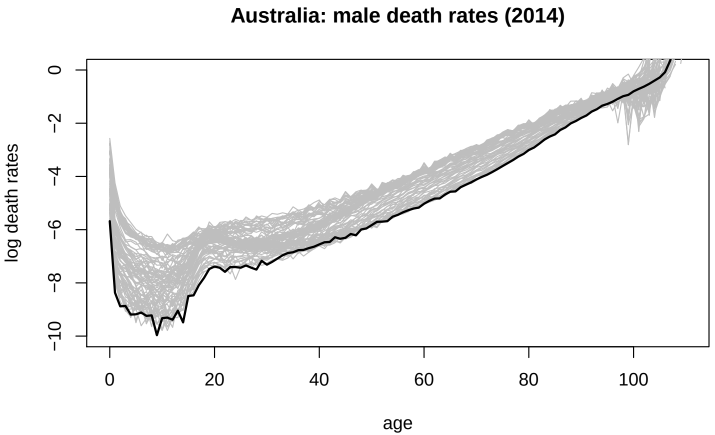

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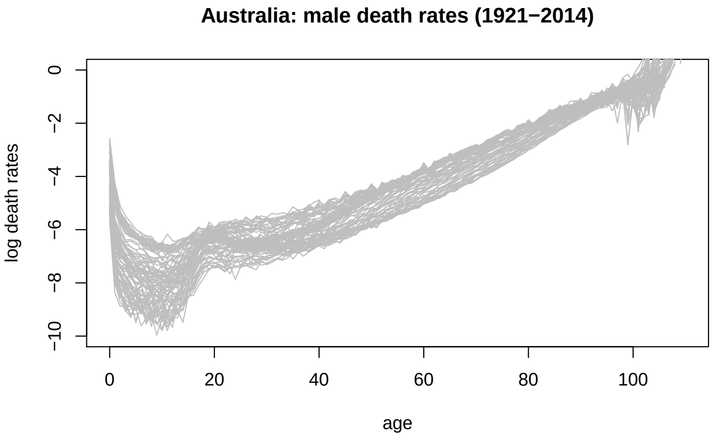
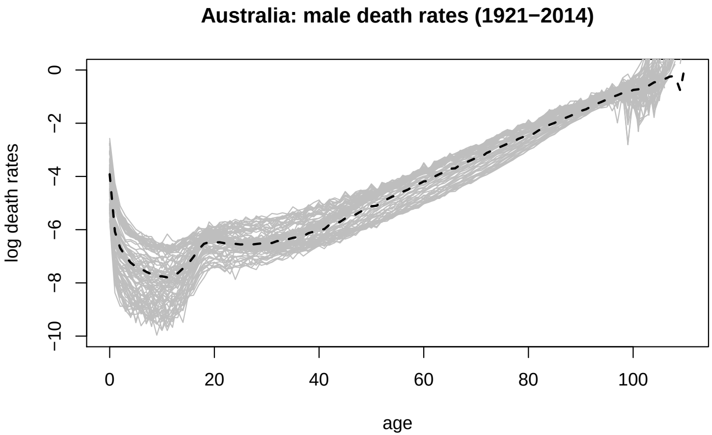


$$\log \mu_{xt} \quad =$$

Lee-Carter model



$$\log \mu_{xt} \quad = \quad \alpha_x$$

Lee-Carter model

$$\log \mu_{xt} \quad = \quad \alpha_x \quad + \quad \qquad \kappa_t$$

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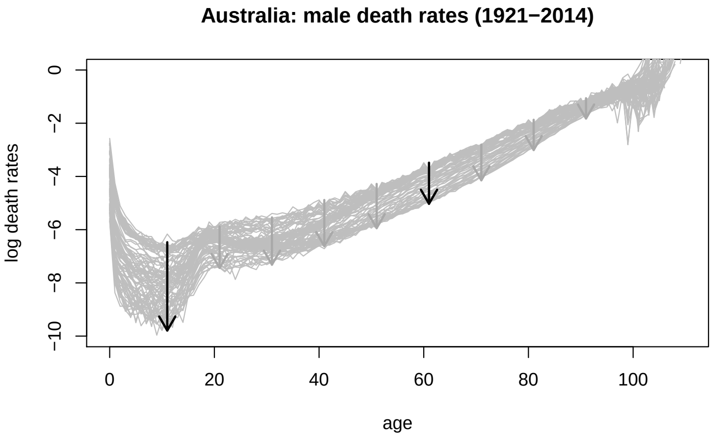

$$\log \mu_{xt} = \alpha_x + \beta_x \kappa_t$$

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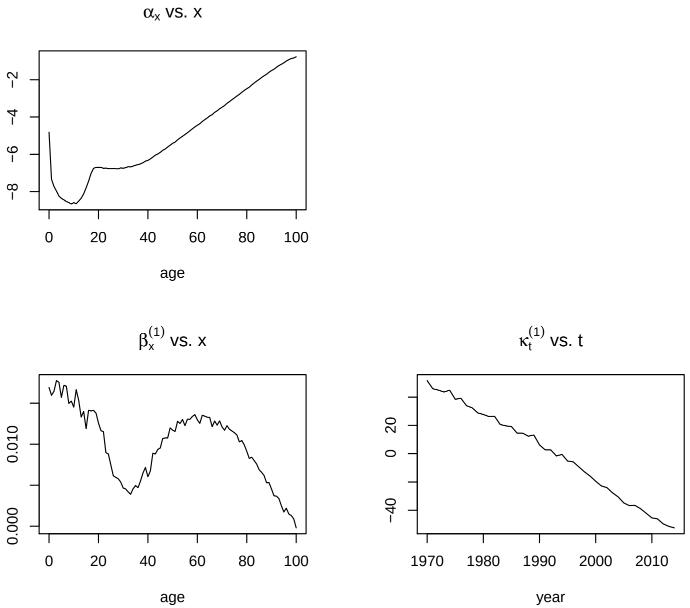
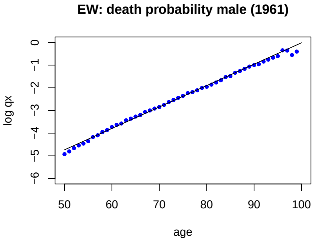
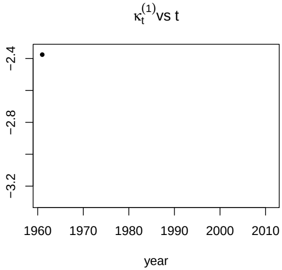
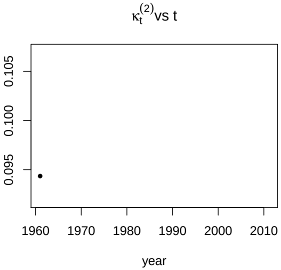


Cairns-Blake-Dowd model

$$\text{logit } q_{xt} = \kappa_t^{(1)} + (x - \bar{x})\kappa_t^{(2)}$$







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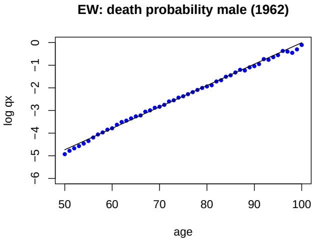

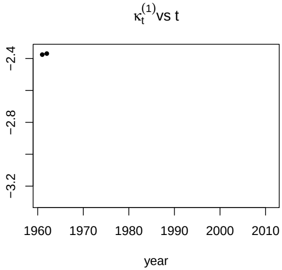

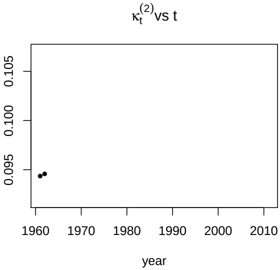

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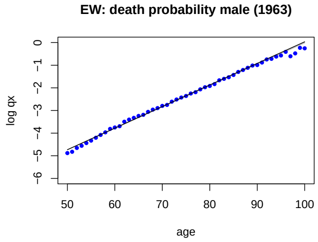

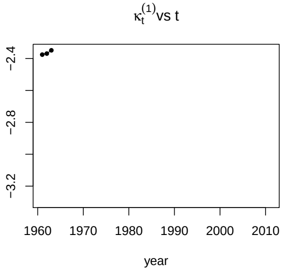

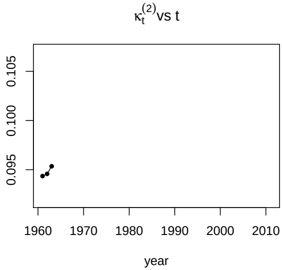

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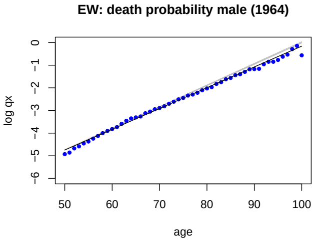

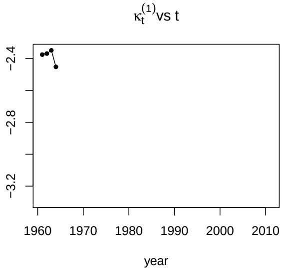

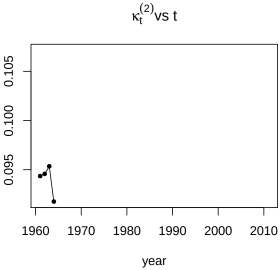

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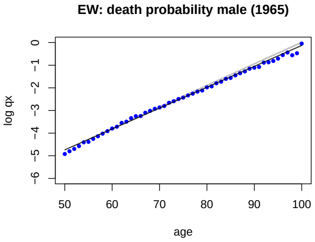

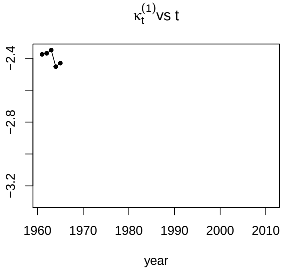

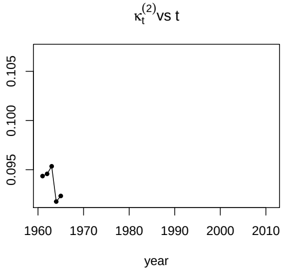

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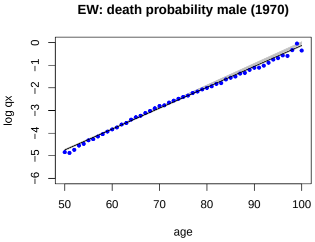

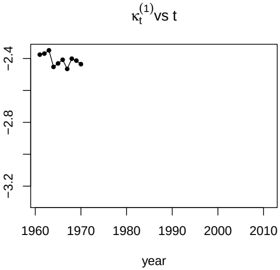

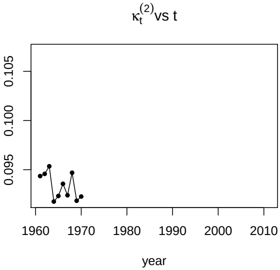

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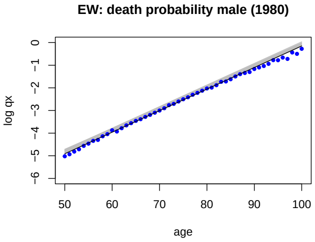

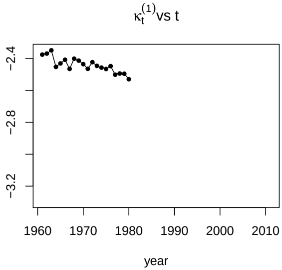

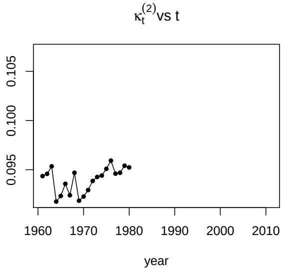

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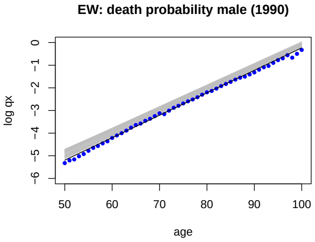

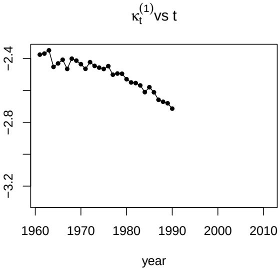

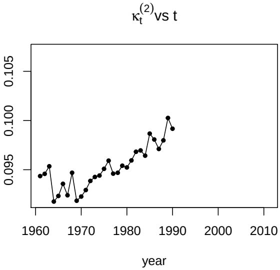

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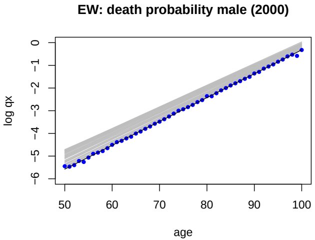

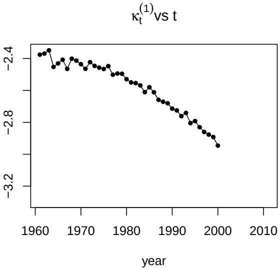

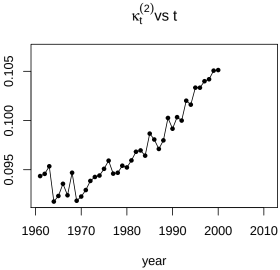

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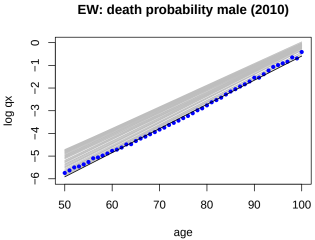
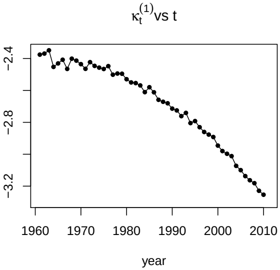
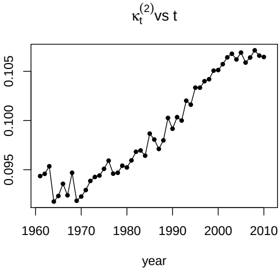
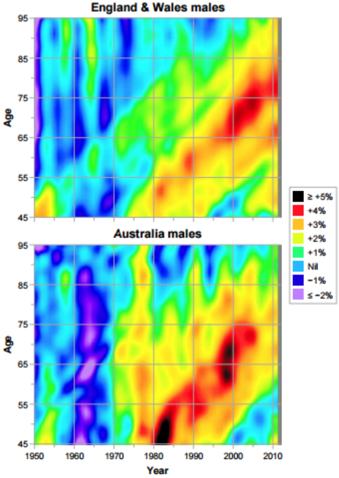






Cohort effects



Source: Human Mortality Database, Aon Hewitt

- Statistical evidence $\rightsquigarrow$ both period and cohort effect have an impact on mortality improvements (Willets 1999, 2004)
- Period effects approximate contemporary factors
  - General health status of the population
  - Healthcare services available
  - Critical weather conditions
- Cohort effects approximate historical factors
  - World War II
  - Diet
  - Welfare State (in the UK)
  - Smoking habits

Age-Period-Cohort models: Classic APC model

- A widely used structured used in medicine, psychology and demography (Hobcraft et al. (1982), Wilmoth (1990))

$$\log \mu_{xt} = \alpha_x + \kappa_t + \gamma_{t-x}$$

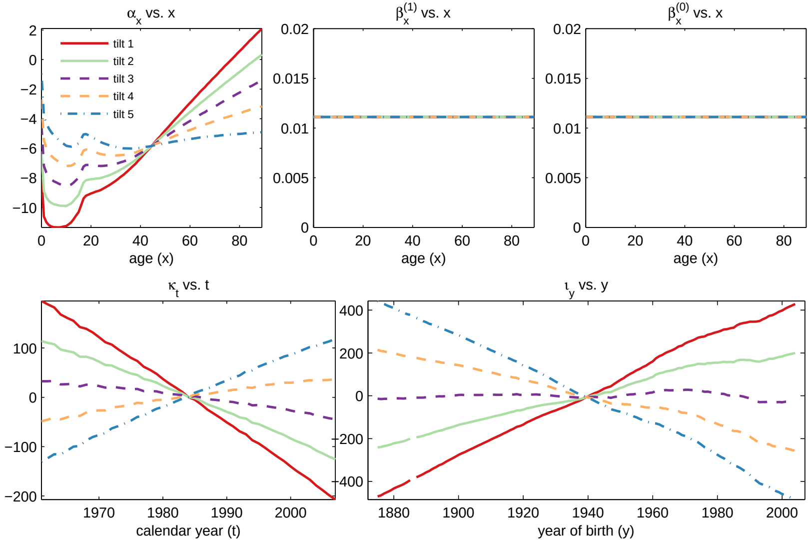
- No unique set of parameters resulting in optimal fit due to $c = t - x$

$$(\alpha_x, \kappa_t, \gamma_{t-x}) \to (\alpha_x + \phi_1 - \phi_2 x, \kappa_t + \phi_2 t, \gamma_{t-x} - \phi_1 - \phi_2(t-x))$$
$$(\alpha_x, \kappa_t, \gamma_{t-x}) \to (\alpha_x + c_1, \kappa_t - c_1, \gamma_{t-x})$$

- Impose constraints

$$\sum_t \kappa_t = 0, \qquad \sum_c \gamma_c = 0, \qquad \sum_c c\gamma_c = 0$$

Age-Period-Cohort models: Identifiability



Other stochastic mortality models

- Lee-Carter extensions
  - Add cohorts

$$\log \mu_{xt} = \alpha_x + \beta_x^{(1)} \kappa_t^{(1)} + \beta_x^{(0)} \gamma_{t-x}$$

- CBD extensions
  - M6: Add cohorts

$$\text{logit}\, q_{xt} = \eta_{xt} = \kappa_t^{(1)} + (x - \bar{x}) \kappa_t^{(2)} + \gamma_{t-x}$$

  - M7: Add cohorts and quadratic age effect

$$\text{logit}\, q_{xt} = \eta_{xt} = \kappa_t^{(1)} + (x - \bar{x}) \kappa_t^{(2)} + \left((x - \bar{x})^2 - \hat{\sigma}_x^2 \right) \kappa_t^{(3)} + \gamma_{t-x}$$

- Plat model combines the Lee-Carter and the CBD

$$\log \mu_{xt} = \alpha_x + \kappa_t^{(1)} + (\bar{x} - x) \kappa_t^{(2)} + (\bar{x} - x)^+ \kappa_t^{(3)} + \gamma_{t-x}$$

Generalised Age-Period-Cohort stochastic mortality models

Recent research has proposed a unifying framework discrete stochastic mortality models

- General Age-Period-Cohort model structure Hunt and Blake (2015a)
- Generalised (non-)linear model Currie (2016)
- R Implementation of GAPC models Villegas et al. (2018)

Generalised Age-Period-Cohort stochastic mortality models

1. Random Component:

$$D_{xt} \sim \text{Poisson}(E^c_{xt}\mu_{xt}) \quad \text{or} \quad D_{xt} \sim \text{Binomial}(E_{xt}, q_{xt})$$

2. Systematic Component:

$$\eta_{xt} = \alpha_x + \sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} + \beta_x^{(0)} \gamma_{t-x}$$

   - Lee-Carter type $\rightsquigarrow \beta_x^{(i)}$, non-parametric
   - CBD type $\rightsquigarrow \beta_x^{(i)} \equiv f^{(i)}(x)$, pre-specified parametric function

3. Link Function:

$$g\left(\mathbb{E}\left(\frac{D_{xt}}{E_{xt}}\right)\right) = \eta_{xt}$$

   - log-Poisson: $\eta_{xt} = \log \mu_{xt}$
   - logit-Binomial: $\eta_{xt} = \operatorname{logit} q_{xt}$

Generalised Age-Period-Cohort stochastic mortality models

4. Set of parameter constraints:
   - Need parameters constraints to ensure identifiability
5. Forecasting and simulation
   - Period indexes: Multivariate random walk with drift

$$\boldsymbol{\kappa}_t = \boldsymbol{\delta} + \boldsymbol{\kappa}_{t-1} + \boldsymbol{\xi}_t^{\kappa}, \qquad \boldsymbol{\kappa}_t = \begin{pmatrix} \kappa_t^{(1)} \\ \vdots \\ \kappa_t^{(N)} \end{pmatrix}, \qquad \boldsymbol{\xi}_t^{\kappa} \sim N(\mathbf{0}, \Sigma),$$

   - Cohort effect: ARIMA(p, q, d) with drift

$$\Delta^d \gamma_c = \delta_0 + \phi_1 \Delta^d \gamma_{c-1} + \cdots + \phi_p \Delta^d \gamma_{c-p} + \epsilon_c + \delta_1 \epsilon_{c-1} + \cdots + \delta_q \epsilon_{c-q}$$

GAPC models can be implemented with the R package StMoMo (http://cran.r-project.org/package=StMoMo)

Other key areas in single population discrete mortality models I

- Parameter estimation:
  - Poisson (Brouhns et al., 2002)
  - Negative-Binomial (Delwarde et al., 2007a, Li et al. (2009))
  - Bayesian and state-space setting (Czado et al., 2005, Pedroza (2006), Kogure et al. (2009), Fung et al. (2016))
- Parameter Smoothing (Delwarde et al., 2007b) and functional data approach (Hyndman and Ullah, 2007)
- Bootstrapping and parameter uncertainty
  - Semiparametric (Brouhns et al., 2005)
  - Parametric (Koissi et al., 2006)
  - Comparison of methods (Renshaw and Haberman, 2008)
- Modelling of errors: residual dependence (Debón, 2008, Debón et al. (2010), Mavros et al. (2017))
- Identifiability

Other key areas in single population discrete mortality models II

- Age-Period Models (Hunt and Blake, 2015b)
- APC models (Hunt and Blake, 2015c)
- Impact on estimation (Hunt and Villegas, 2015)

- Modelling and projecting of period and cohort factors
  - Optimal calibration period (Booth et al., 2002, Denuit2005)
  - Regime-switching (Milidonis et al., 2011)
  - Structural changes (Coelho and Nunes, 2011, van Berkum et al. (2014))
- Selection criteria of most appropriate model
  - Goodness-of-fit (Cairns et al., 2009, Dowd et al. (2010b))
  - Backtesting (Dowd et al., 2010a)
  - Qualitative properties of forecasts (Cairns et al., 2011b)
  - Overall performance for England and Wales and the USA (Haberman and Renshaw, 2011).

Mortality improvement rate models

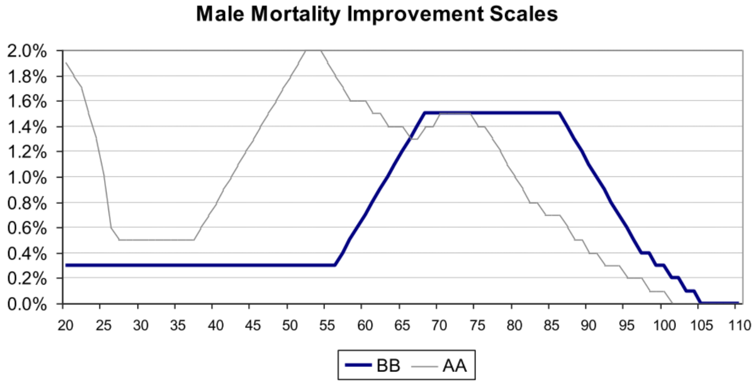
Discussion based on:

Hunt, A., & Villegas, A. M. (2017). Mortality Improvement Rates: Modeling and Parameter Uncertainty. In Living to 100, Society of Actuaries International Symposium. Retrieved from https://www.soa.org/essays-monographs/2017-living-to-100/2017-living-100-monograph-hunt-villegas-paper.pdf

Two parallel approaches to modelling and projecting mortality

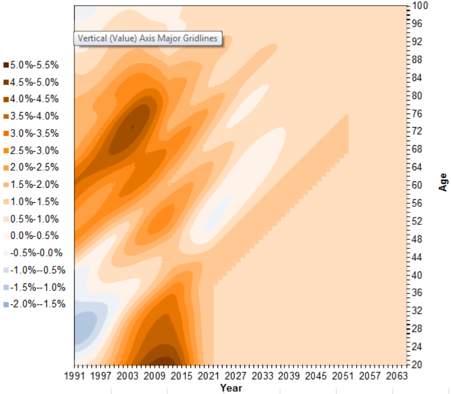
	Mortality Rates	Improvement Rates
What?	$q_{x,t}$, $\mu_{x,t}$, $m_{x,t}$	$1 - \frac{q_{x,t}}{q_{x,t-1}}$, $-\ln(\frac{m_{x,t}}{m_{x,t-1}})$
Who?	Lee and Carter (1992)	CMIB (1978)
	Cairns et al. (2006)	CMI (2002, 2009, 2016)
	Brouhns et al. (2005)	SOA (1995, 2012)
How?	$\ln m_{x,t} = \alpha_x + \sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} + \gamma_{t-x}$	$q_{x,T+n} = q_{x,T}(1 - R_x)^n$

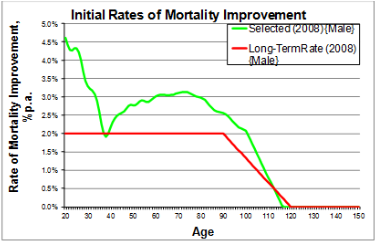
Improvement rates in actuarial practice – AA and BB scales

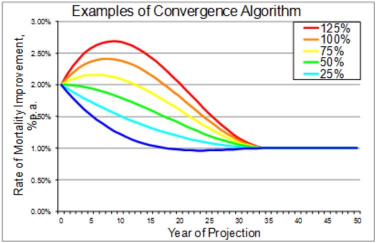


$$q_{x,T+n} = \underbrace{q_{x,T}}_{\text{Base table}} \times \underbrace{(1 - AA_x)^n}_{\text{Reduction factor}}$$

Improvement rates in actuarial practice – CMI model





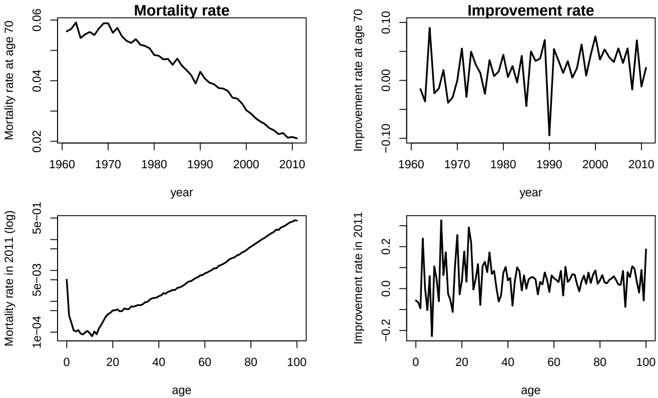


Latest CMI projection model uses an APC model on improvement rates (CMI Working paper 90): $-\Delta \ln m_{x,t} = \alpha_x + \kappa_t + \gamma_{t-x}$

Recent academic interest on improvement rate modelling

- Mitchell et al. (2013)
  - $\ln\left(\frac{\hat{m}_{x,t+1}}{\hat{m}_{x,t}}\right) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}$
  - Benefits of detrending
  - Estimation with singular value decomposition
- Haberman and Renshaw (2012)
  - $\frac{\hat{m}_{x,t-1} - \hat{m}_{x,t}}{0.5(\hat{m}_{x,t-1} + \hat{m}_{x,t})} = \eta_{x,t}$
  - Predictor structures $\eta_{x,t}$ borrowed from mortality rate modelling
  - Duality between mortality rate and improvement rate modelling
  - Estimation using Gaussian model with variable dispersion
- Haberman and Renshaw (2013), Plat (2011), Danesi et al. (2015), Chuliá et al. (2016), Njenga and Sherris (2011)

Complications with improvement rate modelling



- Patterns are not that clear
- Non-standard distribution
- Heteroscedasticity
- Parameter uncertainty

Two alterantive approaches to modelling improvement rates

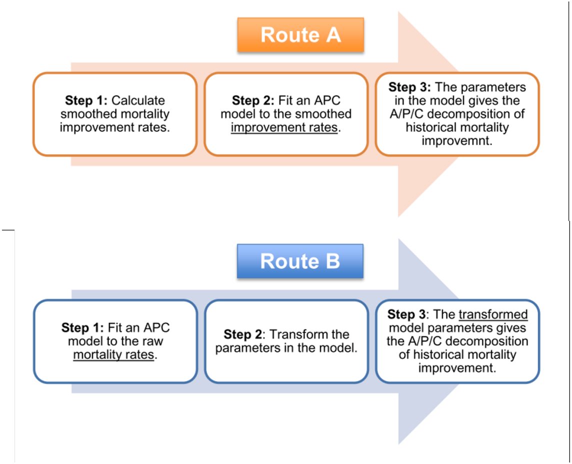


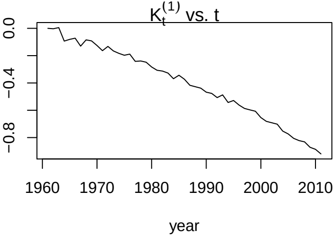
Diagram source: Li et al. (2017)

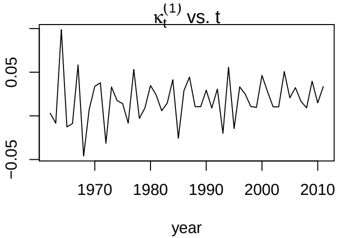
- **Route A (Direct)**: Direct modelling of improvement rates
- **Route B (Indirect))** Derive improvement rates from mortality rate model

Route B: Estimation and equivalent mortality rate structure

$$\ln(m_{x,t}) = A_x - \alpha_x t + \sum_{i=1}^{N} \beta_x^{(i)} K_t^{(i)} + \Gamma_{t-x}$$

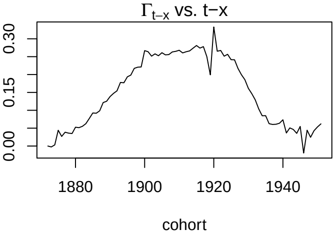
$$-\Delta \ln m_{x,t} = \alpha_x + \sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} + \gamma_{t-x}$$



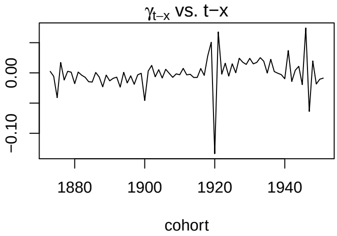


$$\kappa_t^{(i)} = -\Delta K_t^{(i)}$$

$$\Longleftrightarrow$$



$$\gamma_c = -\Delta \Gamma_c$$



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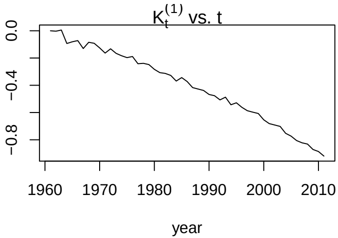
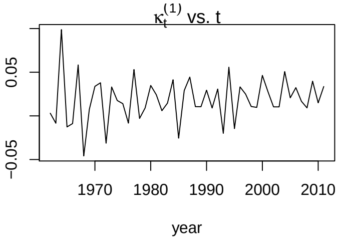
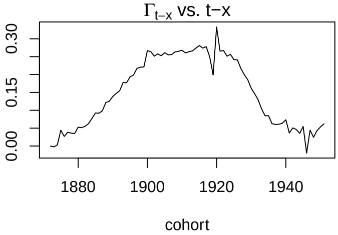


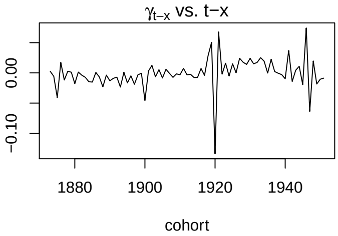


$$\kappa_t^{(i)} = -\Delta K_t^{(i)}$$

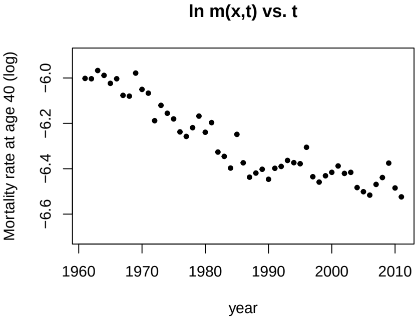
$$\Longleftrightarrow$$

$$\gamma_c = -\Delta \Gamma_c$$





Estimation routes illustration: $\eta_{x,t} = \alpha_x$



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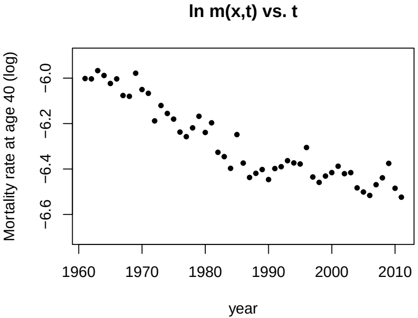
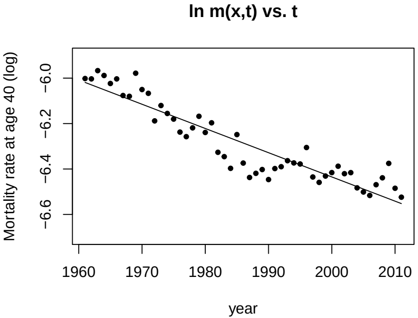


Route 2: "Indirect"

$$\ln m_{x,t} = A_x - \alpha_x t + \epsilon_{x,t}$$

Estimation routes illustration: $\eta_{x,t} = \alpha_x$



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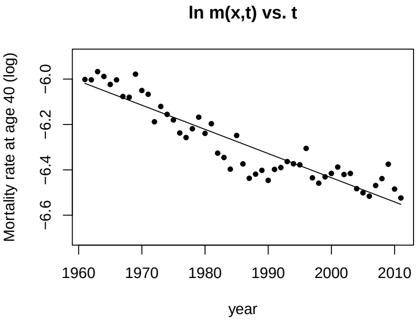

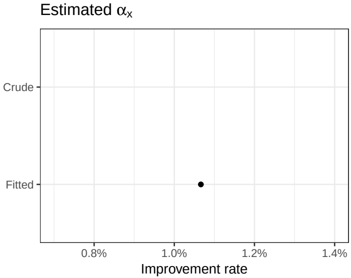

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$$\alpha_x = \frac{\sum_{t=0}^{T} (t - \bar{t})(\ln \hat{m}_{x,t} - \overline{\ln \hat{m}_{x,t}})}{\sum_{t=0}^{T} (t - \bar{t})^2}$$

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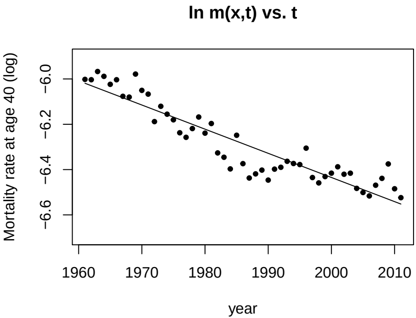

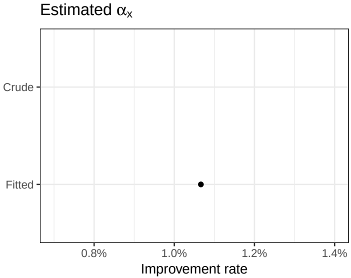

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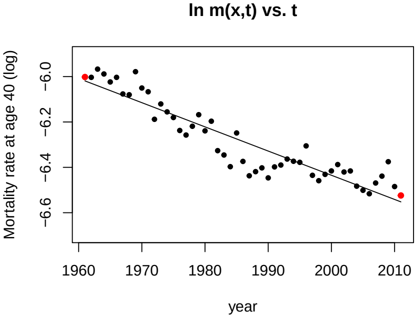

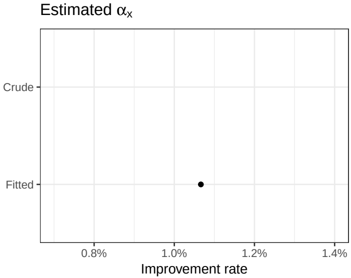

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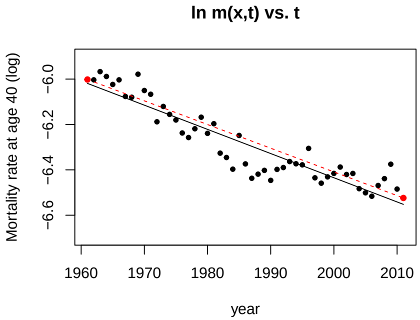

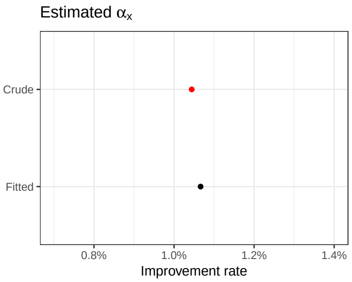

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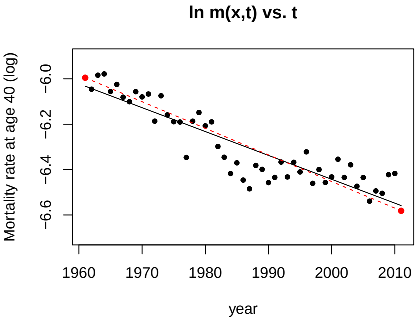

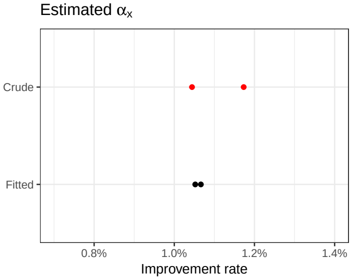

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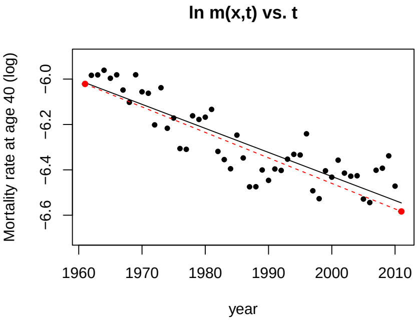

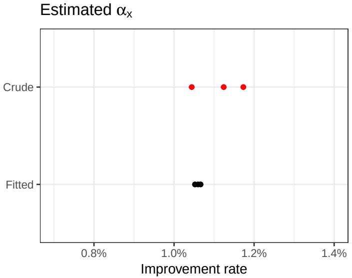

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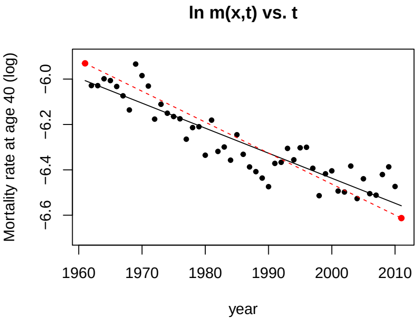

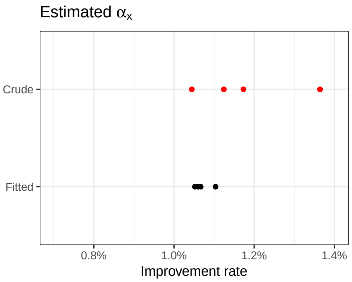

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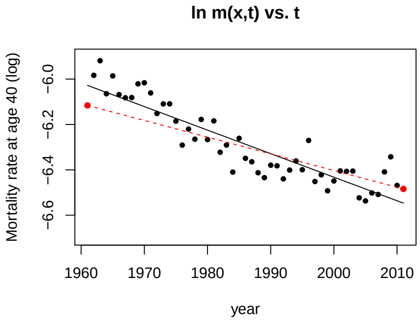
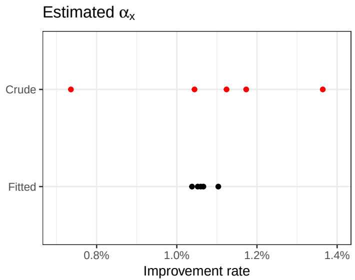
$$\alpha_x = \frac{\ln \hat{m}_{x,0} - \ln \hat{m}_{x,T}}{T}$$

Estimation routes illustration: $\eta_{x,t} = \alpha_x$





Route 2: "Indirect"

$$\ln m_{x,t} = A_x - \alpha_x t + \epsilon_{x,t}$$

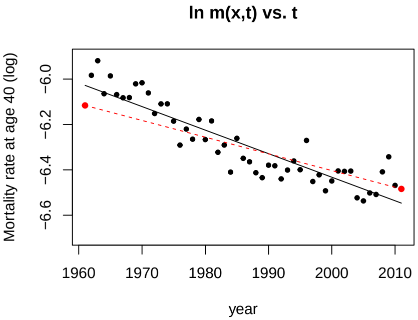
$$\alpha_x = \frac{\sum_{t=0}^{T} (t - \bar{t})(\ln \hat{m}_{x,t} - \overline{\ln \hat{m}_{x,t}})}{\sum_{t=0}^{T} (t - \bar{t})^2}$$

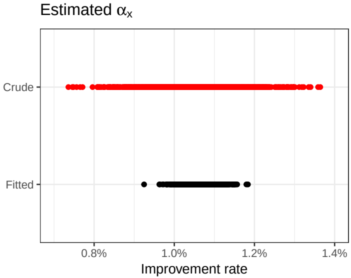
Route 1: "Direct"

$$\alpha_x = \frac{1}{T} \sum_{t=1}^{T} -\Delta \ln \hat{m}_{x,t}$$

$$\alpha_x = \frac{\ln \hat{m}_{x,0} - \ln \hat{m}_{x,T}}{T}$$

Estimation routes illustration: $\eta_{x,t} = \alpha_x$





Route 2: "Indirect"

$$\ln m_{x,t} = A_x - \alpha_x t + \epsilon_{x,t}$$

$$\alpha_x = \frac{\sum_{t=0}^{T} (t - \bar{t})(\ln \hat{m}_{x,t} - \overline{\ln \hat{m}_{x,t}})}{\sum_{t=0}^{T} (t - \bar{t})^2}$$

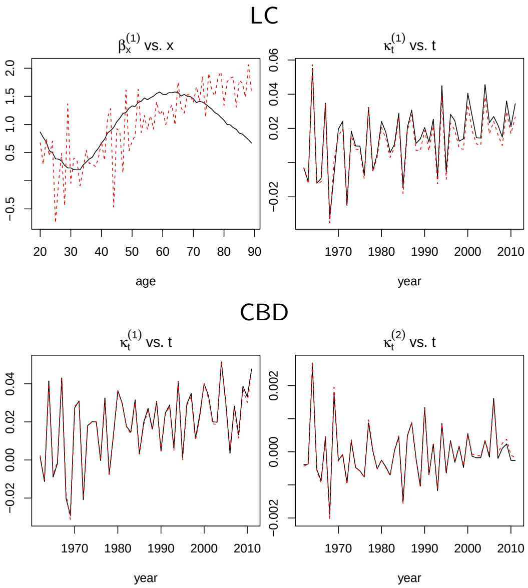
Route 1: "Direct"

$$\alpha_x = \frac{1}{T} \sum_{t=1}^{T} -\Delta \ln \hat{m}_{x,t}$$

$$\alpha_x = \frac{\ln \hat{m}_{x,0} - \ln \hat{m}_{x,T}}{T}$$

Parameter estimates: LC and CBD

Black lines: ''Indirect'' approach, Red lines: "Direct" approach



Key message

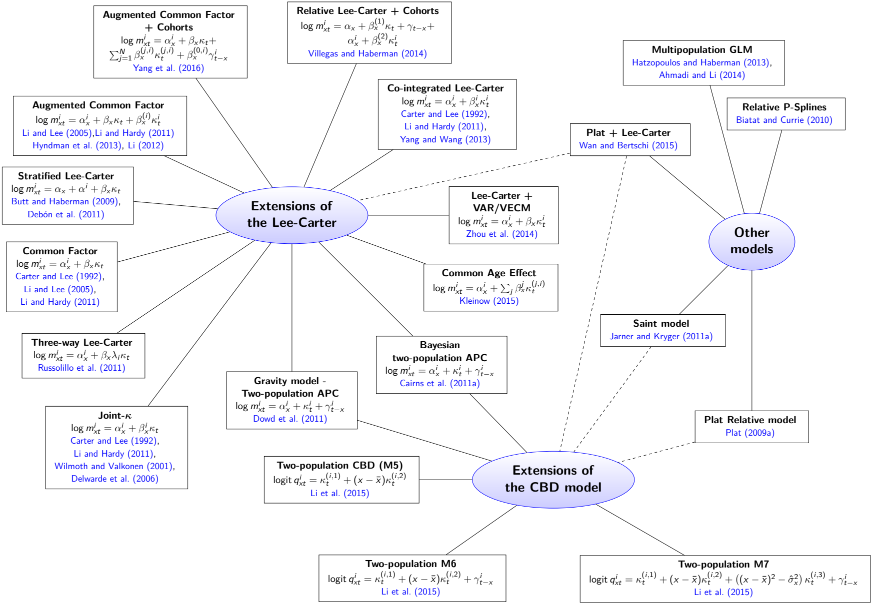
- Improvement rates are an **intuitive** and **natural** way to interpret mortality data
- Compelling reasons for **formulating** and **communicating** projection models in terms of **improvement rates**
- Important **differences** between approached to fitting improvement rate models
  - Considerable **parameter uncertainty**
  - Implication for **projections** and **robustness**
- Compelling reasons for **estimating** projection models in terms of **mortality rates**

Multipopulation (discrete) mortality models

Discussion based on:

Villegas, A. M., Haberman, S., Kaishev, V. K., & Millossovich, P. (2017). A Comparative Study of Two-Population Models for the Assessment of Basis Risk in Longevity Hedges. ASTIN Bulletin, 47(03), 631–679. https://doi.org/10.1017/asb.2017.18

Universe of Multipopulation Models



Coherent forecasts for multiple populations

- See for example Li and Lee (2005), Hyndman et al. (2013)

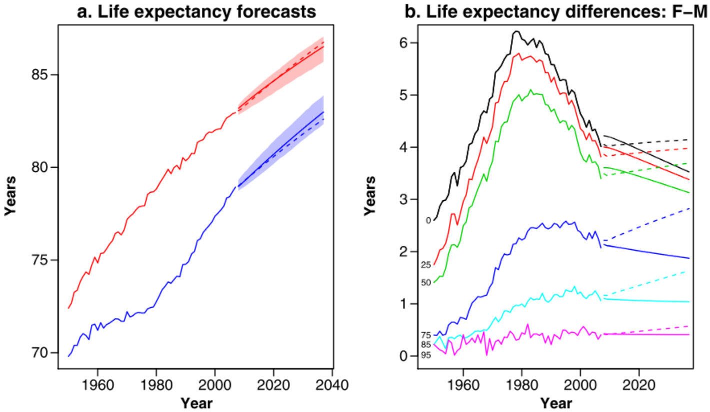
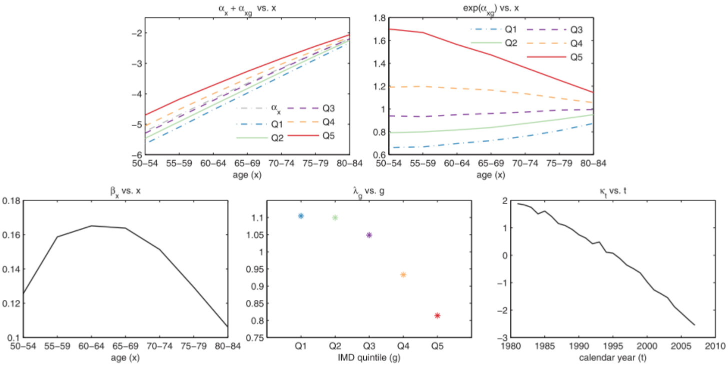


Fig. 5 (Color figure online) Thirty-year life expectancy forecasts for males and females in Sweden using coherent models (solid lines) and independent models (dotted lines). Blue is used for males, and red is used for females

Source: Hyndman et al. (2013)

Quantifying socio-economic differences in mortality

- See for example Villegas and Haberman (2014), Cairns et al. (2016)

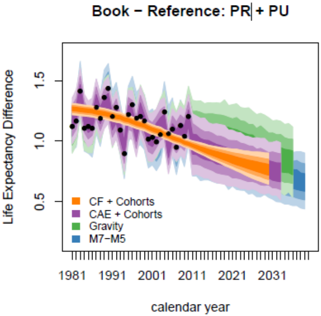


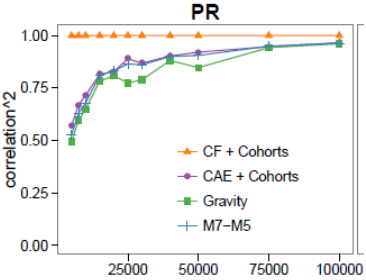
Source: Villegas and Haberman (2014)

$$\log m^i_{xt} = \alpha_x + \alpha^i_x + \beta_x \lambda_i \kappa_t$$

Assessing longevity basis risk

- See for example Li and Hardy (2011), Villegas et al. (2017), Li et al. (2018)

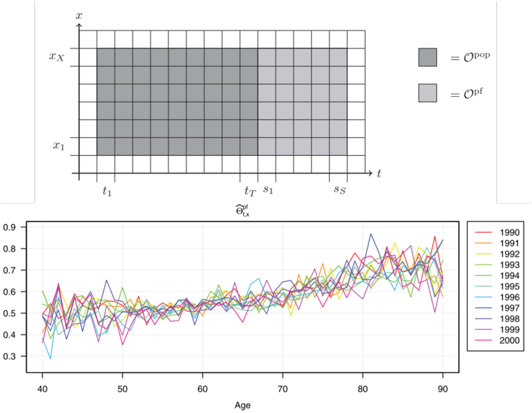




Source: Villegas et al. (2017)

Borrowing strength from a larger population

- Derive the trend from the larger population and model the spread (ratio) between the larger and the smaller population (Jarner and Kryger, 2011b; van Berkum et al., 2017)



Source: van Berkum et al. (2017)

Other developments in multipopulation modelling

- Ensuring consistency between subpopulation and total population projections (Shang and Hyndman, 2017; Shang and Haberman, 2017)
- Modelling of period effects in multipopulation models (Zhou et al., 2014; Li et al., 2015)
- Clustering of mortality trends in multiple-populations (Debón et al., 2017)

Recent developments and outlook

Affine Continuos time mortality models

- Satisfy important requirements for financial applications
- Continuous time mortality modelling framework for insurance application Dahl (2004); Biffis (2005)
- Affine mortality models
  - m factor-model Schrager (2006)
  - Consistent 3-factor model Blackburn and Sherris (2013)
  - Affine processes and multi-cohort factors Xu et al. (2018)
  - two population multiple cohorts (Sherris et al., 2018)

Recent developments

- Applications of statistical machine learning techniques to mortality modelling
  - Sparse Vector Autoregression (Li and Lu, 2017)
  - High Dimensional VAR with elastic nets (Guibert et al., 2017)
  - Gaussian processes (Ludkovski et al., 2016)
  - Neural networks (Hainaut, 2018)
  - Random fields (Doukhan et al., 2017)
- Scope for applying more of these techniques
  - Incorporation of additional information (other populations, macro-economic data, etc)
  - Evaluation and selection of models

Outlook

- Mortality modelling remains an issue of current interest
- Saturation in the traditional Lee-Carter style-approaches
- But many topics remain relevant
  - Modelling of populations of small populations or with scarce data
  - Mortality trends at older ages
  - Understanding of trends in causes of death
- Modelling of dependence between ages, cohorts and populations
- Quantification of trends differences between sociology-economic groups
- Incorporation of individual level data
- Integration between financial models and mortality models (continuous time approaches)

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