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Yang Shen[1], Michael Sherris[2] and Jonathan Ziveyi[3]

[1]Department of Mathematics and Statistics, Tornoto, Canada, email: skyshen87@gmail
[2] School of Risk and Actuarial Studies, UNSW Business School and Chief Investigator, CEPAR, UNSW Australia, email: m.sherris@unsw.edu.au
[3]School of Risk and Actuarial Studies, UNSW Business School, email: j.ziveyi@unsw.edu.au

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Yang Shen[†], Michael Sherris[‡] and Jonathan Ziveyi[§]

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Abstract

We present a numerical approach to the pricing of guaranteed minimum maturity benefits embedded in variable annuity contracts in the case where the guarantees can be surrendered at any time prior to maturity that improves on current approaches. Surrender charges are important in practice and are imposed as a way of discouraging early termination of variable annuity contracts. We formulate the valuation framework and focus on the surrender option as an American put option pricing problem and derive the corresponding pricing partial differential equation by using hedging arguments and Itô's Lemma. Given the underlying stochastic evolution of the fund, we also present the associated transition density partial differential equation allowing us to develop solutions. An explicit integral expression for the pricing partial differential equation is then presented with the aid of Duhamel's principle. Our analysis is relevant to risk management applications since we derive an expression for the sensitivity of the guarantee fees with respect to changes in the underlying fund value (called the "delta"). We provide algorithms for implementing the integral expressions for the price, the corresponding early exercise boundary and the delta of the surrender option. We quantify and assess the sensitivity of the prices, early exercise boundaries and deltas to changes in the underlying variables including an analysis of the fair insurance fees.

JEL Classification: C63, G12, G22, G23

Keywords: Guaranteed minimum maturity benefits, Surrender options, Numerical integration

[†]skyshen87@gmail.com; Department of Mathematics and Statistics, Toronto, Ontario, Canada, M3J 1P3

[‡]m.sherris@unsw.edu.au; CEPAR and School of Risk and Actuarial Studies, UNSW, Sydney, NSW 2052, Australia.

[§]Corresponding Author: j.ziveyi@unsw.edu.au; School of Risk and Actuarial Studies, UNSW, Sydney, NSW 2052, Australia.

1 Introduction

A variable annuity is a contract between a policyholder and an insurance company where the policyholder agrees to pay either a single premium or a stream of periodic premium payments during the accumulation phase in return for minimum guaranteed payments from the insurer during the annuitization phase. Variable annuities are long-term insurance contracts designed to meet retirement and other long-range goals. The guarantees embedded in these contracts offer protection against the possibility of the policyholders outliving their assets. These guarantees exhibit financial option-like features. There are two major classes of guarantees: guaranteed minimum death benefits (GMDBs) and guaranteed minimum living benefits (GMLBs).

GMDBs are usually offered during the accumulation phase and they provide guaranteed payments of the accumulated value of premiums to beneficiaries in the event of untimely death of the policyholder. GMLBs provide principal and/or income guarantees to protect the policyholder's income from declining during the annuitization phase. GMLBs can be further categorized into three subclasses, namely, GMxB, where "x" stands for maturity (M), income (I) and withdrawal (W). A GMMB guarantees the return of the premium payments made by the policyholder or a higher stepped-up value at the end of the accumulation period. On the other hand, a GMIB guarantees a lifetime income stream when a policyholder annuitizes the GMMB regardless of the underlying investment performance. A GMWB guarantees a stream of income payments, regardless of the contract account value and payments can be guaranteed for a specified period or for the lifetime of the policyholder.

Insurance companies usually charge proportional fees on variable annuity contracts as a way of funding the guarantees. If such fees are too high relative to the performance of the fund, the policyholder has an incentive to surrender the contract, or the guarantee, prior to maturity in return for a surrender benefit. The benefits are usually net of surrender/penalty charges which is a way of discouraging early termination of the contracts. In this paper we will focus on the valuation of the GMMB rider embedded in a variable annuity contract in the case where the guarantee can be surrendered anytime prior to maturity. This valuation problem has received little attention in the literature. Shen and Xu (2005) consider the fair valuation of equity-linked policies with interest rate guarantees in the presence of surrender options using the partial differential equation approach. The valuation problem is reduced to a free boundary problem

which can be solved using a variety of numerical techniques such as finite difference schemes. The authors also derive explicit Black-Scholes type solutions (see also Black and Scholes (1973)) for the case where there are no surrender options. Constabile et al. (2008) consider a similar valuation problem and devise a binomial tree approach to determine fair premium values.

Bauer et al. (2008) provide a general framework for consistent pricing of various types of guarantees embedded in variable annuities which are currently traded in the market. They present an extensive analysis of the guarantees by incorporating the possibility of surrendering the contracts anytime prior to maturity. Bacinello (2013) considers the pricing of participating life insurance policies with surrender options using a recursive binomial tree approach. Bernard et al. (2014) use techniques developed in Carr et al. (1992) to derive the representation of the optimal surrender strategy for a variable annuity contract embedded with guaranteed minimum accumulation benefits (GMAB). In deriving the pricing framework, Bernard et al. (2014) treat the entire variable annuity contract (the mutual fund plus the GMAB) as a single underlying asset and then derive the corresponding pricing formulas. Advances in the valuation of GMWBs embedded in variable annuities and surrender options in participating life insurance policies are also found in Milevsky and Salisbury (2006), Hyndman and Wenger (2014), Siu (2005) and references therein.

In this paper, we provide new and alternative derivations and representations of the GMMB embedded in a variable annuity contract where the policyholder can surrender the guarantee anytime prior to maturity. In contrast to the approach used in Bernard et al. (2014), who treat the variable annuity contract as a single product, we decompose the annuity contract into a mutual fund and a guarantee. We then focus on valuing the guarantee, and in so doing, the impact of various parameters on the GMMB can be explicitly assessed. We then provide a detailed numerical analysis of the guarantee and compare it with American put options. This approach readily allows insurance companies and annuity providers to compare variable annuity contracts with traditional mutual funds.

The early exercise feature on the GMMB makes the valuation problem similar to that encountered in American put option pricing. This leads us to presenting the valuation problem of the option embedded in the GMMB as an optimal stopping time problem. Using well established arguments developed in Jacka (1991), Myneni (1992) and El Karoui and Karatzas (1993), we

transform the optimal stopping time problem into a free-boundary problem leading to an equivalent representation in Shen and Xu (2005). By incorporating Jamshidian (1992)'s techniques for transforming the free-boundary problem to a non-homogeneous partial differential equation (PDE), we derive the general integral solution of the PDE with the aid of Duhamel's principle. This differs from the probabilistic approach adopted by Bernard et al. (2014). We can readily derive expressions for the corresponding early exercise boundary and the delta, which is the sensitivity of the option price with respect to changes in the fund value.

Numerical results quantifying the early exercise boundary profiles, premiums to be charged per guarantee and the corresponding delta profiles are presented, in contrast to Bernard et al. (2014) where only the optimal exercise boundary and fair insurance charges are given in their numerical examples.

We confirm that when surrender charges are relatively high, it is optimal to delay exercising the guarantee early as a significant amount of surrender benefits can be consumed by early termination charges. We perform numerical comparisons between standard American put options and surrender options by assessing the impact of continuously compounded insurance charges and surrender fees on the premium to be levied on guarantees. We quantify the extent that premium values for surrender options are consistently higher than the corresponding American put option prices reflecting the effects of surrender charges and the fees levied for providing variable annuities. This highlights the higher premiums that insurers can earn from the sale of variable annuity contracts as compared to premium proceeds from selling standard American put options.

The rest of the paper is structured as follows. Section 2 sets up the model dynamics and relates the valuation of GMMB to American option pricing. The general integral solution of the valuation problem is presented in Section 3 together with expressions for the corresponding early exercise boundary and delta of the option. Algorithms for numerically implementing the valuation expressions are presented in Section 4. Numerical results are then presented in Section 5 followed by concluding remarks in Section 6. Lengthy derivations and proofs have been relegated to the Appendices.

2   Problem Statement

Let $[0, T]$ be a finite horizon and $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space carrying a one-dimensional standard Brownian motion $W = (W_t)_{0 \leq t \leq T}$. Here, $\mathbb{Q}$ is the risk-neutral probability measure. Throughout this paper, we denote by $\mathbb{E}[\cdot]$ the expectation under $\mathbb{Q}$. Let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be the natural filtration which is generated by the standard Brownian motion W and satisfies the usual conditions. We consider a variable annuity contract embedded with a guaranteed minimum maturity benefit (GMMB) rider where the policyholder can choose to surrender the guarantee anytime prior to maturity. The policyholder's premium is invested in a fund consisting of units of an underlying asset, $S = (S_t)_{0 \leq t \leq T}$, whose risk-neutral evolution is governed by a geometric Brownian motion model as

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

where $r > 0$ and $\sigma > 0$ are the risk-free interest rate and the volatility of the underlying asset, respectively. The fund value at time t is denoted as

$$F_t = e^{-ct} S_t, \tag{2}$$

where c is the continuously compounded insurance charge levied on the fund value by the variable annuity provider (see Milevsky and Salisbury (2001)). It can be shown that the risk-neutral dynamics of the fund value, $F = (F_t)_{0 \leq t \leq T}$, satisfies

$$dF_t = (r - c)F_t dt + \sigma F_t dW_t. \tag{3}$$

Using the risk-neutral arguments, initial value of the variable annuity contract, X_0, net of initial expense charges can be represented as the expected discounted value of the terminal payoff, that is

$$\begin{aligned} X_0 &= \mathbb{E}\left[e^{-rT} \max(G_T, F_T) \right] \\ &= \mathbb{E}\left[e^{-rT} F_T \right] + \mathbb{E}\left[e^{-rT} \max(G_T - F_T, 0) \right], \end{aligned} \tag{4}$$

where F_T is the fund value at maturity time T, given that the insurance fee charged during $t \in [0, T]$ is equal to c, and G_T is the guaranteed value at maturity of the contract Equation (4) is made up of two components: the first being the expected discounted value of the terminal

fund value and the second being a put option which is equivalent to a guarantee rider to be exercised only if the terminal fund value is below the guaranteed amount, G_T.

Remark 2.1. *Bernard et al. (2014) are concerned with finding the fair insurance charge c^* such that the pricing equation $F_0 = \mathbb{E}\left[e^{-rT} \max(G_T, F_T^{c^*})\right]$, where $F_T^{c^*}$ denotes the fund value at T when the insurance charge is c^*, is fulfilled. Our paper focuses on calculating the price of the variable annuity contract for a given insurance charge c. Indeed, the problem of finding the fair insurance charge can be nested in the framework of our paper. That is, setting $X_0 \equiv F_0$ in equation (4), we can determine the fair insurance charge endogenously similar to determining implied volatility from option price data.*

Whilst equation (4) is akin to a standard European put option, the policyholder may find it optimal to exercise the guarantee prior to maturity. Among such events include the guarantee being deep out-of-the-money. In such a case, the policyholder will be better off not to hold the guarantee as the probability of it ending up in-the-money will be very low. The second case is when the continuation value of the guarantee is equal to the immediate exercise value prior to maturity. In the event of the guarantee being exercised prior to maturity, early termination/ surrender charges will be applied such that the fund value to be used for computing premiums for the guarantee reduces to

$$(1 - \kappa_t)F_t, \tag{5}$$

with κ_t being the penalty percentage charged for exercising the guarantee at time t. Milevsky and Salisbury (2001) interpret κ_t as an incentive to remain in the variable annuity contract and as a mechanism for funding the guarantee. As in Bernard et al. (2014), we assume that κ_t is exponentially decreasing and is equal to $1 - e^{-\kappa(T-t)}$. This implies that if the policyholder surrenders the guarantee at $t \in [0, T]$, the resulting benefits can be represented as

$$e^{-\kappa(T-t)}F_t. \tag{6}$$

For optimality in exercising the guarantee (surrender option) early, we will assume that the inequality $\kappa < c < r$ holds otherwise the option will be held to maturity. As will be explained below, this inequality ensures that the surrender charges will not exceed/erode the benefits of exercising the option early. From equation (4), the variable annuity account at maturity can be

represented as

$$F_T + \max(G_T - F_T, 0), \tag{7}$$

and at any time prior to maturity[1], this can be represented as

$$X_t = e^{-r(T-t)}\mathbb{E}\left[F_T|\mathcal{F}_t\right] + \operatorname*{ess\,sup}_{t \leq \tau^* \leq T} e^{-r(\tau^*-t)}\mathbb{E}\left[\max(G_T - e^{-\kappa(T-\tau^*)}F_{\tau^*}, 0)|\mathcal{F}_t\right], \tag{8}$$

where the supremum is taken over all stopping times, τ^*, over $[t, T]$. The first component on the right-hand side of equation (8) is the discounted expectation of the maturity value of the fund which can be trivially solved since the dynamics of F is governed by the GBM as presented in (3). The second component is a typical American put option which we reproduce here as

$$P(t, F) = \operatorname*{ess\,sup}_{t \leq \tau^* \leq T} e^{-r(\tau^*-t)}\mathbb{E}\left[\max(G_T - e^{-\kappa(T-\tau^*)}F_{\tau^*}, 0)|\mathcal{F}_t\right]. \tag{9}$$

The valuation problem in equation (9) is essentially an optimal stopping time problem.

Proposition 2.2. *The fair value of the American put option in* (9) *is a unique strong solution of the free boundary problem*

$$\frac{\partial P}{\partial t} + (r - c)F\frac{\partial P}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 P}{\partial F^2} - rP = 0, \tag{10}$$

where $B_t < e^{-\kappa(T-t)}F < \infty$ with B_t being the optimal early exercise boundary below which the put option will be exercised. The partial differential equation (PDE) (10) is solved subject to the boundary and terminal conditions

$$P(T, F) = \max(G_T - F, 0), \tag{11}$$

$$\lim_{F \to \infty} P(t, F) = 0, \quad t \in [0, T], \tag{12}$$

$$P(t, B_t) = G_T - B_t, \quad t \in [0, T], \tag{13}$$

$$\lim_{F \to B_t e^{\kappa(T-t)}} \frac{\partial P}{\partial F} = -e^{-\kappa(T-t)}, \quad t \in [0, T]. \tag{14}$$

Proof. Refer to the proof of Proposition 2.7 in Jacka (1991). Also, the PDE can be derived by applying standard hedging arguments and Itô's Lemma to a portfolio consisting of a put option, $P(t, F)$, and optimal units of the underlying fund, F, using the arguments presented in Black and Scholes (1973) where dynamics of F is governed by the SDE presented in equation (3). □

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[1]Due to the possibility of surrendering the contract early.

The underlying asset domain for the PDE (10) is bounded below by the early exercise boundary, B_t. Jamshidian (1992) shows that one can consider an unbounded domain for the underlying asset by noting that at any time, $t \in [0, T]$, below the early exercise boundary

$$P(t, F) = G_T - e^{-\kappa(T-t)}F_t, \tag{15}$$

implying that the following equation holds

$$\frac{\partial P}{\partial t} + (r-c)F\frac{\partial P}{\partial F} + \frac{1}{2}\sigma^2 F^2\frac{\partial^2 P}{\partial F^2} - rP = (c-\kappa)e^{-\kappa(T-t)}F - rG_T. \tag{16}$$

Above the early exercise boundary, the option will be held and the option price will satisfy equation (10). Combining equations (10) and (16), and using the fact that $P(t, F)$ is a continuously differentiable function in F at B_t yield

$$\frac{\partial P}{\partial t} + (r-c)F\frac{\partial P}{\partial F} + \frac{1}{2}\sigma^2 F^2\frac{\partial^2 P}{\partial F^2} - rP + \mathbb{1}_{\{F \le B_t e^{\kappa(T-t)}\}}\left[rG - (c-\kappa)e^{-\kappa(T-t)}F\right] = 0, \tag{17}$$

where $\mathbb{1}_{\{x \le B\}}$ is an indicator function which is equal to one if $x \le B$ or zero otherwise. Equation (17) is defined on the domain $0 < F < \infty$ and this motivates the solution approach we present below. For notational convenience, we have suppressed the dependance of the guarantee on maturity time T in (17). In the rest of the paper, we will also write $G \equiv G_T$ unless stated otherwise.

We now explain the economic intuition of the inhomogenous term in equation (17). Suppose that at time t, $e^{-\kappa(T-t)}F_t < G$ implying that it is optimal to exercise the option. If the option is not exercised now, it can still be exercised at the next instant $t + dt$ because the fund value function is a continuous-time process. By not exercising now, the policyholder will lose the instantaneous interest $rGdt$, but would save on the early termination charges $(c-\kappa)e^{-\kappa[T-(t+dt)]}F_{t+dt}dt$ such that the total net loss would be

$$[rG - (c-\kappa)e^{-\kappa[T-(t+dt)]}F_{t+dt}]dt. \tag{18}$$

However if the variable annuity provider were to compensate the policyholder with an equivalent amount, then the policyholder will be indifferent to delaying exercise to the next instant. Suppose that the two counterparties agree to prohibit exercise until maturity, and in return, the policyholder is continuously compensated by the amount equivalent to equation (18) for delaying optimal exercise when it is optimal to do so. Whenever $e^{-\kappa(T-t)}F_t$ is above the early exercise

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boundary, the compensation will be zero since exercising is not optimal. This leads to the conclusion that the guaranteed minimum maturity benefit embedded in a variable annuity with a surrender option is a typical American put option consisting of a European option component plus a contract that pays a continuous cashflow presented in equation (18). An equivalent quantity to the continous cashflow in Jamshidian (1992) is termed the "cost-of-carry" of an option, which is compensation for delayed exercise.

In what follows, we let $\tau = T - t$ denote the time-to-maturity. Associated with the SDE (3) is the corresponding transition density function denoted here as $H(\tau, F; F_0)$. This function represents the probability of passage from state F at time-to-maturity, τ, to state F_0 at maturity of the variable annuity contract. The transition density function satisfies the backward Kolmogorov PDE

$$\frac{\partial H}{\partial \tau} = (r - c)F\frac{\partial H}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 H}{\partial F^2}, \tag{19}$$

where $0 \leq F < \infty$. Equation (19) is solved subject to the terminal condition

$$H(0, F; F_0) = \delta(F - F_0), \tag{20}$$

where $\delta(\cdot)$ is a Dirac delta function.

Now let $F = e^x$ and $P(\tau, e^x) \equiv V(\tau, x)$ such that the PDE in (17) transforms to

$$\frac{\partial V}{\partial \tau} = \phi\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - rV + \mathbb{1}_{\{x \leq \ln B + \kappa\tau\}} \left[rG - (c - \kappa)e^{-\kappa\tau}e^x \right], \tag{21}$$

where $\phi = (r - c - \frac{1}{2}\sigma^2)$.

Applying similar transformations to the transition density PDE and letting $H(\tau, e^x) \equiv U(\tau, x)$ equation (19) becomes

$$\frac{\partial U}{\partial \tau} = \phi\frac{\partial U}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2}. \tag{22}$$

Equation (22) is solved subject to the terminal condition

$$U(0, x; x_0) = \delta(x - x_0), \tag{23}$$

where $x_0 = \ln F_0$.

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3 Main Results

Having outlined the problem statement in Section 2, we now present the integral representations of premium values of surrender options and the corresponding early exercise boundary which needs to be determined as part of the solution. We also present the delta expression for the surrender option which quantifies the sensitivity of the premium values to changes in the underlying fund.

Proposition 3.1. *The general solution of equation (21) can be represented as*

$$V(\tau, x) = V_E(\tau, x) + V_P(\tau, x), \tag{24}$$

where

$$V_E(\tau, x) = e^{-r\tau} \int_{-\infty}^{\infty} (G - e^w)^+ U(\tau, x; w) dw, \tag{25}$$

and

$$V_P(\tau, x) = \int_0^\tau e^{-r(\tau - \xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau - \xi)} [rG - (c - \kappa)e^{-\kappa(\tau - \xi)}e^w] U(\tau - \xi, x; w) dw d\xi. \tag{26}$$

The first term on the right hand side of equation (24) is the European option component and the second term is the early exercise premium component. The function, $U(\tau, x; w)$ is the univariate normal transition density function, that is

$$U(\tau, x; w) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{ -\frac{(x - w + \phi\tau)^2}{2\tau\sigma^2} \right\}, \tag{27}$$

which is a solution to the transition density function PDE (22).

Proof. The proof proceeds by substituting equation (24) into the the PDE (21) and using Duhamel's principle, and then proceeding as detailed in Appendix B of Chiarella and Ziveyi (2014). □

Proposition 3.2. *The explicit solution of equation (24) can be represented as*

$$V(\tau, x) = V_E(\tau, x) + V_P(\tau, x), \tag{28}$$

where

$$V_E(\tau, x) = Ge^{-r\tau}\mathcal{N}(-d_2(\tau, x, G)) - e^x e^{-c\tau}\mathcal{N}(-d_1(\tau, x, G)), \tag{29}$$

and

$$V_P(\tau, x) = rG \int_0^\tau e^{-r(\tau-\xi)} \mathcal{N}\left(-d_2\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi$$

$$- (c - \kappa)e^x \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi, \tag{30}$$

with $\mathcal{N}(d)$ being a cumulative normal distribution function and

$$d_1(\tau, x, G) = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2(\tau, x, G) = d_1(\tau, x, G) - \sigma\sqrt{\tau}. \tag{31}$$

Proof. Refer to Appendix A.1. □

The early exercise premium component in equation (30) is implicitly dependent on the early exercise boundary, B_τ, and hence needs to be determined as part of the solution. By using the value-matching condition presented in equation (13), the early exercise boundary is the solution to the implicit Volterra integral equation

$$G - B_\tau = Ge^{-r\tau}\mathcal{N}(-d_2(\tau, \ln B_\tau + \kappa\tau, G)) - B_\tau e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau, \ln B_\tau + \kappa\tau, G))$$

$$+ rG \int_0^\tau e^{-r(\tau-\xi)} \mathcal{N}\left(-d_2\left(\tau - \xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi \tag{32}$$

$$- (c - \kappa)B_\tau e^{\kappa\tau} \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau - \xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi.$$

Proposition 3.3. *The exercise boundary at maturity is*

$$B_0 = \min\left(1, \frac{r}{c - \kappa}\right) G. \tag{33}$$

Proof. Refer to Appendix A.2 □

From our earlier assumption that $\kappa < c < r$, it turns out that

$$B_0 = G, \tag{34}$$

which is the guaranteed fund value at maturity. At every other instant prior to maturity the early exercise boundary is determined by solving equation (32) recursively.

The sensitivity of the guarantee premium values to changes in the underlying factors is crucial when dealing with variable annuities. In option pricing, a family of such sensitivities is termed "Greeks". The most commonly used Greek in option pricing is the delta, which measures the

degree to which an option is exposed to shifts in the price of the underlying asset. In our current situation, we will interpret the delta as the sensitivity of the premium values to changes in the fund value. A negative delta implies that the guarantee fee decreases for every dollar increase in the fund value, so one is effectively short the fund through a long put option position. Delta works best for short-term options and does not tell the probability of how often the fund will hit the early exercise boundary before expiration time, but merely the probability that it expires in the money. We present the delta expression in the next proposition.

Proposition 3.4. *The delta of the surrender option can be represented as*

$$D(\tau, x) = D_E(\tau, x) + D_P(\tau, x), \tag{35}$$

where

$$D_E(\tau, x) = -e^{-c\tau} \mathcal{N}(-d_1(\tau, x, G)), \tag{36}$$

and

$$
\begin{aligned}
D_P(\tau, x) = \ & -\frac{rG}{\sigma e^x} \int_0^\tau e^{-r(\tau-\xi)} n\left(-d_2\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) \frac{1}{\sqrt{\tau-\xi}} d\xi \\
& - (c - \kappa) \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi \\
& + \frac{c-\kappa}{\sigma} \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} n\left(-d_1\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) \frac{1}{\sqrt{\tau-\xi}} d\xi.
\end{aligned}
\tag{37}
$$

with $n(d)$ being a density function of the standard normal distribution.

Proof. Refer to Appendix A.3. ☐

4 Numerical Implementation

Having formulated the guarantee equations as presented in (29) and (30), together with the early exercise boundary equation in (32) and the corresponding delta in (35) , we now outline numerical techniques for solving this system of equations.

We adopt the numerical integration techniques developed in Huang et al. (1996) who implement the American put option pricing framework developed in Kim (1990). Similar techniques have also been employed in Chiarella and Ziveyi (2014) when pricing American spread call options

where the dynamics of the underlying assets evolve under the influence of geometric Brownian motion processes.

The European option component $V_E(\tau, x)$ involves a cumulative normal distribution function which can be easily handled by a variety of in-built software packages. However, such software packages cannot be readily applied to the early exercise premium component, $V_P(\tau, x)$, as this term involves the entire history of the early exercise boundary, B_τ, which needs to be iteratively solved at each instant.

The early-exercise premium component also involves an integral with respect to the running time-to-maturity, ξ, which also makes use of the entire history of the early exercise boundary at each point in time.

To implement equations (29), (30), (32) and (35), we first discretise the time domain, τ, into M equally spaced subintervals of length $h = T/M$ and apply the extended Simpson's rule.

The numerical algorithm is initiated at maturity, $\tau_0 = 0$ where the exercise boundary is equal to the guarantee value presented in equation (34). This serves as the starting value for tracking the early exercise boundary backwards in time. We denote the time-steps as $\tau_m = mh$, for $m = 1, 2, \cdots, M$. The discretised version of the variable annuity guarantee is then represented as

$$V(mh, x) = V_E(mh, x) + V_P(mh, x), \tag{38}$$

where

$$V_E(mh, x) = Ge^{-r(mh)}\mathcal{N}(-d_2(mh, x, G)) - e^x e^{-c(mh)}\mathcal{N}(-d_1(mh, x, G)), \tag{39}$$

and

$$
\begin{aligned}
V_P(mh, x) = {} & hrG \sum_{j=0}^{m} e^{-r(m-j)h} \mathcal{N}\left(-d_2\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right) w_j \\
& - h(c - \kappa)e^x \sum_{j=0}^{m} e^{-(c+\kappa)(m-j)h} \mathcal{N}\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right) w_j. \tag{40}
\end{aligned}
$$

Here, w_j are the weights of Simpson's rule for integration in the ξ direction while h is the corresponding step size. At each time step, we need to implicitly determine the early exercise boundary $B(mh)$ which also depends on its entire history up to the current time step. Root

finding techniques are employed to accomplish this task. The discretised version of the value-matching condition equation can be shown to be

$$B(mh) = G - V(mh, x),\tag{41}$$

where $V(mh, x)$ is presented in equation (38). Likewise, the discretised version of the delta presented in equation (35) can be represented as

$$D(mh, x) = D_E(mh, x) + D_P(mh, x),\tag{42}$$

where

$$D_E(mh, x) = -e^{-c(mh)}\mathcal{N}(-d_1(mh, x, G)),\tag{43}$$

and

$$
\begin{aligned}
D_P(mh, x) = &-\frac{hrG}{\sigma e^x}\sum_{j=0}^{m} e^{-r(m-j)h}n\left(-d_2\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)\frac{1}{\sqrt{(m-j)h}}w_j \\
&- h(c-\kappa)\sum_{j=0}^{m}e^{-(c+\kappa)(m-j)h}\mathcal{N}\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)w_j \\
&+ \frac{h(c-\kappa)}{\sigma}\sum_{j=0}^{m}e^{-(c+\kappa)(m-j)h}n\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)\frac{1}{\sqrt{(m-j)h}}w_j.
\end{aligned}\tag{44}
$$

5   Numerical Results

We now present numerical results obtained from implementing the framework presented in Section 4. For all numerical experiments that follow, we use the parameter set presented in Table 1 unless stated otherwise. Our choice of parameters is consistent with those used in Bernard et al. (2014) for their numerical experiments. Numerical experiments help in shedding light on the sensitivities of the early exercise boundary and guarantee fees to changes in the underlying variables. The time domain has been discretised into 100 time steps, implying that $h = 0.15$ years when $\tau = 15$.

Figure 1 shows the impact of varying the surrender charge, κ, on the early exercise boundary. We note that the early exercise boundary increases as the level of κ increases. Increasing levels of κ result in higher guarantee fees, making it prohibitively expensive to surrender the guarantee early as further revealed in Table 2. From this table, we note that when $c = 3\%$ for instance,

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Parameter	G	τ	σ	r	c	κ
Value	100	15	0.20	0.05	0.03	0.01

Table 1: Parameters for the GMMB Rider.

varying the surrender charges from $\kappa = 0$ to $\kappa = 3\%$ (last three columns of the table) results in a gradual increase of guarantee fees. Hence, when surrender charges are relatively high, it is advisable to delay exercising the guarantee early as a significant amount of surrender benefits may end up being used to settle for the early termination charges.
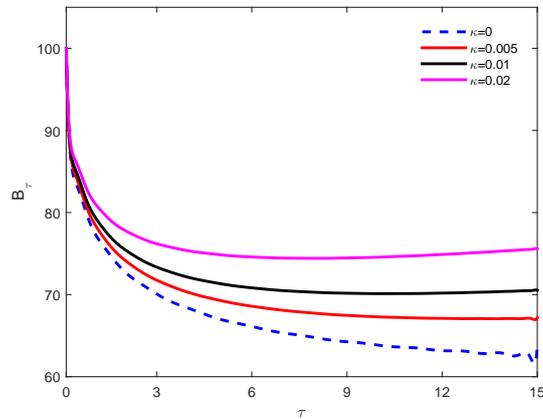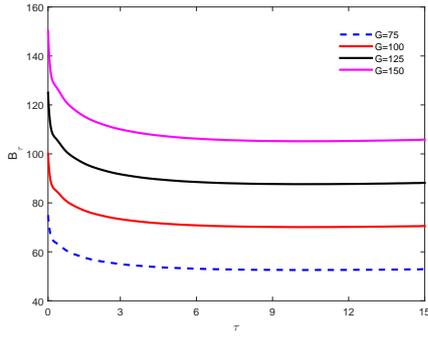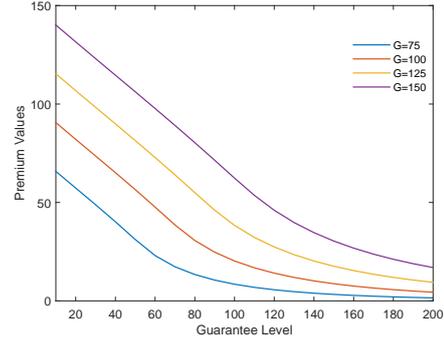

Figure 1: The impact of varying the surrender charges on the early exercise boundary.

We next assess the impact of varying the guarantee level on the early exercise boundary and the corresponding effects on the premiums to be charged for provision of such guarantees in Figure 2. In this figure, we vary the guarantee level and keep all other parameters constant as presented in Table 1. Increasing the guarantee levels result in higher early exercise boundaries as revealed in Figure 2(a), that is, when the guarantee level is increased, the corresponding exercise boundary is shifted upwards. Likewise, as the minimum guarantee level increases, the associated premiums also increase as presented in Figure 2(b). It naturally makes sense for insurers to charge higher premiums for increasing levels of minimum guarantees so that they can use the proceeds to devise appropriate hedging strategies. Such strategies can be used to offset the increased exposure levels in the event of the guarantees ending up being exercised by the policyholders.

(a) Assessing how the early exercise boundary changes by varying the minimum guarantee.
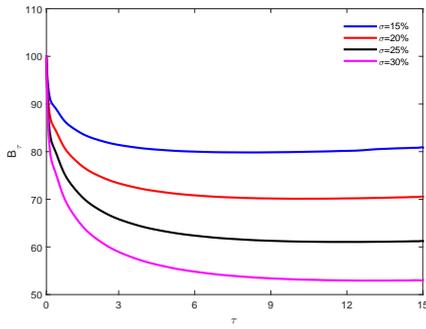

(b) Assessing the effects of varying the guarantee level on the premium values

Figure 2: Assessing the impacts of varying the guarantee levels on the early exercise boundary and premium values. All other parameters are as provided in Table 1

One of the most important concepts when trading options is volatility. Volatility measures how fast and how much prices of the underlying asset move. As such it is important to understand how premiums to be levied on guarantees respond to changes in volatility. Figure 3 shows the effects of increasing volatility on the early exercise boundary and the premium values. We note from Figure 3(a) that as volatility increases, the corresponding early exercise boundary decreases. In option pricing theory, it is also well established that an increase in volatility results in an increase in option prices. This is depicted in Figure 3(b) where premium values for near in-the-money and out-of-the-money guarantees increase with increasing levels of volatility. On the other hand, since the option is more valuable as volatility increases, the early exercise boundary will be decreased so that the proceed brought from the early termination, i.e. $G - B_\tau$ can compensate for the higher option value.

It is also of interest to investigate how the early exercise boundary and the corresponding guarantee premiums respond to changes in interest rates. From Figure 4(a) we note that as the level of interest rates is increased, early exercise boundary also increases. The rule of thumb when trading put options is that higher risk free interest rates mean cheaper put option prices, all things being equal. This is revealed in Figure 4(b) where we note a decrease in premiums as interest rates gradually increase from 3% to 5% for near at-the-money and out-of-the-money options. The interest rates are fully priced for deep in-the-money guarantees hence the convergence of premiums as depicted in the figure. Contrary to the case of volatility as

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(a) The impact of varying the volatility level on the early exercise boundary.
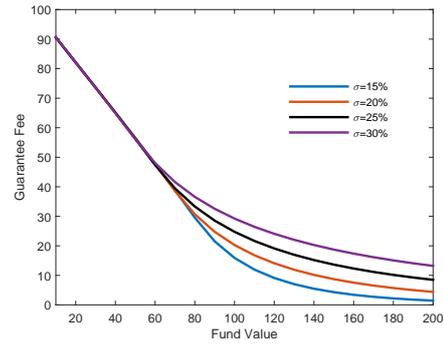
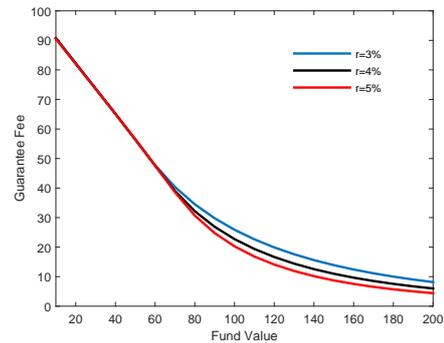
(b) Surrender option premiums for varying σ.

Figure 3: The effects of varying the volatility on the early exercise boundary and premium values. All other parameters are as presented in Table 1

shown in Figures 3(a) and 3(b), the decreased option value will lead to the increased early exercise boundary (see Figure 4(a)) as interest rate is increased.
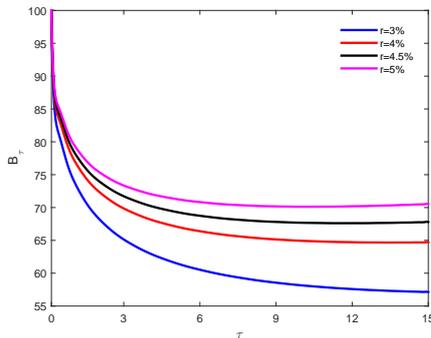
(a) The impact of varying interest rates on the early exercise boundary.

(b) The impact of varying interest rates on guarantee premiums.

Figure 4: The effects of varying interest rates on the early exercise boundary and premium values. All other parameters are as presented in Table 1

It is of interest to analyze the premium differences between the surrender option and a standard American put option. The formula for a standard put option on a non-dividend paying stock can be recovered by setting κ and c equal to zero in equation (28). In our analysis we subtract the implied standard American put option values from the associated guarantee values obtained by using the parameter set in Table 1. The standard American put option prices have been

generated by implementing the algorithm devised in Kallast and Kivinukk (2003). The results of this analysis are presented in Figure 5. We note that at-the-money guarantee premiums are consistently higher than the corresponding standard American put option prices under the Black and Scholes (1973) framework. Surrender options are more expensive than standard American put options, reflecting the effects of surrender charges and continuously compounded insurance charges levied on the fund value. This quantifies the extent that insurers realize higher premiums from selling variable annuity contracts as compared to premiums from equivalent standard American put options traded in the financial markets.

In Table 2 we further elaborate how premium values change for various combinations of κ and c. As pointed above, we note that prices corresponding to the standard American put option case ($c = 0$ & $\kappa = 0$) are consistently lower than cases where we have non-zero fees and surrender charges.
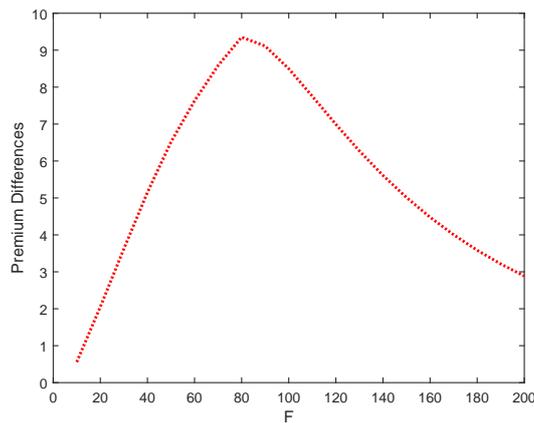
Figure 5: Premium Differences which is equal to the surrender option value minus the standard American call option value.

In Table 3, we present the sensitivities of guarantee premiums to changes on the underlying fund value for maturities ranging from 6 months to 15 years. We note that deltas for deep in-the-money guarantees with shorter maturities are very close to -1 implying that for every \$1 increase in the fund value, the guarantee premium will decrease by \$1. For deep-in-the-money guarantees, the deltas gradually drift from -1 with increasing maturities. This behaviour is reversed for out-of-the-money guarantees whose deltas become more negative with increasing

maturities. The negative delta predicts how much value the insurers will gain in guarantee premiums if the fund value falls a dollar in value. This implies that insurers will collect more premiums when selling guarantees with deltas close to -1 and less for deltas close to zero.

Table 2: Premium values when $G = 100$ with all other parameters as presented in Table 1.

Fund Value	$c = 0, \kappa = 0$	$c = 0.01, \kappa = 0.01$	$c = 0.03, \kappa = 0$	$c = 0.03, \kappa = 0.02$	$c = 0.03, \kappa = 0.03$
40	60	65.2470	60	70.2339	74.7314
50	50	56.6159	50	62.8030	68.4884
60	40	47.7468	40	54.8843	61.6104
70	30	38.5665	30.9631	46.3405	53.8201
80	21.3315	29.6139	24.5171	37.7474	45.2010
90	15.6349	22.5599	19.8098	30.6772	37.2821
100	11.7707	17.5265	16.2646	25.2593	31.0000
110	9.0604	13.8665	13.5269	21.0425	26.0470
120	7.1019	11.1369	11.3704	17.7024	22.0809
130	5.6521	9.0587	9.6440	15.0185	18.8631
140	4.5569	7.4486	8.2431	12.8354	16.2234
150	3.7152	6.1828	7.0934	11.04101	14.0371
160	3.0588	5.1749	6.1406	9.5526	12.2110

To sum up this section, we report the fair insurance charge c^* implied by the pricing equation (8) for the variable annuity as in Remark 2.1. As reported in Table 4, for given levels of the surrender charge, κ, and volatility, σ, the fair fee decreases with increasing maturity. Also the fair insurance fee increases with the increasing level of volatility. However, as the level of κ is gradually increased from zero, the fair insurance fee decreases. This quantifies how the surrender charges and insurance charges received by the insurer interact.

Finally, in Table 5 we compare the accuracy of the fair insurance charges obtained using our approach and that in Bernard et al. (2014). When $\kappa = 0$, $r = 0.03$, $\sigma = 0.20$ and $T = 15$, the fair insurance charge reported in Bernard et al. (2014) is $c^* = 0.0091$. With all parameters being the same as those of Bernard et al. (2014), the fair insurance charge from our approach is $c^* = 0.014082$. By substituting the values of c^* into equation 8 and set $t = 0$, it is shown in Table 5 that the corresponding fund value is $X_0 = 105.1758$ when $c^* = 0.0091$, while $X_0 = 100.0053$ when $c^* = 0.014082$. The fair insurance fees computed using our approach differ from those in

Table 3: Delta values when $G = 100$ with all other parameters as presented in Table 1.

Fund Value	$T = 0.5$	$T = 1$	$T = 5$	$T = 10$	$T = 15$
20	-0.99484791	-0.989725298	-0.949623193	-0.897362991	-0.845196661
40	-0.99484791	-0.989723709	-0.943734302	-0.892194197	-0.853653871
60	-0.994806799	-0.988795379	-0.952461492	-0.926987922	-0.903160608
80	-0.994181179	-0.945265697	-0.734921426	-0.691619866	-0.686952558
100	-0.456332196	-0.440967322	-0.397656989	-0.38784721	-0.39079854
120	-0.076696552	-0.135035882	-0.21952068	-0.233500335	-0.241847212
140	-0.005844514	-0.029891602	-0.122438974	-0.147318762	-0.158547139
160	-0.000263757	-0.005346292	-0.068960555	-0.096189125	-0.108387754

Bernard et al. (2014) but since the resulting X_0 should be close to 100, provide more accurate numerical computations. Also, we note from Table 4 that the resulting guarantee premium (a portion of the variable annuity used to fund the GMMB) is higher using our approach[2].

Table 4: Fair insurance charges for varying κ, σ and T.

$T = 10$			$T = 15$		
κ	σ	c^*	κ	σ	c^*
	0.20	0.020340		0.20	0.014082
0	0.25	0.028939	0	0.25	0.020147
	0.30	0.038128		0.30	0.026602
	0.20	0.019009		0.20	0.013203
0.005	0.25	0.027330	0.005	0.25	0.019066
	0.30	0.036337		0.30	0.025454
	0.20	0.017855		0.20	0.012419
0.01	0.25	0.025885	0.01	0.25	0.018125
	0.30	0.034685		0.30	0.024369

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[2]This is computed from equation (38) with $c^* = 0.014082$ and also using $c^* = 0.0091$ as reported in Bernard et al. (2014)

Table 5: Accuracy of fair insurance charges

	Bernard et al. (2014)	Our Approach
c^*	0.0091	0.014082
X_0	105.1758	100.0053
Premium	17.93514	19.0465

6 Conclusion

In this paper we have presented an approach to valuing the surrender option in a guaranteed minimum maturity benefit (GMMB) rider embedded in a variable annuity contract using numerical integration techniques. Our approach differs from and improves on those currently used. It formulates the valuation problem as an optimal stopping problem and then presented a systematic approach to transforming the optimal stopping time problem into a free-boundary problem. We employ Jamshidian (1992)'s techniques to transform the homogenous free-boundary problem to a non-homogeneous partial differential equation (PDE) whose general integral solution can readily be found by using Duhamel's principle. Semi-closed form integral expressions for the guarantee premium, the early exercise boundary and the corresponding delta of the GMMB are derived and implemented using Simpson's rule.

We present numerical results that quantify the impact of surrender fees and insurance charges on the guarantee premiums, the free-boundary and the delta of the underlying surrender option. We also analyze the behaviour of the fair insurance charges for varying levels of surrender charges, volatility and time to maturity for the guarantee. Numerical comparisons are provided between the surrender option value and that for valuing standard American put options as presented in Kallast and Kivinukk (2003). Surrender fees and charges result in premiums for GMMBs that are higher than for standard American put options.

This paper has focused on valuing the guarantee component of a variable annuity contract and presenting a comprehensive analysis of this option value. This differs from the work presented in Bernard et al. (2014) who present their results for the full variable annuity contract as a single product. By taking this approach, the impact of the GMMB in a variable annuity contract

is explicitly assessed in terms of its price, delta, the insurance charge and the surrender fee. This provides clearer guidance to practitioners for the risk management of these guarantees. In particular analysis of the delta receives little attention in the valuation of embedded options for annuity and insurance products.

The theoretical and numerical results, particularly for the risk management of these guarantees where the delta is important, is an important new contribution in the analysis of the GMMB rider benefit for variable annuities. We also provide a numerical methodology that allows the study of sensitivity of other guarantees in these insurance contracts with respect to underlying risk factors.

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A Appendices

A.1 Proof of Proposition 3.2

We first derive the explicit form of the European option component, $V_E(\tau, x)$, as follows

$$V_E(\tau, x) = e^{-r\tau} \int_{-\infty}^{\infty} (G - e^w)^+ U(\tau, x; w) dw$$

$$= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\ln G} (G - e^w) \exp\left\{-\frac{[x - w + \phi\tau]^2}{2\sigma^2\tau}\right\} dw$$

$$\equiv A_1(\tau, x) - A_2(\tau, x), \tag{A1}$$

where

$$A_1(\tau, x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\ln G} G \exp\left\{-\frac{[x - w + \phi\tau]^2}{2\sigma^2\tau}\right\} dw, \tag{A2}$$

and

$$A_2(\tau, x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\ln G} e^w \exp\left\{-\frac{[x - w + \phi\tau]^2}{2\sigma^2\tau}\right\} dw. \tag{A3}$$

In simplifying $A_1(\tau, x)$, we let $y = \frac{x - w + \phi\tau}{\sigma\sqrt{\tau}}$ such that $dw = -\sigma\sqrt{\tau}dy$. Also

$$w = \ln G \Rightarrow y = \frac{x - \ln G + \phi\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad w = -\infty \Rightarrow y = \infty.$$

Equation (A2) then becomes

$$A_1(\tau, x) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} G e^{-\frac{y^2}{2}} dy = G e^{-r\tau} \mathcal{N}\left(-d_2\left(\tau, x, G\right)\right), \tag{A4}$$

where $\mathcal{N}\left(-d_2\left(\tau, x, G\right)\right)$ is a cumulative Normal distribution function with

$$d_2(\tau, x, K) = \frac{x - \ln G + \phi\tau}{\sigma\sqrt{\tau}}. \tag{A5}$$

The second component, $A_2(\tau, x)$ is simplified by first re-writing it as follows

$$A_2(\tau, x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln G} \exp\left\{w - \frac{(x - w + \phi\tau)^2}{2\sigma^2\tau}\right\} dw.$$

By completing the square and simplifying the above equation we obtain

$$A_2(\tau, x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln G} \exp\left\{\frac{[x + \phi\tau]^2}{-2\sigma^2\tau}\right\} \exp\left\{\frac{w^2 - 2w[x + \{r - c + \frac{1}{2}\sigma^2\}\tau]}{-2\sigma^2\tau}\right\} dw, \tag{A6}$$

which can also be represented as

$$A_2(\tau, x) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln G} \exp\left\{\frac{[x + \phi\tau]^2}{-2\sigma^2\tau}\right\} \exp\left\{\frac{[x + (r - c + \frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau}\right\} \exp\left\{\frac{[x - w + (r - c + \frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\} dw$$

$$= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln G} e^{x + (r-c)\tau} \exp\left\{\frac{[x - w + (r - c + \frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\} dw$$

$$= \frac{e^{-c\tau} e^x}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\ln G} \exp\left\{\frac{[x - w + (r - c + \frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\} dw. \tag{A7}$$

Now, we let

$$y = \frac{x - w + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \Rightarrow dw = -\sigma\sqrt{\tau}.$$

As for the integral limits, $w = \ln G \Rightarrow y = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and $w = -\infty \Rightarrow y = \infty$, hence

$$A_2(\tau, x) = \frac{e^{-c\tau}e^x}{2\pi} \int_{-d_1}^{\infty} e^{-\frac{y^2}{2}} dy = e^{-c\tau}e^x \mathcal{N}(-d_1(\tau, x, G)), \tag{A8}$$

with

$$d_1(\tau, x, G) = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{A9}$$

By comparing equations (A9) and (A5) it can be shown that

$$d_2(\tau, x, G) = d_1(\tau, x, G) - \sigma\sqrt{\tau}. \tag{A10}$$

Combining the results in equations (A4) and (A8) yield the European option component presented in equation (29) of Proposition 3.2.

Next we derive the explicit form of the early exercise premium by simplifying the expression presented in equation (26) which we reproduce here as

$$V_P(\tau, x) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} [rG - (c-k)e^{-\kappa(\tau-\xi)}e^w] U(\tau - \xi, x; w) dw d\xi. \tag{A11}$$

The derivations proceed as those for the European option component case. We split the above equation in two parts by letting

$$V_P(\tau, x) = I(\tau, x) - II(\tau, x), \tag{A12}$$

where

$$I(\tau, x) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} rG \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{-\frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\right\} dw d\xi, \tag{A13}$$

and

$$II(\tau, x) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} (c-k)e^{-\kappa(\tau-\xi)}e^w \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{-\frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\right\} dw d\xi. \tag{A14}$$

In simplifying the first component, $I(\tau, x)$, we let $y = \frac{x - w + \phi(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}$, such that $dw = -\sigma\sqrt{\tau-\xi}dy$. Also

$$w = \ln B + \kappa(\tau - \xi) \Rightarrow y = \frac{x - \ln B - \kappa(\tau-\xi) + \phi(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}$$

and $w = -\infty \Rightarrow y = \infty$. Equation (A13) then becomes

$$I(\tau, x) = \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} rG e^{-\frac{y^2}{2}} dy d\xi = rG \int_0^\tau e^{-r(\tau-\xi)} \mathcal{N}\left(-d_2\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi. \tag{A15}$$

The second component, $II(\tau, x)$, is simplified by first rearranging it as follows

$$II(\tau, x) = \int_0^\tau \frac{(c-\kappa)e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B + \kappa(\tau-\xi)} e^{-\kappa(\tau-\xi)} \exp\{w - \frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\} dw d\xi.$$

By completing the square and simplifying the above equation we obtain

$$II(\tau, x) = \int_0^\tau \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B + \kappa(\tau-\xi)} \exp\left\{\frac{[x + \phi(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\}$$
$$\times \exp\left\{\frac{w^2 - 2w[x + \{r - c + \frac{1}{2}\sigma^2\}(\tau-\xi)]}{-2\sigma^2(\tau-\xi)}\right\} dw d\xi, \tag{A16}$$

which can also be represented as

$$
\begin{aligned}
II(\tau,x) &= \int_0^\tau \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\left\{\frac{[x+\phi(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} \exp\left\{\frac{[x+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\} \\
&\quad \times \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi \\
&= \int_0^\tau \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\left\{x+(r-c)(\tau-\xi)\right\} \\
&\quad \times \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi \qquad\qquad (A17) \\
&= \int_0^\tau \frac{(c-\kappa)e^x}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\left\{-(c+\kappa)(\tau-\xi)\right\} \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi.
\end{aligned}
$$

Now, we let

$$
y = \frac{x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)}{\sigma\sqrt{(\tau-\xi)}} \Rightarrow dw = -\sigma\sqrt{(\tau-\xi)}.
$$

When $w = \ln B + \kappa(\tau-\xi) \Rightarrow y = \frac{x-\ln B-\kappa(\tau-\xi)+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)}{\sigma\sqrt{(\tau-\xi)}}$ and $w = -\infty \Rightarrow y = \infty$, hence

$$
\begin{aligned}
II(\tau,x) &= \int_0^\tau \frac{(c-\kappa)e^x}{2\pi} \int_{-d_1}^\infty e^{(c+\kappa)(\tau-\xi)} e^{-\frac{y^2}{2}} dyd\xi \\
&= (c-\kappa)e^x \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi. \qquad (A18)
\end{aligned}
$$

Combining equations (A15) and (A18) yields the results presented in (30).

A.2 Proof of Proposition 3.3

Rearrange equation (32) as

$$
\begin{aligned}
\frac{B_\tau}{G} &= \left[e^{-r\tau}\mathcal{N}(-d_2(\tau,\ln B_\tau+\kappa\tau,G)) + r\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1\right] \\
&\quad \times \left[e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau,\ln B_\tau+\kappa\tau,G)) + (c-\kappa)e^{\kappa\tau}\right. \\
&\quad\quad \left. \times \int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1\right]^{-1}. \qquad (A19)
\end{aligned}
$$

For simplicity, we let

$$
M_1(\tau) = e^{-r\tau}\mathcal{N}(-d_2(\tau,\ln B_\tau+\kappa\tau,G)) + r\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1, \quad (A20)
$$

and

$$
\begin{aligned}
M_2(\tau) &= e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau,\ln B_\tau+\kappa\tau,G)) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A21) \\
&\quad + (c-\kappa)e^{\kappa\tau}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1.
\end{aligned}
$$

Next we wish to find $\lim_{\tau \to 0} \frac{B_\tau}{G}$ with the aid of l'Hôpital's rule. To this end, we first calculate the derivatives of $M_1(\tau)$ and $M_2(\tau)$ as

$$
\begin{aligned}
M_1'(\tau) = & -re^{-r\tau}\mathcal{N}(-d_2(\tau, \ln B_\tau + \kappa\tau, G)) - e^{-r\tau}\mathcal{N}'(-d_2(\tau, \ln B_\tau + \kappa\tau, G))\frac{\partial}{\partial\tau}d_2(\tau, \ln B_\tau + \kappa\tau, G) \\
& + r\mathcal{N}\left(-d_2\left(0, \ln B_\tau + \kappa\tau, B_\tau\right)\right) + r\int_0^\tau \left\{ -re^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau - \xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) \right. \\
& \left. - e^{-r(\tau-\xi)}\mathcal{N}'\left(-d_2\left(\tau-\xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial\tau}d_2\left(\tau-\xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right) \right\}d\xi, \quad \text{(A22)}
\end{aligned}
$$

and

$$
\begin{aligned}
M_2'(\tau) = & -(c-\kappa)e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau, \ln B_\tau + \kappa\tau, G)) - e^{-(c-\kappa)\tau}\mathcal{N}'(-d_1(\tau, \ln B_\tau + \kappa\tau, G))\frac{\partial}{\partial\tau}d_1(\tau, \ln B_\tau + \kappa\tau, G) \\
& + (c-\kappa)e^{\kappa\tau}\mathcal{N}\left(-d_1\left(0, \ln B_\tau + \kappa\tau, B_\tau\right)\right) \quad \text{(A23)} \\
& + (c-\kappa)e^{\kappa\tau}\int_0^\tau \left\{ -ce^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) \right. \\
& \left. - e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}'\left(-d_1\left(\tau-\xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial\tau}d_1\left(\tau-\xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right) \right\}d\xi.
\end{aligned}
$$

For $i = 1, 2$, we notice that

$$
\lim_{\tau \to 0} d_i(\tau, \ln B_\tau + \kappa\tau, G) = \begin{cases} 0, & \text{if } B_0 = G, \\ \infty, & \text{if } B_0 > G. \end{cases} \quad \text{(A24)}
$$

Thus if $B_0 > G$, we have

$$
\lim_{\tau \to 0} M_1'(\tau) = \frac{r}{2}, \quad \text{(A25)}
$$

and

$$
\lim_{\tau \to 0} M_2'(\tau) = \frac{c - \kappa}{2}. \quad \text{(A26)}
$$

Therefore, taking limit in (A19) and using l'Hôpital's rule yield

$$
B_0 = \min\left(1, \frac{r}{c - \kappa}\right)G. \quad \text{(A27)}
$$

A.3   Proof of Proposition 3.4

The derivation for $D_E(\tau, x)$ is the same as that for delta of a European put option. We only derive $D_P(\tau, x)$. Differentiating $V_P(\tau, x)$ with respect to the underlying fund value yields

$$
\begin{aligned}
D_P(\tau, x) = & -rG\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}'\left(-d_2\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial F}d_2\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)d\xi \\
& -(c-\kappa)\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi \\
& +(c-\kappa)e^x\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}'\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial F}d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)d\xi \\
= & -\frac{rG}{\sigma F}\int_0^\tau e^{-r(\tau-\xi)}n\left(-d_2\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi \\
& -(c-\kappa)\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi \\
& +\frac{c-\kappa}{\sigma}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}n\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi. \quad \text{(A28)}
\end{aligned}
$$