Modeling Australian Population Mortality Heterogeneity with Frailty and Markov Ageing Models

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Motivation and Background

- Superannuation and Life annuity markets in Australia
  - Accumulation in DC funds (Corporate Funds, Industry Funds, Retail Funds) with limited longevity products
  - Reviews (Henry Tax Review, Cooper Review) recommend retirement options to include longevity products
  - Heterogeneity in population and impact on longevity products not quantified or well understood
- Heterogeneity in mortality risks in providing (mandated) longevity products for a population
  - Need for use of rating factors for fairness, particularly different industry funds
  - Limited support for Government provision of annuities and community rating for retirement savings

Research Objective

- Comparing models and quantifying mortality heterogeneity of Australian population
  - Frailty model (Vaupel et al. 1979)
    - Heterogeneity modelled by the distribution of frailty factor of individuals at each age
  - Markov ageing model (Lin and Liu 2007)
    - Heterogeneity modelled by the distribution of health states individuals occupy at each age
- Projection of mortality rates at higher ages, taking into account heterogeneity
- Potential impact of heterogeneity on life annuity pricing

Frailty Factor

- Unobserved mortality risk factor fixed at birth, mathematically defined in terms of force of mortality:

$$\mu(x,z) = z \cdot \mu(x,1)$$

- Assumptions: standard force of mortality and frailty distribution
  - Standard force of mortality

$$\mu(x,1) = \alpha \cdot e^{\beta x}$$

  - Frailty distribution
    - Gamma

$$f_Z(z) = \frac{\lambda^k}{\Gamma(k)} \cdot z^{k-1} \cdot e^{-\lambda \cdot z}$$

    - Inverse Gaussian

$$f_Z(z) = (\frac{\delta}{\pi})^{\frac{1}{2}} \cdot e^{\sqrt{4\delta\theta}} \cdot z^{-\frac{3}{2}} \cdot e^{-\theta z - \frac{\delta}{z}}$$

Frailty Model

- Distribution of frailty at age x
  - Gamma distribution

$$f_{Z|X}(z|X=x) = \frac{(\lambda(x))^k}{\Gamma(k)} \cdot z^{k-1} \cdot e^{-\lambda(x)\cdot z}$$

with

$$\lambda(x) = \lambda + H(x,1), \text{and } E[z] = \frac{k}{\lambda(x)}, \quad Var[z] = \frac{k}{(\lambda(x))^2}$$

  - Inverse Gaussian distribution

$$f_{Z|X}(z|X=x) = (\frac{\delta}{\pi})^{\frac{1}{2}} \cdot e^{\sqrt{4\delta\theta(x)}} \cdot z^{-\frac{3}{2}} \cdot e^{-\theta(x)\cdot z - \frac{\delta}{z}}$$

with

$$\theta(x) = \theta + H(x,1), \text{and } E[z] = (\frac{\delta}{\theta(x)})^{\frac{1}{2}}, \quad Var[z] = \frac{1}{2}\sqrt{\frac{\delta}{(\theta(x))^3}}$$

Model Estimation

- Mean frailty approach
  - Assumes the average force of mortality is the cohort force of mortality

$$\bar{\mu}_x = \mu(x,1) \cdot \bar{z}_x$$

  - Number of deaths follows $Poisson(\bar{\mu}_x E_x)$
- Normal approximation for sample mean mortality rates
  - The observed cohort is a sample of size E_x of the population
  - Sample mean force of mortality is approximately normally distributed with
    - Under Gamma distributed frailty

$$E[\hat{\mu}_x] = \mu(x,1) \cdot \frac{k}{\lambda + H(x,1)}, \quad Var[\hat{\mu}_x] = \frac{(\mu(x,1))^2 \cdot k}{E_x \cdot (k + H(x,1))^2}$$

    - Under Inverse Gaussian distributed frailty

$$E[\hat{\mu}_x] = \mu(x,1) \cdot (\frac{\delta}{\theta + H(x,1)})^{\frac{1}{2}}, \quad Var[\hat{\mu}_x] = \frac{(\mu(x,1))^2}{2 \cdot E_x} \sqrt{\frac{\delta}{(\theta + H(x,1))^3}}$$

Markov Ageing Model

- Ageing process modelled in terms of changes in physiological functions
- Studies of human body functions reveal that functional variables decline roughly linearly after age 30
- Physiological age: represents the degree of ageing
  - For any given age there is a range of physiological ages (representing heterogeneity)
  - Higher mortality rates for higher physiological ages

Markov Ageing Model

- Markov process with n transient states and 1 absorbing death state, describing the aging process of human beings

$$\mathbf{\Lambda} = \begin{pmatrix} -(\lambda_1 + q_1) & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & -(\lambda_k + q_k) & \lambda_k & \cdots & 0 \\ 0 & \cdots & 0 & -(\lambda + \gamma + \alpha e^{\beta(k+1)}) & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -(\alpha + e^{\beta n}) \end{pmatrix}$$

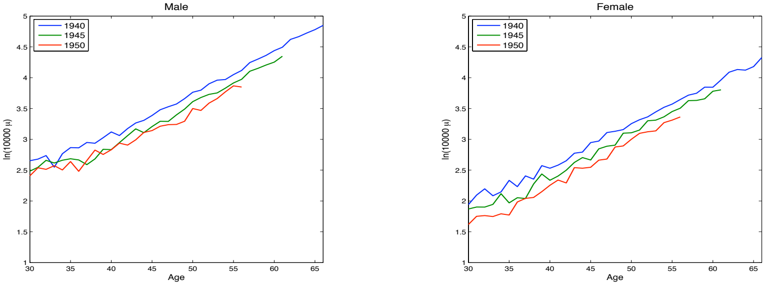
- 4 developmental periods
- $\lambda_i = \lambda$ for $i = 5, 6, ..., n-1$
- Death rates for $i = 5, 6, ...n$

$$q_i = \begin{cases} \gamma + \gamma_1 + \alpha e^{\beta i} & : \text{for } i_1 < i < i_2 \\ \gamma + \alpha e^{\beta i} & : \text{otherwise} \end{cases}$$

- Time to death follows phase-type distribution with $\hat{S}(t) = \alpha exp(\Lambda t)e$
- $\hat{q}_x = \frac{\hat{S}_x - \hat{S}_{x+1}}{\hat{S}_x}$
- Weighted least squares estimation: $\sum_x (q_x - \hat{q}_x)^2 \cdot w_x$

Data for Frailty Model: Human Mortality database

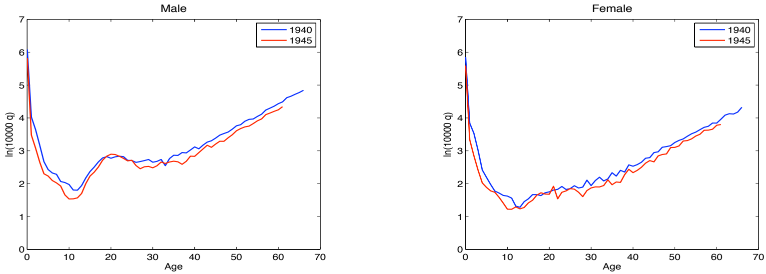
Figure: Observed Cohort Force of Mortality: Log Transform



Note: Cohort force of mortality is estimated by the central death rate for birth cohorts 1940, 1945, and 1950, both male and female

Data for Markov Aging Model: Human Mortality Database

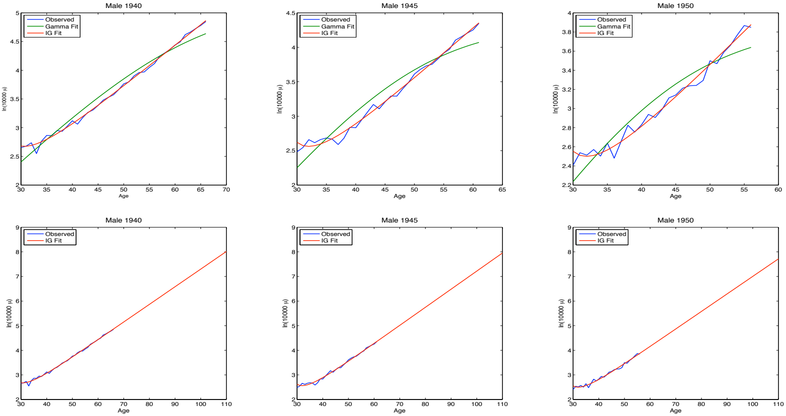
Figure: Observed Cohort Death Probability: Log Transform



Note: Observed death probability estimated from central death rates $q_x = \frac{m_x}{1+1/2m_x}$ for birth cohort 1940, and 1945, both male and female

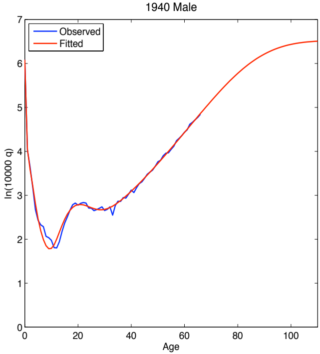
Fitting and Projection for Frailty Model

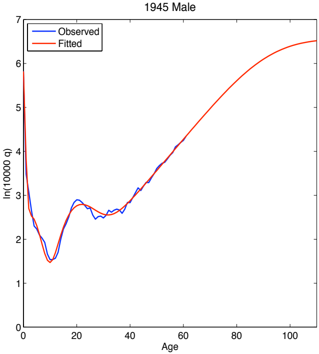
Figure: Observed versus Fitted Cohort Average Force of Mortality: Male



Fitting and Projection for Markov Ageing Model

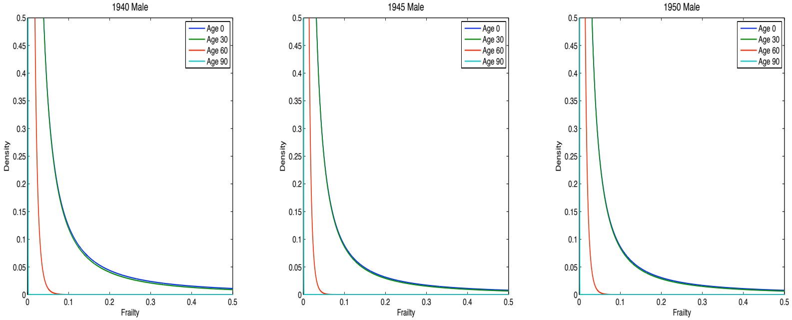
Figure: Observed versus Fitted Death Probability with Projection at Higher Ages





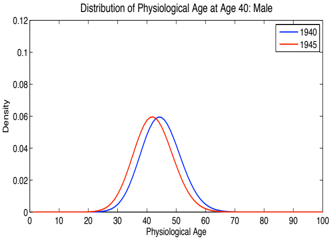
Distribution of Frailty

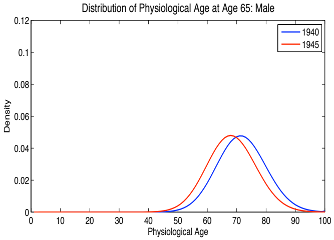
Figure: Distribution of Frailty at different ages



Distribution of Physiological Age

Figure: Distribution of Physiological Age: Male





Annuity Rates for Heterogeneous Individuals: 1940 Male

Table: Annuity Rate for Individuals with Differing Frailty

Frailty	Cohort	0.00005	0.0001	0.0002	0.0005	0.001	0.01
q_{65}	0.012	0.001	0.002	0.004	0.010	0.021	0.189
Whole Life Annuity	\$14.31	\$18.12	\$16.36	\$14.38	\$11.49	\$9.18	\$2.59
Deferred Life Annuity	\$2.36	\$4.32	\$2.94	\$1.66	\$0.47	\$0.08	\$0.00
F(z)		19.40%	38.26%	56.95%	76.49%	86.70%	99.59%

Table: Annuity Rates for Individuals with Differing Physiological Age

Physiological Age j	64	68	73	77	81	94
q_{65}	0.006	0.008	0.011	0.015	0.019	0.047
Whole Life	\$19.44	\$18.31	\$16.83	\$15.63	\$14.44	\$11.15
Deferred Life	\$5.34	\$4.55	\$3.63	\$2.99	\$2.46	\$1.50
F(j)	19.47%	35.49%	59.01%	75.81%	87.83%	99.60%

Note: Whole life annuity prices for individual aged 65. Deferred life annuity prices for individuals aged 65 with first payment at age 85. Real interest rate of 3%.

Summary of Results

- Variability in mortality rates from both heterogeneity and stochastic variation: only heterogeneity considered
- Frailty and Markov Ageing models fitted and compared with Australian population data
- Frailty model
  - Depends heavily on cohort mortality and frailty distribution assumptions
  - Older age improvement from earlier deaths of more frail
  - Frailty distribution less variable at older ages
- Markov Ageing Model
  - Physiological age more readily calibrated to ageing process
  - Heterogeneity increases with age
  - Provides reasonable basis for quantifying heterogeneity
- Markov Ageing model preferred over Frailty model